Preliminaries

T : a complete consistent theory, in language L (countable) <u>with infinite models.</u>

that is, $T = Th(M)$, M : L-structure, infinite

L denotes also the set of formulas of language L

$M = (|M|; \ldots)$, but M also denotes $|M|$ (for brevity)

$\emptyset \neq$ universe of M

usually we omit $|\cdot|$ in $|M|$.

$M \supseteq A$: a set of parameters.

$L_n(A) = \{\varphi(x_1, \ldots, x_n, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in L,\ \bar{a} \subseteq A\}$

$L(A) = \bigcup_n L_n(A)$, also $L(A)$: language L extended by names for elements of A.

$L_n(A)$: Lindenbaum algebra.

[formally : on $L_n(A)$: $\varphi \sim \psi \iff T(A) \vdash \varphi \leftrightarrow \psi$
$\iff M \models \varphi \leftrightarrow \psi$

here : $\underbrace{T(A)}$ $= Th(M, a)_{a \in A}$

a complete theory in language $L(A)$.

$L_n(A)/_\sim$: a Boolean algebra
(Lindenbaum algebra)

$[\varphi]_\sim \wedge [\psi]_\sim = [\varphi \wedge \psi]_\sim$ etc.

shortly : $L_n(A)$ denotes also $L_n(A)/_\sim$.

$S_n(A) = \{$complete n-types over A, in $M\}$.
in variables $x_1, \dots, x_n$

consistent n-type over A $\rightsquigarrow$ proper filter in $L_n(A)$

An n-type $p(\bar{x})$ over A is <u>complete</u> if
$\cap$
$L_n(A)$
- $p(\bar{x})$: consistent type
- $\forall \varphi(\bar{x}) \in L_n(A)$ $(\varphi(\bar{x}) \in p$ or $(\neg\varphi(\bar{x})) \in p)$

$S(A) := S_1(A)$
(default)

$S_n(A)$: topological space :
for $\varphi(\bar{x}) \in L_n(A)$

$$[\varphi] = \{p \in S_n(A) : \varphi \in p\}$$

basic open set [clopen]
$\shortparallel$
closed and open

$S_n(A)$: compact Hausdorff space, 0-dimensional
(i.e. basis of clopen sets)

complete n-types$/A$ $\rightsquigarrow$ ultrafilters in $L_n(A)$

So $S_n(A) = S(L_n(A))$, the Stone space of ultrafilters in $L_n(A)$

- the ~~topo~~ topology on $S_n(A)$ $=$ the Stone space topology.

For $p(\bar{x}) \in S_n(A)$

realizes p

$p(M) = \{ \bar{a} \in M^n : \underbrace{\bar{a} \text{ satisfies } p}_{} \}$

$\bar{a} \models p$, i.e. $M \models \varphi(\bar{a})$ for every $\varphi(\bar{x}) \in p(\bar{x})$

- The same notation for arbitrary type (also incomplete)
- A formula $\varphi(\bar{x}) \in L(M)$: a special case of a type $\{\varphi(\bar{x})\}$.

$\varphi(M) = \ldots$

- When $p \in S_n(A)$, $\bar{a} \subseteq M$ and $\bar{a} \models p$, then

$$p = tp^M(\bar{a}/A) = tp(\bar{a}/A) = \{\varphi(\bar{x}) \in L_n(A) : M \models \varphi(\bar{a})\}.$$

<u>Example</u> Assume $p(\bar{x})$: a consistent type over M.

Then $\exists N \succcurlyeq M$, $\underbrace{p \text{ is realized in } N}_{}$

i.e. $p(N) \neq \emptyset$.

From now on "a type" means "a consistent type".

Def A type $p(\bar{x})$ over A is <u>isolated</u>, if:

$\exists\, \varphi(\bar{x}) \in L_n(A)$
① $\varphi(\bar{x})$ is consistent (with T), i.e. $\varphi(M) \neq \emptyset$ $(\Leftrightarrow T(A) \vdash \exists \bar{x}\, \varphi(\bar{x}))$

(2) ~~$\varphi(M)$~~ $\forall \psi(\bar{x}) \in p(\bar{x})$ $\varphi(M) \subseteq \psi(M)$
$\Updownarrow$
$T(A) \vdash \varphi(\bar{x}) \to \psi(\bar{x})$

symbolically: $\varphi(\bar{x}) \vdash p(\bar{x})$

- When $p(\bar{x})$: a complete type over A, then:

$p(\bar{x})$ is isolated $\Leftrightarrow$ p is isolated in $S_n(A)$ in the topological sense (i.e. $\{p\}$ is open)

<u>Tarski–Vaught test</u>

Assume $A \subseteq M$. Then $A = |N|$ for some $N \prec M$ <u>iff</u>

$$\forall \varphi(x) \in L_1(A)\, [\varphi(M) \neq \emptyset \Rightarrow \varphi(M) \cap A \neq \emptyset]$$

Construction of an elementary submodel of M containing A:

- $A_n \subseteq M$, $n < \omega$, increasing chain of sets

recursive construction:

$A_0 = A$

$A_n \subseteq A_{n+1} \subseteq M$ such that $\forall \psi(x) \in L_1(A_n)$
$[\psi(M) \neq \emptyset \Rightarrow \psi(M) \cap A_{n+1} \neq \emptyset]$

$A_\infty = \bigcup_{n<\omega} A_n$ satisfies TV-test.

Omitting types theorem

Assume $p_n(\bar{x}_n)$, $n<\omega$: a family of non-isolated types in theory T, over $\emptyset$. Then:

$(\exists M \models T)$ M omits every p_n [i.e. $p_n(M)=\emptyset$]

Assume $\underset{A}{\overset{\cup\!\shortmid}{M}}, N \models T$

Def. $f: A \to N$ is <u>elementary</u> ($f: A \xrightarrow{\equiv} N$) if:

$$\forall \bar{a} \subseteq A \ \forall \varphi(\bar{x}) \in L \ \left(M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a})) \right.$$

$$\left. (\Leftrightarrow tp^M(\bar{a}) = tp^N(f(\bar{a}))\right)$$

Elementary diagram of $A \subseteq M$:

$D_e(A) = T(A) = Th(M, a)_{a \in A}$

<u>Remark</u> $f: A \to N$ is elementary $\Leftrightarrow (N, f(a))_{a\in A} \models T(A)$

Atomic diagram of $A \subseteq M$:

$D_{at}(A) = \{\varphi \in D_{el}(A) : \varphi$ is a quantifier free sentence$\}$

$= \{\varphi(\bar{a}) \in L(A) : M \models \varphi(\bar{a})$ and $\varphi(\bar{a})$: q.f.-sentence$\}$

<u>Remark</u> $f: M \to N$ is a monomorphism (i.e: $f: M \xrightarrow{\cong} f(M) \subseteq N$)
↑substructure

$\Leftrightarrow (N, f(a))_{a \in M} \models D_{at}(M)$.

Here always $f: M \to N$ denotes a <u>monomorphism</u>.

$M \subseteq N$: M is a submodel (substructure) of N

$M \prec N$: M is an elementary submodel of N, i.e:

$$M \subseteq N \text{ and } id_M : M \xrightarrow{\equiv} N$$

<u>Remark</u> Assume $M \prec N$, $A \subseteq M$.

(1) Assume $p(\bar{x}) \subseteq L_n(A)$. Then

$p(\bar{x})$ is a consistent type in $M \Leftrightarrow p(\bar{x})$ is a consistent type in N

(2) Assume $A \subseteq B \subseteq M$

- If $p(\bar{x})$: a type over B, then $p\restriction_A \overset{def}{=} p(\bar{x}) \cap L(A)$ a type over A
- Let $r: S_n(B) \to S_n(A)$, $r(p) \overset{def}{=} p\restriction_A$.
  Then r: continuous and "onto".

(3) If $p(\bar{x})$: a type over A, then $\exists q(\bar{x}) \in S_n(A)$ $p(\bar{x}) \subseteq q(\bar{x})$

Saturation, universality, (strong) homogeneity.

Let $\varkappa \in CN$, $\varkappa \geqslant \aleph_0$.

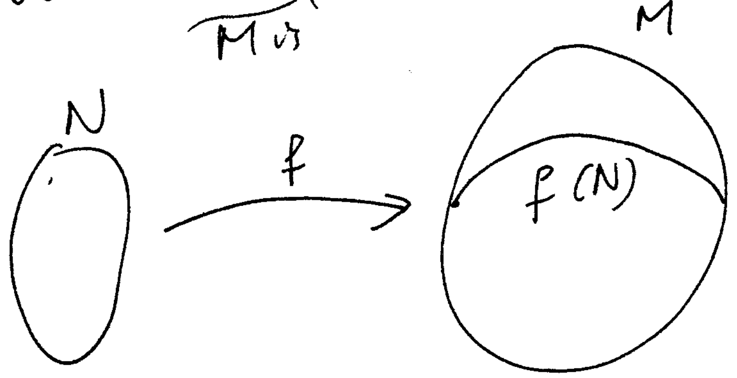
<u>Def.</u> (1) M $\varkappa$-<u>saturated</u> if $\forall A \subseteq M$ $\forall p \in S_1(A)$ $p(M) \neq \emptyset$
(nasycony) $|A| < \varkappa$

M is <u>saturated</u> if M is $\|M\|$-saturated

(2) M is $\varkappa$-universal if $\forall N \equiv M$ $(\|N\| \leqslant \varkappa \Rightarrow \exists f: N \xrightarrow{\equiv} M)$

elementarily equivalent
i.e. $Th(N) = Th(M)$

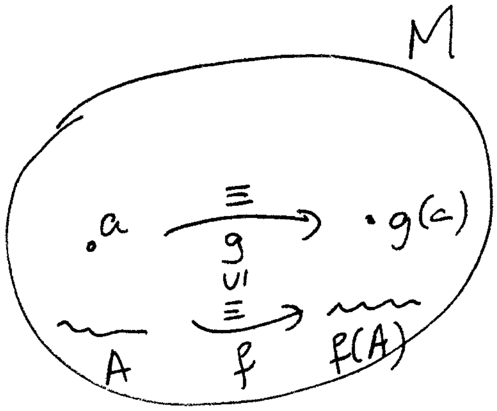
M: universal $\underset{M \text{ is}}{\iff}$ $\|M\|$-universal



$f: N \xrightarrow{\cong} f(N) \prec M$

(3) M : κ-homogeneous if $\underset{|A|<\kappa}{\forall A \subseteq M}\ \forall a \in M\ \forall f: A \xrightarrow{\equiv} M$
$\exists \underset{g}{\overset{\cap}{f}}: A \cup \{a\} \xrightarrow{\equiv} M$

homogeneous = $\|M\|$-homogeneous.



4. M strongly κ-homogeneous if $\underset{|A|<\kappa}{\forall A \subseteq M}\ \forall f: A \xrightarrow{\equiv} M$
$\exists \underset{g}{\overset{\cap}{f}}: M \xrightarrow{\cong} M$

strongly homogeneous =: strongly $\|M\|$-homogeneous.

5. M is κ-compact if ($\forall$ 1-type $p(x)$ over M)
$(|p| < \kappa \Rightarrow p(M) \neq \emptyset)$

Elementary chains of structures

Def $\langle M_\alpha : \alpha < \mu \rangle$, $\mu \in \mathrm{Ord}$, : an elementary chain of structures if $(\forall \alpha < \beta < \mu)\ M_\alpha \prec M_\beta$.

Union of chain (when $\mu \in \mathrm{Lim}$)

$$M_\mu = \bigcup_{\alpha < \mu} M_\alpha \ :$$

- $|M_\mu| := \bigcup\limits_{\alpha<\mu}^{\nearrow} |M_\alpha|$
- $c \in L$ constant symbol $\quad c^{M_\mu} = c^{M_\alpha}$ for $\alpha<\mu$
- P : relation symbol

$$\underset{\substack{\cap \\ |M_\mu|}}{P^{M_\mu}(a_1,\dots,a_n)} \Leftrightarrow M_\alpha \models P(a_1,\dots,a_n) \text{ for } \alpha<\mu$$

sufficiently large
[so that $\bar{a} \subseteq M_\alpha$]

- $f^{M_\mu}(\bar{a}) = b \Leftrightarrow M_\alpha \models f(\bar{a}) = b$ for $\alpha<\mu$ sufficiently large

Fact (Tarski) $M_\alpha \prec M_\mu$ for all $\alpha<\mu$.

<u>Proof</u> (1) $M_\alpha \subseteq M_\mu$ (substructure) : exercise

(2) $\forall \varphi(\bar{x}) \in L \;\forall \alpha<\mu \;\forall \bar{a} \subseteq M_\alpha \left(M_\alpha \models \varphi(\bar{a}) \Leftrightarrow M_\mu \models \varphi(\bar{a})\right)$

(a) φ atomic : $M_\alpha \subseteq M_\mu$ ✓

(b) $\varphi = \psi_1 \wedge \psi_2$, $\varphi = \neg\psi$: easy

(c) $\varphi(\bar{x}) = \exists y\, \psi(\bar{x}, y)$

$$M_\alpha \models \varphi(\bar{a}) \Rightarrow M_\alpha \models \psi(\bar{a}, b) \text{ for some } b \in M_\alpha$$

$$\Downarrow \text{ ind. assumption for } \psi$$

$$M_\mu \models \psi(\bar{a}, b)$$

$$\Downarrow$$

$$M_\mu \models \varphi(\bar{a})$$

$$\begin{array}{ll} M_\mu \models \varphi(\bar{a}) \Rightarrow M_\mu \models \psi(\bar{a},b) \text{ for some } b \in M_\mu & \\ \varphi(\bar a) = \exists y\, \psi(\bar{a},y) & \Downarrow \\ \quad \Downarrow \text{ ind. assumption} & b \in M_\beta \text{ for some } \alpha \le \beta < \mu \\ M_\beta \models \psi(\bar{a},b) & \\ \quad \Downarrow & \\ M_\beta \models \varphi(\bar{a}) & \\ \quad \Downarrow\ M_\alpha \prec M_\beta & \\ M_\alpha \models \varphi(\bar{a}) & \end{array}$$

<u>Elementary directed systems of structures:</u>

Let $(I, \leq)$: a <u>directed set</u>, i.e.:

(1) $\leq$: partial order on I

(2) $(\forall a, b \in I)(\exists c \in I)(a \leq c \wedge b \leq c)$

<u>Example</u> J : a set $\rightsquigarrow$ $([J]^{<\omega}, \subseteq)$: directed set.

Directed system of structures:

- $\mathcal{M} = (M_i, f_{ij})_{i \leq j \in I}$

f_{ij} : connecting functions $f_{ij} : M_i \longrightarrow M_j$, $f_{ii} = id_{M_i}$ such that

$(\forall i \leq j \leq t \in I)$ $f_{it} = f_{jt} \circ f_{ij}$ (compatibility)

$$\begin{tikzcd} & M_t \\ M_i \arrow[ur, "f_{it}"] \arrow[r, "f_{ij}"'] & M_j \arrow[u, "f_{jt}"'] \arrow[ul, phantom, "\#"] \end{tikzcd}$$

- System $\mathcal{M}$ is <u>elementary</u> if all f_{ij} are elementary.

Example Elementary chain $(M_\alpha)_{\alpha<\mu}$

$\mathcal{M} = (M_\alpha, f_{\alpha\beta})_{\alpha\le\beta<\mu}$ $f_{\alpha\beta} = id_{M_\alpha} : M_\alpha \xrightarrow{\equiv} M_\beta$

elementary directed system of structures

Direct limit of a directed system $\mathcal{M}$: $M_\infty = \varinjlim \mathcal{M}$.

$(M_\infty, f_{i\infty})_{i\in I}$, where $f_{i\infty} : M_i \longrightarrow M_\infty$ such that

(1) $\forall i \le j \in I$ $f_{i\infty} = f_{j\infty} \circ f_{ij}$ [compatible with connecting functions]

$$
\begin{tikzcd}
M_j \arrow[dr, "f_{j\infty}"] & \\
 & M_\infty \\
M_i \arrow[uu, "f_{ij}"] \arrow[ur, "f_{i\infty}"'] \arrow[u, phantom, "\#"] &
\end{tikzcd}
$$

(2) $(\forall (M'_\infty, f'_{i\infty})_{i\in I}$ satisfying (1)$)$ $\exists ! f : M_\infty \to M'_\infty$
(universality) $(\forall i\in I)$ $f'_{i\infty} = f \circ f_{i\infty}$

$$
\begin{tikzcd}
 & M_\infty \arrow[dd, "f"] \\
M_i \arrow[ur, "f_{i\infty}"] \arrow[dr, "f'_{i\infty}"'] \arrow[r, phantom, "\#"] & {} \\
 & M'_\infty
\end{tikzcd}
$$

Fact M_∞ exists (and is unique up to $\cong$).
If $\mathcal{M}$ is elementary, then $f_{i,\infty} : M_i \xrightarrow{\equiv} M_\infty$.

Proof 1. Construction of M_∞:

$S := \dot{\bigcup}_{i\in I} |M_i|$: formally disjoint union.

$\sim$ on S : an equivalence relation

$$\underset{\substack{\Downarrow\\ x}}{M_i} \sim \underset{\substack{\Downarrow\\ y}}{M_j} \overset{\text{def}}{\iff} f_{it}(x) = f_{jt}(y) \text{ for some } (\equiv \text{every}) \ t \geq i, j$$

exercise: $\sim$ is transitive.

$|M_\infty| := S/\sim$

- $\sim \restriction_{|M_i|}$: the equality (because f_{ij} : 1-1 (monomorphism)
- $f_{i\infty}(\underset{\substack{\Uparrow \\ M_i}}{x}) = x/\sim$, $\quad f_{i\infty} : |M_i| \xrightarrow{1-1} |M_\infty|$.

L-structure on $|M_\infty|$:

- $c^{M_\infty} = c^{M_i}/\sim$
- $p^{M_\infty}(a_{i_1}/\sim, \ldots, a_{i_n}/\sim) \iff M_t \models P(f_{i_1 t}(a_{i_1}), \ldots, f_{i_n t}(a_{i_n}))$

  $a_{i_j} \in M_{i_j}$ $\qquad$ for $t \geq i_1, \ldots, i_n$

- f^{M_∞} : similarly

the rest is an exercise.

How to extend elementary mappings?

~~Bat.~~ BAlg : Category of Boolean algebras

Comp$_0$: — // — of compact Hausdorff 0-dimensional spaces

$F : BAlg \longrightarrow Comp_0$ $\quad$ $G : Comp_0 \longrightarrow BAlg$

$F(A) = S(A)$ $\quad$ $G(X) = Clopen(X)$.

F, G : contravariant functors "inverse" to each other

~~(F, G) is a duality of categories... (look it up)~~

Categories BAlg and Comp$_0$ are dually equivalent.

A, B : Boolean algebras

$f : A \longrightarrow B$ homomorphism $\Rightarrow$ $\underset{f^*}{F(f)} : S(B) \longrightarrow S(A)$

$F(f)(p) = f^{-1}[p]$

continuous.

Assume $f : A \xrightarrow{\equiv} B$, $T \models M \supseteq A$, $N \supseteq B$, $N \models T$. Then $\hat{f} : L_n(A) \longrightarrow L_n(B)$

$\hat{f}(\varphi(\bar{x}, \bar{a})) = \varphi(\bar{x}, f(\bar{a}))$

homomorphism of Boolean algebras.

even: monomorphism.

We skip ^ in $\hat{f}$, so:

$f : L_n(A) \longrightarrow L_n(B)$ monomorphism

$\Downarrow$

$f^* : S_n(B) \longrightarrow S_n(A)$ epimorphism in Comp$_0$

i.e: continuous onto

Lemma (on extensions of elementary mappings)

Assume $M, N \models T$, $A \subseteq M$, $B \subseteq N$, $f : A \xrightarrow{\equiv} B$ "onto".

Assume $\underset{a}{\rotatebox{90}{$\in$}}$, $\underset{b}{\rotatebox{90}{$\in$}}$, $p = tp(a/A)$, $q = tp(b/B)$.

Then $f \cup \{\langle a, b\rangle\}$ is elementary $\iff$ $f^*(q) = p$.

[here $f^* : S(B) \xrightarrow{\approx} S(A)$] homeomorphism

Proof. exercise.

Def. M is $(<\aleph_0)$-universal $\iff \forall n\, \forall p \in S_n(\emptyset)\; p(M) \neq \emptyset$.

Remark $M : \varkappa$-universal $\Rightarrow M : (<\aleph_0)$-universal.

Proof Let $p \in S_n(\emptyset)$.

Choose a countable $N \models T$ with $p(N) \neq \emptyset$.

$M : \varkappa$-universal $\Rightarrow \exists f : N \xrightarrow{\equiv} M$

$\bar{a} \models p \overset{f}{\rightsquigarrow} f(\bar{a}) \models p$.

Thm. (1) $M : \varkappa$-saturated $\Rightarrow M : \varkappa$-homogeneous and $\varkappa$-universal.

(2) $M : \varkappa$-~~universal~~ homogeneous and $(<\aleph_0)$-universal $\Rightarrow$ $M : \varkappa$-saturated.

Proof. (1) $\varkappa$-homogeneity of M:

Assume $f : A \xrightarrow{\equiv} M$, $A \subseteq M$, $|A| < \varkappa$, $a \in M$.

We seek $b \in M$ s.t. $g = f \cup \{\langle a, b\rangle\}$ elementary

$$\Updownarrow \text{ Lemma}$$

$$f^*(tp(b/B)) = tp(a/A).$$

Let $\underset{\substack{\uparrow \\ S_1(A)}}{p} = tp(a/A)$, $q = (f^*)^{-1}(p) \in S_1(B)$

Let $b \in M$ (exists by κ-saturation) of M

good.

- ~~κ-universality~~ of M:

Assume $N \equiv M$, $\|N\| \leq \kappa$.

We seek $f : N \xrightarrow{\equiv} M$.

Let $\{a_\alpha : \alpha < \mu\}$: an enumeration of N, $\underset{\kappa}{\overset{}{\mu}} = \|N\|$. ($\mu \leq \kappa$)

We define $f(a_\alpha)$ by induction on $\alpha < \mu$:

- Suppose $f(a_\beta)$ defined for all $\beta < \alpha$ so that

$$f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} M$$

Want to find $f(a_\alpha)$ so that

$$f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} M.$$

~~By the Lemma it is enough that~~

Let $p = tp(a_\alpha / \{a_\beta : \beta < \alpha\})$.

By the lemma it is enough to find $f(a_\alpha) \in M$ so that $f^*(tp(f(a_\alpha)/\{f(a_\beta) : \beta < \alpha\})) = p$.

So let $q = (f^*)^{-1}(p) \in S_1(\underbrace{\{f(a_\beta) : \beta < \alpha\}}_{\text{power} < \varkappa})$

M $\varkappa$-saturated $\Rightarrow$ q realized in M.

Let $f(a_\alpha) \in M$ s.t. $f(a_\alpha) \models q$.

(2) Assume M is $\varkappa$-homogeneous & $(< \aleph_0)$-~~saturated~~ universal.

Want: M: $\varkappa$-saturated.

So: Let $A \subseteq M$, $|A| < \varkappa$, $p \in S_1(A)$. Show: $p(M) \neq \emptyset$.

Induction on $|A|$.

Case (a): $|A| < \aleph_0$.

$\exists N \succ M$ $p(N) \neq \emptyset$. So let $b \models p$.

$$\text{Let } A^* = \underset{\{a_1,\dots,a_k\}}{\underset{\shortparallel}{A}} \cup \{b\}$$

Let $q = tp^N(a_1, \dots, a_k, b) \in S_{k+1}(\emptyset)$

q is realized in M (by $(< \aleph_0)$-universality),

by $\langle \underbrace{a_1', \dots, a_k'}_{A'}, b' \rangle$

Let $g : A \cup \{b\} \longrightarrow A' \cup \{b'\}$, $g(a_i) = a_i'$, $g(b) = b'$.

g: elementary.

$\Rightarrow g\restriction_A : A \xrightarrow{\equiv} A'$

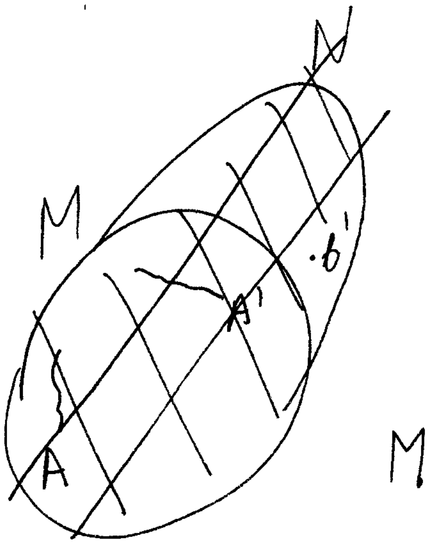
$\Downarrow$

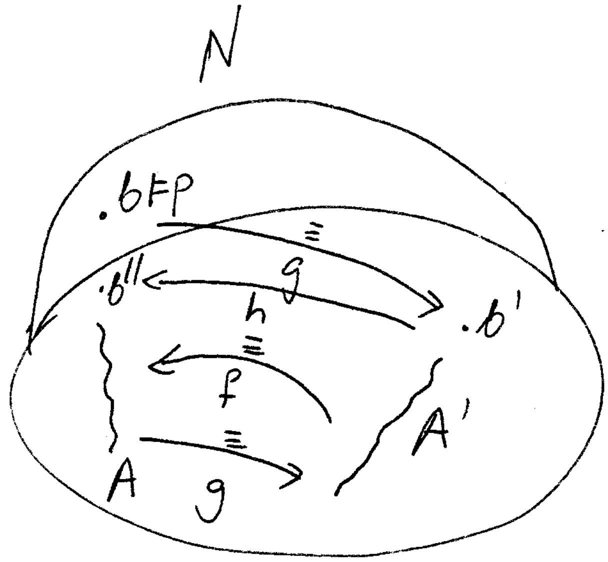
$f := (g\restriction_A)^{-1} : A' \xrightarrow{\equiv} A$

$\cap|$

M: κ-homogeneous $\Rightarrow \exists\ h : A' \cup \{b'\} \xrightarrow{\equiv} A \cup \{b''\}$

for some $b'' \in M$.





$$\begin{tikzcd} Ab \arrow[rr, "\equiv", "h\circ g"'] \arrow[dr, "g"', "\equiv"] & & Ab'' \\ & A'b' \arrow[ur, "\equiv", "h"'] & \end{tikzcd}$$

Let $s = h \circ g$

$s\restriction A = \underbrace{(h\restriction_{A'})}_{= (g\restriction_A)^{-1}} \circ (g\restriction_A) = id_A$.

$\Downarrow$

$\cancel{s^*(tp(b''/A)) = tp(b}$

$s\restriction_A = id_A \Rightarrow \underset{= id_{S(A)}}{s^*} : S(A) \xrightarrow{\cong} S(A)$

hence: $p = tp(b/A) \underset{\text{Lemma}}{=} s^*(tp(b''/A)) \underset{s^* = id_{S(A)}}{=} tp(b''/A)$

and $b'' \models p$

Case (b) $|A| = \mu$, $\aleph_0 \leq \mu < \kappa$.

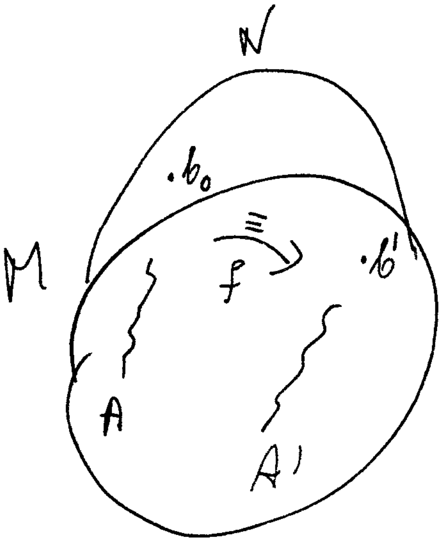
$A = \{a_\alpha : \alpha < \mu\}$, $p \in S_1(A)$.

$p\restriction\emptyset \in S_1(\emptyset) \implies \exists\, b' \in M \quad b' \models p\restriction\emptyset$.

M: $(<\aleph_0)$-universal

$\exists\, N \succ M \quad \exists\, \cancel{b}\, b_0 \in N$

$\models p$



Will find $A' = \{a'_\alpha : \alpha < \mu\} \subseteq M$

s.t. $f : Ab_0 \longrightarrow A'b'$

given by $f(a_\alpha) = a'_\alpha$

$f(b_0) = b'$

is <u>elementary</u>!

We find a'_α, $\alpha < \mu$ by induction on $\alpha < \mu$.

So suppose $\alpha < \mu$ and a'_β already defined for all $\beta < \alpha$

so that $\underbrace{f\restriction_{\{a_\beta : \beta<\alpha\} b_0}}_{=:\, f_0} : \{a_\beta : \beta < \alpha\} b_0 \xrightarrow{\equiv} \{a'_\beta : \beta < \alpha\} b'$

We look for a'_α.

Let $q = tp(a_\alpha / \{a_\beta : \beta < \alpha\} \cup \{b_0\})$

Then $(f_0^*)^{-1}(q) \in S(\underbrace{\{a'_\beta : \beta < \alpha\} \cup \{b'\}}_{\text{power } < \mu \leq |A|})$

By the lemma it is enough that $a'_\alpha \models (f_0^*)^{-1}(q)$.

M.

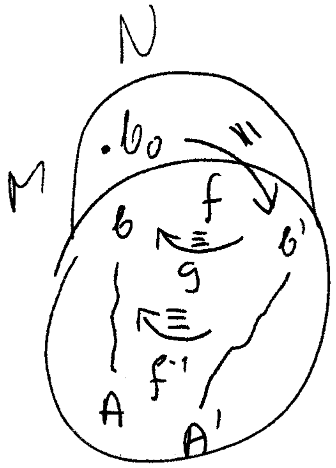
~~But M: κ-~~

~~Let A~~ By the inductive assumption on A:

$(f_0^*)^{-1}(q)$ is realized in M, so we are done with constructing A'.

Now: $f^{-1} : A' \xrightarrow{\equiv} A$ in $M \leftarrow \kappa$-homogeneous, so

$\exists\ \hat{g}^{-1} : A'b' \xrightarrow{\equiv} Ab$ for some $b \in M$.



~~Let~~ Let $s = g \circ f$

$s : Ab_0 \xrightarrow{\equiv} Ab$

$$s\restriction_A = \underbrace{(g\restriction_{A'})}_{= (f\restriction_A)^{-1}} \circ (f\restriction_A) = id_A,$$

so $p = tp(b_0/A) = tp(b/A)$

and p is realized in M.

<u>Corollary</u>

M is κ-saturated $\iff$ M is κ-homogeneous and ~~κ-universal~~ and κ-universal.

<u>Proof</u> $\Rightarrow$ by Thm (1).

$\Leftarrow$ κ-homogeneous + κ-universal $\Rightarrow$

κ-homogeneous + $(<\aleph_0)$-universal $\underset{\text{Thm (2)}}{\Rightarrow}$ κ-saturated

Properties of saturated models.

Thm. Assume $M, N \models T$ saturated models of the same power. Then $M \cong N$.

Proof $M = \{m_\alpha : \alpha < \varkappa\}$, $N = \{n_\alpha : \alpha < \varkappa\}$,

$\varkappa = \|M\| = \|N\|$. We find $f : M \xrightarrow{\cong} N$

back-and-forth method:

$$f = \bigcup_{\alpha < \varkappa}^{\nearrow} f_\alpha \qquad f_\alpha : M \xrightarrow[\circ]{\equiv} N \quad \text{s.t.:}$$

partial, elementary

(1) $m_\alpha \in \operatorname{Dom} f_{\alpha+1}$

$n_\alpha \in \operatorname{Rng} f_{\alpha+1}$, $|f_\alpha| \leq 2 \cdot |\alpha|$

(2) $f_0 = \emptyset$

(3) For $\delta \in \mathrm{Lim}$, $f_\delta = \bigcup_{\alpha<\delta}^{\nearrow} f_\alpha$.

(4) $f_{\alpha+1} = f_\alpha \cup \{\langle m_\alpha, \underset{N}{\underset{\Uparrow}{n}} \rangle, \langle \underset{M}{\underset{\Uparrow}{m}}, n_\alpha \rangle\}$

Inductive step:

Suppose we have f_α. Want: $f_{\alpha+1}$.

Let $A_\alpha = \operatorname{Dom} f_\alpha \subseteq M$, $B_\alpha = \operatorname{Rng} f_\alpha \subseteq N$.

$$f_\alpha : A_\alpha \xrightarrow{\equiv} B_\alpha$$

$$\Downarrow$$

$$f_\alpha^{*} : S(B_\alpha) \xrightarrow{\approx} S(A_\alpha).$$

"forth": Find $n \in N$ s.t. $f_\alpha \cup \{\langle m_\alpha, n\rangle\}$ elementary

$$\Updownarrow$$

$$(f_\alpha^*)^{-1}(tp(m_\alpha/A_\alpha)) = tp(n/B_\alpha).$$

Let $p = tp(m_\alpha/A_\alpha)$.

So $(f_\alpha^*)^{-1}(p) \in S(B_\alpha)$ is realized in N by some n.

"~~forth~~ back": similarly.

<u>Thm</u> Assume $M, N \models T$ are homogeneous, of the same power and $\forall n<\omega\ \forall p \in S_n(\emptyset)\ (p(M)\neq\emptyset \Leftrightarrow p(N)\neq\emptyset)$. Then $M \cong N$.

<u>Lemma</u> Under the assumptions of the Thm,

$$\forall A \subseteq M\ \exists f : A \xrightarrow{\equiv} N.$$

<u>Proof</u>. Induction on $|A|$.

Case (a) $|A| < \aleph_0$. $A = \{a_1, \dots, a_n\}$.

Let $p = tp(\langle a_1, \dots, a_n\rangle) \in S_n(\emptyset)$. realized in M

$$\Downarrow$$

realized in N

by some $\langle b_1, \dots, b_n\rangle \subseteq N$.

$f(a_i) = b_i$ is <u>good</u>.

Case (b) $|A| = \mu \geq \aleph_0$, $A = \{a_\alpha : \alpha < \mu\}$

We find $f(a_\alpha)$ by induction on $\alpha < \mu$.

Inductive step.

Suppose $\alpha < \mu$ and for every $\beta < \alpha$ we have $f(a_\beta)$

s.t. $f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} N$.

We shall find $f(a_\alpha) \in N$ s.t. $f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} N$.

Let $a_{<\alpha} := \{a_\beta : \beta < \alpha\}$. Likewise $a_{\leq\alpha}$.

$|a_{\leq\alpha}| < \mu = |A|$

By inductive assumption: $\exists g : a_{\leq\alpha} \xrightarrow{\equiv} N$.

Then $f \circ g^{-1} : g(a_{<\alpha}) \xrightarrow{\equiv} f(a_{<\alpha})$

$$g(a_{<\alpha}) \subseteq N, \quad f(a_{<\alpha}) \subseteq N$$

By homogeneity of N: $\exists f(a_\alpha) \in N$ s.t.

$$f \circ g^{-1} : \underbrace{g(a_{<\alpha})\, g(a_\alpha)}_{= g(a_{\leq\alpha})} \xrightarrow{\equiv} f(a_{<\alpha}) f(a_\alpha) = f(a_{\leq\alpha})$$

Then $f = (f \circ g^{-1}) \circ g : a_{\leq\alpha} \xrightarrow{\equiv} N$.

Proof of the theorem $\qquad \varkappa := \|M\| = \|N\|$

$f : M \xrightarrow{\cong} N$ constructed by back-and-forth method

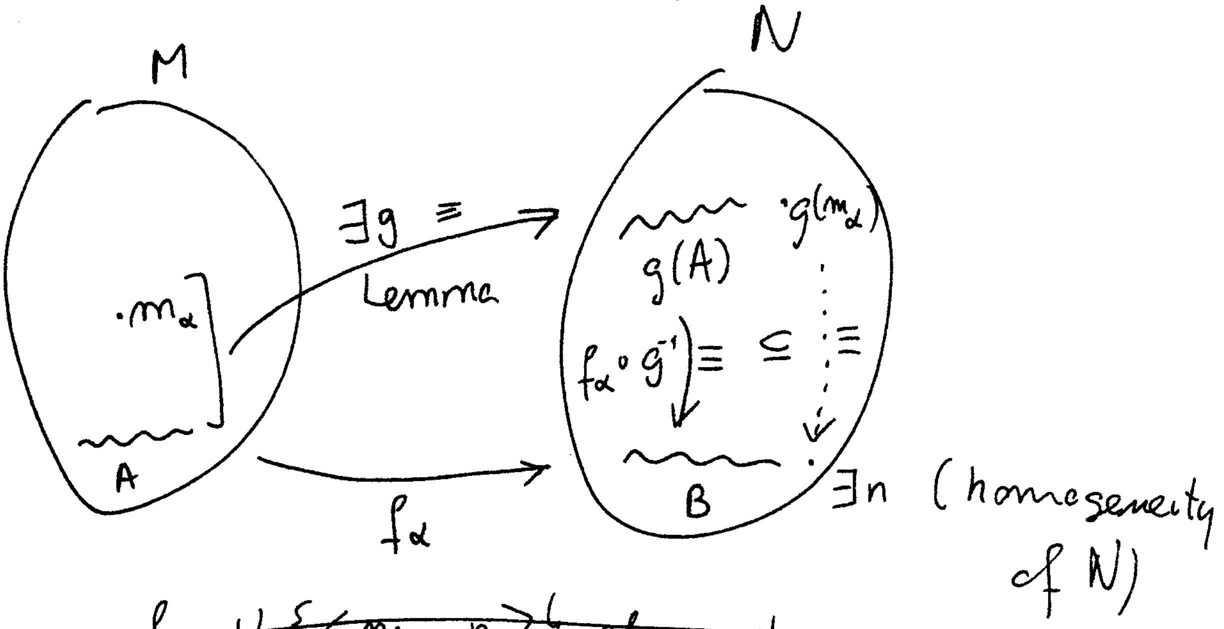
$f = \bigcup^{\uparrow}_{\alpha < \varkappa} f_\alpha$, $f_\alpha : M \xrightarrow{\equiv}_{\circ} N$ (partial elementary), $\alpha < \varkappa$

$|f_\alpha| \leq 2 \cdot |\alpha|$ + the same conditions as in the previous thm.

Inductive step $f_\alpha \longmapsto f_{\alpha+1}$

$A = \mathrm{Dom}\, f_\alpha \qquad B = \mathrm{Rng}\, f_\alpha$

$\boxed{\text{"forth":}}$



~~$f_\alpha \cup \{\langle m_\alpha, n\rangle\}$ elementary~~

$$h = (f_\alpha \circ g^{-1})\restriction_{g(A)} \cup \{\langle g(m_\alpha), n\rangle\} \text{ elementary}$$

$$\underset{\substack{\cup\!\!\mid \\ f_\alpha}}{h \circ g} : A m_\alpha \xrightarrow{\equiv} B n \subseteq N$$

"back": similarly.

Constructions of models:

- Saturated $\Longrightarrow$ • (strongly) homogeneous $\not\Leftarrow$

Thm. $\underbrace{\kappa = 2^{<\kappa},\ \kappa \in Reg}_{\substack{|||\\ \kappa^{<\kappa} = \kappa}}$, $\kappa > \aleph_0 \Longrightarrow \exists M \models T$ saturated, of power κ.

Proof

(∗) $|S_1(A)| \leq 2^{|A| + \aleph_0}$, because: $|L_1(A)| = |A| + \aleph_0$

Here: $|A| < \kappa \Rightarrow |S_1(A)| \leq \kappa$.

Lemma $N \models T$, $\|N\| \leq \kappa \Rightarrow X_N := \bigcup \{S_1(A) : A \subseteq N \ \&\ |A| < \kappa\}$ the set has power $\leq \kappa$.

Pf • $|\{A \subset N : |A| < \kappa\}| \leq \kappa^{<\kappa} = \kappa$.

• $|S_1(A)| \leq \kappa$ for such A.

Proof of the thm.

M_α, $\alpha < \kappa$: elementary chain of models of T of power κ.

- M_0 : whatever
- $M_\delta = \bigcup\limits_{\alpha<\delta}^{\uparrow} M_\alpha$, when $\delta < \kappa$ limit.
- $M_{\alpha+1} \succ M_\alpha$ such that $\forall p \in X_{M_\alpha}\ p(M_{\alpha+1}) \neq \emptyset$:

$T' = Th(M_\alpha, m)_{m \in M_\alpha} \cup \bigcup\limits_{\beta<\kappa} \{\varphi(c_\beta) : \varphi(x) \in p_\beta\}$,

where $X_{M_\alpha} = \{p_\beta : \beta < \kappa\}$ new constant symbols,

and T' in language $L(M_\alpha) \cup \{c_\beta : \beta < \kappa\}$.

T': consistent, has model of power $\varkappa$: $M_{\alpha+1}$
such that $M_\alpha \prec M_{\alpha+1}$.

$M = \bigcup\limits_{\alpha<\varkappa}^{\nearrow} M_\alpha$: of power $\varkappa$, saturated:

Let $A \subseteq M$, $|A| < \varkappa$ and $p \in S_1^M(A)$

$\varkappa \in Reg \Rightarrow A \subseteq M_\alpha$ for some $\alpha < \varkappa$.

proof: $A = \{a_\beta : \beta < \mu\}$ for some $\overset{\text{CN}}{\mu} < \varkappa$.

· $\forall \beta < \mu \ \exists \alpha_\beta < \varkappa \ a_\beta \in M_{\alpha_\beta}$

· $\{\alpha_\beta : \beta < \mu\} \subseteq \varkappa$, $\mu < cf(\varkappa) = \varkappa$

$\Rightarrow \exists \alpha < \varkappa \ \forall \beta < \mu \ \alpha_\beta < \alpha$

$A \subseteq M_\alpha$.

~~Let~~ $M_\alpha \prec M \Rightarrow p \in S_1^{M_\alpha}(A) = S_1^M(A)$

p realized in $M_{\alpha+1}$ by some $a \in M_{\alpha+1}$

$a \models p$ in $M_{\alpha+1} \xRightarrow{M_{\alpha+1} \prec M} a \models p$ in M.

Monster model:

Let $\bar{\varkappa}$: a large cardinal number.

"Ideal model" $\mathcal{M} \models T$: saturated of power $\bar{\varkappa}$

because: $\forall M \models T$ ($\|M\| < \bar{\varkappa} \Rightarrow \exists M' \prec \mathcal{M} \ M \cong M'$).

Advantages of saturated model $\mathcal{M}$:

(i) universality

(ii) strong homogeneity

~~More~~ More Weakly (a bit):

(1) $\bar{\kappa}$-universality

(2) strong $\bar{\kappa}$-homogeneity

$Aut(M)$: the group of automorphisms of M

$Aut(M/A) = \{ f \in Aut(M) : f\restriction_A = id_A \}$: automorphisms of M over A.
$A \subseteq M$

Lemma Assume M is strongly κ-homogeneous, κ-saturated, $A \subseteq M$, $|A| < \kappa$. Then:

(1) For $a, b \in M$ ($tp(a/A) = tp(b/A) \iff a, b$ are in the same orbit of $Aut(M/A)$ on M.

(2) $[\text{orbits } Aut(M/A) \text{ on } M^n] \xleftrightarrow[\text{onto}]{1\text{-}1} S_n(A)$

Proof (1) $\Leftarrow$: $f \in Aut(M/A)$, $f(a) = b$

$$\Downarrow$$

$$tp(a/A) = tp(b/A).$$

$\Rightarrow$: $tp(a/A) = tp(b/A) \Rightarrow f : Aa \xrightarrow{\equiv} Ab$
$f\restriction_A = id_A$, $f(a) = b$

strong κ-homogeneity
$|A| < \kappa$ $\Rightarrow f \subseteq g \in Aut(M)$, $g \in Aut(M/A)$
$g(a) = b$: a, b in the same orbit of $Aut(M/A)$

(2) $M^n \supseteq \mathcal{O} \xrightarrow[\varphi]{(1)} p_{\mathcal{O}} \in S_n(A)$

$\mathcal{O}$: orbit of $\mathrm{Aut}(M/A)$

$p_{\mathcal{O}}$ = common type $tp(a/A)$ for $a \in \mathcal{O}$.

Φ : {orbits of $\mathrm{Aut}(M/A)$ on M^n}
$\downarrow \varphi$
$S_n(A)$

$\mathcal{O}_1 \neq \mathcal{O}_2 \overset{(1)}{\Rightarrow} p_{\mathcal{O}_1} \neq p_{\mathcal{O}_2}$ $\boxed{\text{so } \varphi : 1\text{-}1}$

[if $p_{\mathcal{O}_1} = p_{\mathcal{O}_2}$ then let $a \in \mathcal{O}_1$, $b \in \mathcal{O}_2 \Rightarrow \exists g \in \mathrm{Aut}(M/A)$
$g(a) = b$ ↯.]

M: κ-saturated $\Rightarrow \varphi$: "onto".

<u>Def</u> Let $\bar{\kappa}$: a (large) cardinal number,
$\mathcal{M} \models T$ <u>monster model</u> (w.r. to $\bar{\kappa}$), if $\mathcal{M}$: $\bar{\kappa}$-saturated, strongly $\bar{\kappa}$-homogeneous

Thm. Assume $\aleph_0 \leq \kappa \in CN$. Then
$\exists M$: κ-saturated ~~&~~ strongly $\bar{\kappa}$-saturated.

<u>Prof</u> $M = \bigcup^{\uparrow}_{\alpha < \kappa^+} M_\alpha$: union of elementary chain s.t.:

(1) $M_0 \models T$ any

(2) $M_\delta = \bigcup_{\alpha < \delta} M_\alpha$ if $\delta \in Lim$,

(3) $M_{\alpha+1} \succ M_\alpha$ s.t:

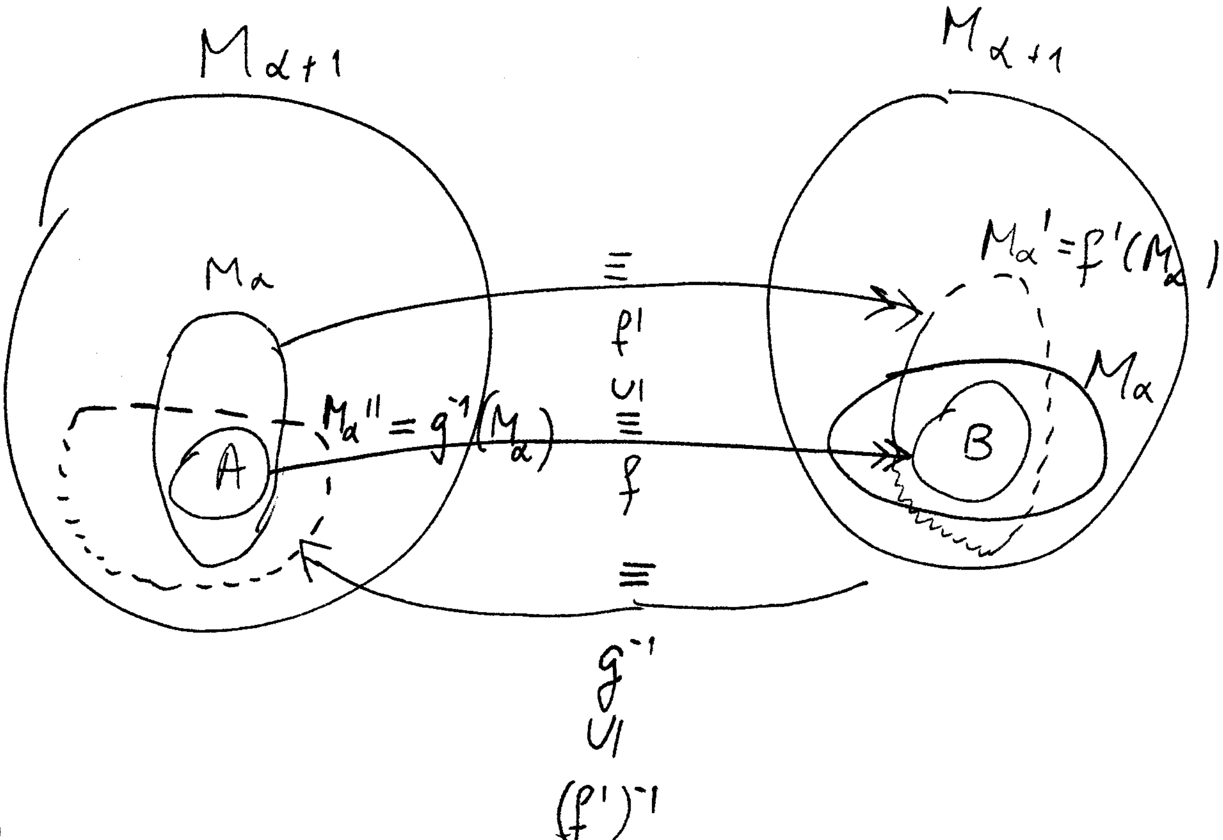
(a) $\forall p \in S_1(M_\alpha)$ p realized in $M_{\alpha+1}$

(b) $\Big(\forall f: \underset{M_\alpha}{\underset{\cap}{A}} \xrightarrow{\equiv} \underset{M_\alpha}{\underset{\cap}{B}}\Big)(\exists\, g \supseteq f)\Big(g: \underset{M_\alpha}{\underset{\cup}{A'}} \xrightarrow{\equiv} \underset{M_\alpha}{\underset{\cup}{B'}} \text{ in } M_{\alpha+1}\Big)$

It is enough that $M_{\alpha+1}$ is $\|M_\alpha\|^+$-saturated,

To satisfy (a), (b). $\quad M_{\alpha+1} \succ M_\alpha$

Proof of (b) for such $M_{\alpha+1}$:



I. M: κ-saturated: clear

II M strongly κ-homogeneous:

Assume $A \subset M$, $|A| < \kappa$. Then $A \subseteq M_\alpha$, $B \subseteq M_\alpha$

$f: A \xrightarrow{\equiv} M$ $\quad$ for some $\alpha < \kappa$.

$B = f[A]$

- we construct

a sequence f_β, $\alpha \le \beta < \varkappa^+$:

- increasing $f_\beta : M \xrightarrow{\overset{\equiv}{\theta}} M$ partial elementary
- $f \subseteq f_\alpha$

(*) $M_\beta \subseteq \operatorname{dom} f_\beta \cap \operatorname{rng} f_\beta$.

- f_α constructed according to 3)(b)

$$f_\alpha : M_{\alpha+1} \xrightarrow{\overset{\equiv}{\theta}} M_{\alpha+1}$$

- $f_\beta : M_{\beta+1} \xrightarrow{\overset{\equiv}{\theta}} M_{\beta+1}$, as in 3.(b) when β : successor.
- $f_\delta = \bigcup_{\beta<\delta} f_\beta$ when δ limit. , still $f_\delta : M_\delta \xrightarrow{\overset{\equiv}{\theta}} M_\delta$.

$f_\infty = \bigcup^{\nearrow}_{\alpha \le \beta < \varkappa^+} f_\beta$, $f_\beta \in \operatorname{Aut}(M)$, $f \subseteq f_\beta$.

<u>Assumptions</u> Let $\bar{\varkappa}$: a cardinal number <u>large enough</u> so that:

(1) We consider only small models of T
$\qquad$ small $\Updownarrow$ of power $< \bar{\varkappa}$, or even $\ll \bar{\varkappa}$

(2) We work within a monster model $\mathcal{M} \models T$ (w.r.to $\bar{\varkappa}$)

(3) We consider only <u>small</u> models $M \prec \mathcal{M}$
$\qquad$ small $\Updownarrow$ of power $< \bar{\varkappa}$, or even $\ll \bar{\varkappa}$,

Consequences:

(1) For $M, N \prec \mathcal{M}$, $M \subseteq N \iff M \prec N$

(2) Convention: For $\bar{a} \subseteq \mathcal{M}$

$\models \varphi(\bar{a})$ means $\mathcal{M} \models \varphi(\bar{a})$

(3) For $A \subseteq M \prec \mathcal{M}$:

$S_n^M(A) = S_n^{\mathcal{M}}(A) =: S_n(A)$

<u>Notation</u> Assume $p(\bar{x})$, $q(\bar{x})$ types (small, over $\mathcal{M}$)

- $p(\bar{x}) \vdash q(\bar{x}) \iff p(\mathcal{M}) \subseteq q(\mathcal{M})$
  "p implies q"
- $p(\bar{x}) \underset{\substack{\uparrow \\ \text{equivalent}}}{\equiv} q(\bar{x}) \iff p \vdash q \;\&\; q \vdash p$

Special case: $p(\bar{x}) = \{\varphi(\bar{x})\}$.

$\varphi(\bar{x}) \vdash q(\bar{x})$: "φ isolates q".

<u>Remark</u>: Syntactically:

$$\underset{\substack{\nwarrow \\ \text{types over } A}}{p(\bar{x})} \vdash \underset{\nearrow}{q(\bar{x})} \iff \forall \varphi(\bar{x}) \in q \;\; \exists \underset{\text{finite}}{p_0(\bar{x}) \subseteq p(\bar{x})}$$

$$T(A) \vdash \bigwedge p_0(x) \to \varphi(x)$$

Remark (exercise)

$$p(\bar{x}) \vdash q(\bar{x}) \iff \underset{|A|^+\text{-saturated}}{\forall M \models T} \; p(M) \subseteq q(M).$$

Def. (reminder)

Let $p(\bar{x})$: a type over A.

p is isolated over $A \iff \exists \underbrace{\varphi(\bar{x}) \in L(A)}_{\text{consistent (with } T)} \quad \varphi \vdash p$.

Thm (omitting types, Ehrenfeucht)

Assume $p_n(\bar{x}_n)$, $n<\omega$: ~~a form~~ ~~non~~ a family of non-isolated types over $\emptyset$. Then $\exists M \models T \ \forall n \ \underbrace{p_n(M)=\emptyset}_{M \text{ omits } p_n}$.

<u>Lemma</u>

Assume A is ctble, $p_n(\bar{x}_n)$, $n<\omega$: a family of non-isolated types over A, $\varphi(x) \in L_1(A)$, $\underbrace{\varphi(\mathcal{M}) \neq \emptyset}_{\text{i.e. } \varphi \text{: consistent}}$.

Then $\exists c \in \varphi(\mathcal{M}) \ \forall n \ p_n$ non-isolated over $A \cup \{c\}$.

Proof [Lemma $\Rightarrow$ Thm]

By the Lemma: $\exists \underbrace{\{a_n : n<\omega\}}_{= A} \subseteq \mathcal{M}$ s.t.

(1) A satisfies the TV-test

(2) p_n, $n<\omega$: non-isolated over A.

Construction of a_n, $n<\omega$: recursion on n.

Let $\{\varphi_n(x, \bar{y}) : n<\omega\}$: all formulas of L of this form.

Suppose $n<\omega$ and $\{a_i : i<n\} = a_{<n}$ already constructed so that all p_k, $k<\omega$ still non-isolated over $a_{<n}$.

We consider a consistent formula $\varphi_n(x) \in L_1(a_{<n})$

By the Lemma we find $\underset{a_n}{\overset{\shortparallel}{c}} \in \varphi_n(\mathcal{M})$ so that all p_k, $k<\omega$, still non-isolated over $a_{\leq n}$.

- the formulas $\varphi_n(x)$, $n<\omega$ may be chosen so that after ω steps:

$$\underset{\text{consistent}}{\forall \varphi(x) \in L(A)}\ \exists n\ \varphi = \varphi_n.$$

Then $A = \{a_n : n<\omega\}$ satisfies TV-test

$A = M \prec \mathcal{M}$, every p_k still non-isolated over A.

$p_k(M) = \emptyset$ [if not,

some $\underset{M}{\overline{m} \atop \cap} \models p_k$. then $(\overline{x}_k = \overline{m}) \vdash p_k(\overline{x}_k)$ ↯)

Proof of the Lemma.

Let $p(\overline{x})$: one of the types $p_n(\overline{x}_n)$.

Let $h(\overline{x}, y, \overline{a}) \in L(A$

$$h(\overline{x}, c, \overline{a}) \vdash p(\overline{x}) \iff h(\mathcal{M}, c, \overline{a}) \subseteq p(\mathcal{M}).$$

$$\iff \forall \psi(\overline{x}) \in p(\overline{x})\ \ h(\mathcal{M}, c, \overline{a}) \subseteq \psi(\mathcal{M})$$

$$\iff \forall \psi \in p\ \ \mathcal{M} \models \forall \overline{x}\,(h(\overline{x}, c, \overline{a}) \rightarrow \psi(\overline{x}))$$

$\Leftrightarrow \forall \psi \in p \quad \psi_h(y) \in tp(c/A)(y)$

where $\psi_h(y) = \forall \bar{x}\, (h(\bar{x}, y, \bar{a}) \rightarrow \psi(\bar{x}))$

<u>hence</u>:

$tp(c/A) = tp(c'/A) \Rightarrow [h(\bar{x}, c, \bar{a}) \vdash p \Leftrightarrow h(\bar{x}, c', \bar{a}) \vdash p]$

$h(\bar{x}, c, \bar{a})$ consistent $\Leftrightarrow (\exists \bar{x}\, h(\bar{x}, y, \bar{a})) \in tp(c/A)(y)$.

Let $X_{h,p} = \{q \in S_1(A) : \text{For } c \models q,\ h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$ and $h(\bar{x}, c, \bar{a})$ consistent.$\}$

"bad types"

Let $q \in S_1(A)$ then

$$q \in X_{h,p} \Leftrightarrow q(y) \in S_1(A) \cap \underbrace{[\exists \bar{x}\, h(\bar{x}, y, \bar{a})]}_{\text{For } c \models q\ h(\bar{x}, c, \bar{a}) \text{ consistent}} \cap \underbrace{\bigcap_{\psi \in p} [\psi_h(y)]}_{\text{For } c \models q\ h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})}$$

(*) $X_{h,p}$: nowhere dense in $S_1(A)$.

Proof of (*) : (a.a.)

Suppose $\theta(y) \in L_1(A)$ and $\emptyset \neq S_1(A) \cap [\theta] \subseteq X_{h,p}$.

Let $\alpha(\bar{x}) = \exists y (h(\bar{x}, y, \bar{a}) \wedge \theta(y))$

$\alpha \in L(A)$

- $\alpha(\bar{x})$: consistent :

Let $c \in \theta(\mathcal{M})$

$\Downarrow$ $[\theta] \subseteq X_{h,p}$)

$\mathcal{M} \models \exists \bar{x}\ h(\bar{x}, c, \bar{a})$.

Let $\bar{d} \subseteq \mathcal{M}$ s.t. $\mathcal{M} \models h(\bar{d}, c, \bar{a})$.

$\bar{d}$ satisfies in $\mathcal{M}$: $\underbrace{\exists y\, h(\bar{x}, y, \bar{a})}_{\alpha(\bar{x})}$.

- $\alpha(\bar{x}) \vdash p(\bar{x})$, i.e. $\alpha(\mathcal{M}) \subseteq p(\mathcal{M})$.

Let $\bar{d} \in \alpha(\mathcal{M})$. So there is $c \in \mathcal{M}$ st

$\models h(\bar{d}, c, \bar{a}) \wedge \theta(c)$

$\Downarrow$ $[\theta] \subseteq X_{h,d}$

$\forall \psi \in p \quad \models \psi_h(c)$

$\Downarrow$

$\forall \psi \in p \quad h(\bar{x}, c, \bar{a}) \vdash \psi(\bar{x}) \quad \Rightarrow h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$

$\bar{\bar{d}} \quad \Rightarrow \quad \bar{d}$ $\lightning$.

as : $p(\bar{x})$ non-isolated

Let $X = \bigcup_{h, p_n} X_{h,p_n} \subseteq S_1(A)$.

$\uparrow$ meager. Let $q \in S_1(A) \cap [\varphi] \setminus X$

$c \models q$ good.

Pf (a.a) Suppose $p = p_n$ isolated over $A \cup \{c\}$.

$\exists\ h(\bar{x}, c, \bar{a})$ $\quad h(\bar{x}, c, \bar{a}) \vdash p(x) \quad \Rightarrow q \in X_{h,p_n}$ $\lightning$.

Consistent $\qquad q = tp(\bar{c}/A)$