

Preliminaries

MT1/1

T : a complete consistent theory, in language L
with infinite models (countable)

that is, $T = \text{Th}(M)$, M : L -structure.
infinite

L denotes also the set of formulas of language L

$M = (|M|; \dots)$, but M also denotes $|M|$.
 $\emptyset \neq \uparrow$ universe of M (for brevity)

usually we omit $| \cdot |$ in $|M|$.

$M \supseteq A$: a set of parameters.

$L_n(A) = \{ \varphi(x_1, \dots, x_n, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in L, \bar{a} \subseteq A \}$

$L(A) = \bigcup_n L_n(A)$, also $L(A)$: language L
extended by names for elements of A .

$L_n(A)$: Lindenbaum algebra.

[formally: on $L_n(A)$: $\varphi \sim \psi \Leftrightarrow T(A) \vdash \varphi \leftrightarrow \psi$
 $\Leftrightarrow M \models \varphi \leftrightarrow \psi$

here: $T(A) = \text{Th}(M, a)_{a \in A}$

a complete theory
in language $L(A)$.



$L_m(A)/\sim$: a Boolean algebra
(Lindenbaum algebra)

$$[\varphi]_{\sim} \wedge [\psi]_{\sim} = [\varphi \wedge \psi]_{\sim} \text{ etc.}$$

shorthand: $L_m(A)$ denotes also $L_m(A)/\sim$.

$S_m(A) = \{ \text{complete } n\text{-types over } A, \text{ in } \mathcal{M}_n \}$
in variables x_1, \dots, x_n

consistent n -type over $A \mapsto$ proper filter in $L_m(A)$

An n -type $p(\bar{x})$ over A is complete if

$$L_m(A) \begin{cases} \cdot p(\bar{x}) : \text{consistent type} \\ \cdot \forall \varphi(\bar{x}) \in L_m(A) (\varphi(\bar{x}) \in p \text{ or } (\neg\varphi(\bar{x})) \in p) \end{cases}$$

$$S(A) := S_1(A)$$

(default)

$S_m(A)$: topological space :

for $\varphi(\bar{x}) \in L_m(A)$

$$[\varphi] = \{ p \in S_m(A) : \varphi \in p \}$$

basic open set [clopen]

closed and open

$S_m(A)$: compact Hausdorff space, 0-dimensional
(i.e. basis of clopen sets)

complete n -types / $A \rightsquigarrow$ ultrafilters in $L_n(A)$ MT1/3

So $S_n(A) = S(L_n(A))$, the Stone space
of ultrafilters in $L_n(A)$

• the ~~type~~ topology
on $S_n(A)$ = the Stone space topology.

For $p(\bar{x}) \in S_n(A)$

$$p(M) = \{ \bar{a} \in M^n : \underbrace{\bar{a} \text{ satisfies } p}_{\text{realizes } p} \}$$

$\bar{a} \models p$, i.e. $M \models \varphi(\bar{a})$ for
every $\varphi(\bar{x}) \in p(\bar{x})$

• The same notation for
arbitrary type (also incomplete)

• A formula $\varphi(\bar{x}) \in L(M)$: a special case of a
type $\{ \varphi(\bar{x}) \}$.

$$\varphi(M) = \dots$$

• When $p \in S_n(A)$, $\bar{a} \subseteq M$ and $\bar{a} \models p$, then

$$p = \text{tp}^M(\bar{a}/A) = \text{tp}(\bar{a}/A) = \{ \varphi(\bar{x}) \in L_n(A) : M \models \varphi(\bar{a}) \}.$$

Example Assume $p(\bar{x})$: a consistent type over M .

Then $\exists N \supseteq M$ p is realized in N

i.e. $p(N) \neq \emptyset$.



From now on "a type" means "a consistent type". MT1/4

Def A type $p(\bar{x})$ over A is isolated, if:

$$\exists \varphi(\bar{x}) \in L_n(A) \left\{ \begin{array}{l} \textcircled{1} \varphi(\bar{x}) \text{ is consistent (wrt } T), \text{ i.e.} \\ \varphi(M) \neq \emptyset \Leftrightarrow T(A) \vdash \exists \bar{x} \varphi(\bar{x}) \end{array} \right.$$

symbolically: $\varphi(\bar{x}) \vdash p(\bar{x}) \rightarrow$

$$\left[\begin{array}{l} \textcircled{2} \varphi(\bar{x}) \\ \forall \psi(\bar{x}) \in p(\bar{x}) \quad \varphi(M) \subseteq \psi(M) \\ \updownarrow \\ T(A) \vdash \varphi(\bar{x}) \rightarrow \psi(\bar{x}) \end{array} \right.$$

• When $p(\bar{x})$: a complete type over A , then:

$p(\bar{x})$ is isolated $\Leftrightarrow p$ is isolated in $S_n(A)$
in the topological sense
(i.e. $\{p\}$ is open)

Tarski - Vaught test

Assume $A \subseteq M$. Then $A = |N|$ for some $N \prec M$ iff

$$\forall \varphi(x) \in L_1(A) \quad [\varphi(M) \neq \emptyset \Rightarrow \varphi(M) \cap A \neq \emptyset]$$

Construction of an elementary submodel of M containing A :

• $A_n \subseteq M$, $n < \omega$, increasing chain of sets

recursive construction:

$$A_0 = A$$

$$A_n \subseteq A_{n+1} \subseteq M \text{ such that } \forall \psi(x) \in L_1(A_n)$$

$$[\psi(M) \neq \emptyset \Rightarrow \psi(M) \cap A_{n+1} \neq \emptyset]$$

$$A_\infty = \bigcup_{n < \omega} A_n \text{ satisfies TV-test.}$$

Omitting types theorem

MT1/5

Assume $p_n(\bar{x}_n)$, $n < \omega$: a family of non-isolated types in theory T , over \emptyset . Then:

$(\exists M \models T)$ M omits every p_n [i.e. $p_n(M) = \emptyset$]

Assume $M, N \models T$
 $\underset{A}{\cup}$

Def. $f: A \rightarrow N$ is elementary ($f: A \xrightarrow{\equiv} N$) if:

$$\forall \bar{a} \in A \forall \varphi(\bar{x}) \in L (M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a})))$$

$$\Leftrightarrow tp^M(\bar{a}) = tp^N(f(\bar{a}))$$

Elementary diagram of $A \subseteq M$:

$$D_e(A) = T(A) = Th(M, a)_{a \in A}$$

Remark $f: A \rightarrow N$ is elementary $\Leftrightarrow (N, f(a))_{a \in A} \models T(A)$

Atomic diagram of $A \subseteq M$:

$$D_{at}(A) = \{ \varphi \in D_{el}(A) : \varphi \text{ is a quantifier free sentence} \}$$
$$= \{ \varphi(\bar{a}) \in L(A) : M \models \varphi(\bar{a}) \text{ and } \varphi(\bar{a}) : \text{q.f.-sentence} \}$$

Remark $f: M \rightarrow N$ is a monomorphism (i.e.:

$$f: M \xrightarrow{\cong} f(M) \subseteq N$$

↑ substructure

$$\Leftrightarrow (N, f(a))_{a \in M} \models D_{at}(M).$$



Here always $f: M \rightarrow N$ denotes a monomorphism. MT 1/6

$M \subseteq N$: M is a submodel (substructure) of N

$M < N$: M is an elementary submodel of N , i.e.:

$$M \subseteq N \text{ and } \text{id}_M: M \xrightarrow{\equiv} N$$

Remark Assume $M < N$, $A \subseteq M$.

(1) Assume $p(\bar{x}) \subseteq L_n(A)$. Then

$p(\bar{x})$ is a consistent type in $M \Leftrightarrow p(\bar{x})$ is a consistent type in N

(2) Assume $A \subseteq B \subseteq M$

• If $p(\bar{x})$: a type over B , then $p \upharpoonright_A \stackrel{\text{def}}{=} p(\bar{x}) \cap L(A)$
a type over A

Let $r: S_n(B) \rightarrow S_n(A)$, $r(p) \stackrel{\text{def}}{=} p \upharpoonright_A$.

Then r : continuous and "onto".

(3) If $p(\bar{x})$: a type over A , then $\exists q(\bar{x}) \in S_n(A)$ $p(\bar{x}) \subseteq q(\bar{x})$

Saturation, universality, (strong) homogeneity.

Let $\kappa \in \mathbb{C}N$, $\kappa \neq \aleph_0$.

Def. (1) M κ -saturated if $\forall A \subseteq M \forall p \in S_n(A)$ $p(M) \neq \emptyset$
(nasyrony) $|A| < \kappa$

M is saturated if M is $\|M\|$ -saturated

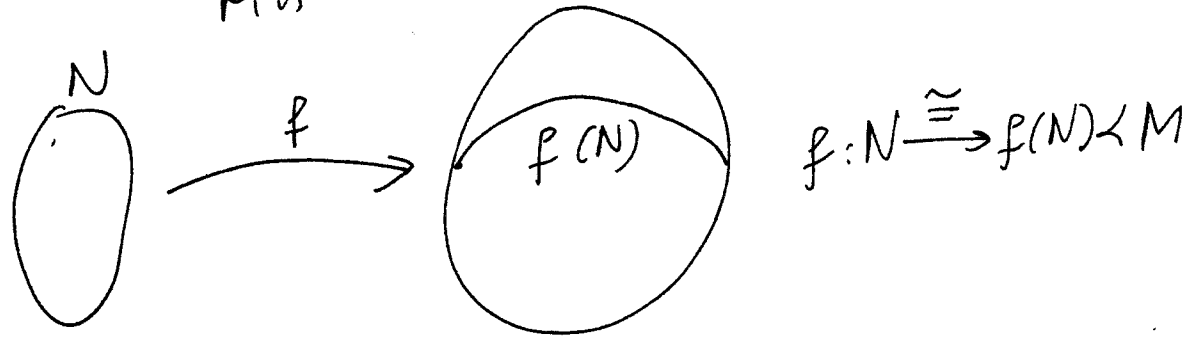
(2) M is κ -universal if $\forall N \equiv M$ ($\|N\| \leq \kappa \Rightarrow \exists f: N \xrightarrow{\equiv} M$)

↑
elementarily equivalent

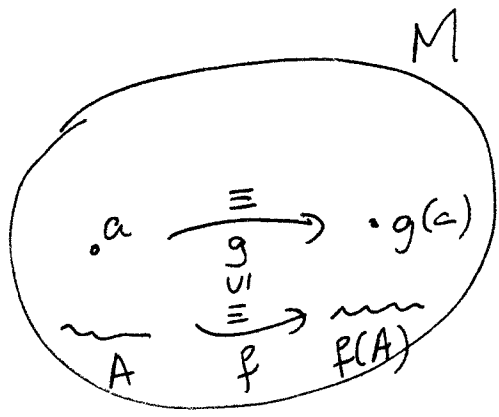
i.e. $\text{Th}(N) = \text{Th}(M)$

M : universal \Leftrightarrow $\|M\|$ -universal

MT1/7



(3) M : κ -homogeneous if $\forall A \subseteq M \forall a \in M \forall f: A \xrightarrow{\cong} M$
 $|A| < \kappa \quad \exists g: A \cup \{a\} \xrightarrow{\cong} M$
 homogeneous = $\|M\|$ -homogeneous.



4. M strongly κ -homogeneous if $\forall A \subseteq M \forall f: A \xrightarrow{\cong} M$
 $|A| < \kappa \quad \exists g: M \xrightarrow{\cong} M$

strongly homogeneous = strongly $\|M\|$ -homogeneous.

5. M is κ -compact if $(\forall 1$ -type $p(x)$ over $M)$
 $(|p| < \kappa \Rightarrow p(M) \neq \emptyset)$

Elementary chains of structures

Def $\langle M_\alpha : \alpha < \mu \rangle, \mu \in \text{Ord}$, : an elementary chain of structures if $(\forall \alpha < \beta < \mu) M_\alpha \prec M_\beta$.

Union of chain (when $\mu \in \text{Lim}$)

$$M_\mu = \bigcup_{\alpha < \mu} M_\alpha \quad ?$$

• $|M_\mu| := \bigcup_{\alpha < \mu} |M_\alpha|$

• $c \in L$ constant symbol

$c^{M_\mu} = c^{M_\alpha}$ for $\alpha < \mu$

• P : relation symbol

$P^{M_\mu}(a_1, \dots, a_n) \Leftrightarrow M_\alpha \models P(a_1, \dots, a_n)$ for $\alpha < \mu$
 $\bigcap_{\alpha < \mu} |M_\alpha|$ sufficiently large
 [so that $\bar{a} \subseteq M_\alpha$]

• $f^{M_\mu}(\bar{a}) = b \Leftrightarrow M_\alpha \models f(\bar{a}) = b$ for $\alpha < \mu$
 sufficiently large

Fact (Tarski) $M_\alpha \prec M_\mu$ for all $\alpha < \mu$.

Proof (1) $M_\alpha \subseteq M_\mu$ (substructure): exercise

(2) $\forall \varphi(\bar{x}) \in L \forall \alpha < \mu \forall \bar{a} \subseteq M_\alpha (M_\alpha \models \varphi(\bar{a}) \Leftrightarrow M_\mu \models \varphi(\bar{a}))$

(a) φ atomic: $M_\alpha \subseteq M_\mu \checkmark$

(b) $\varphi = \psi_1 \wedge \psi_2, \varphi = \neg \psi$: easy

(c) $\varphi(\bar{x}) = \exists y \psi(\bar{x}, y)$

$M_\alpha \models \varphi(\bar{a}) \Rightarrow M_\alpha \models \psi(\bar{a}, b)$ for some $b \in M_\alpha$

\Downarrow ind. assumption for ψ

$M_\mu \models \psi(\bar{a}, b)$

\Downarrow
 $M_\mu \models \varphi(\bar{a})$

$$M_\mu \models \varphi(\bar{a}) \Rightarrow M_\mu \models \psi(\bar{a}, b) \text{ for some } b \in M_\mu$$

$$\exists y \psi(\bar{a}, y)$$

\Downarrow ind. assumption

$$\downarrow \\ b \in M_\beta \text{ for some } \alpha \leq \beta < \mu$$

$$M_\beta \models \psi(\bar{a}, b)$$

\Downarrow

$$M_\beta \models \varphi(\bar{a})$$

$$\Downarrow M_\alpha < M_\beta$$

$$M_\alpha \models \varphi(\bar{a})$$

Elementary directed systems of structures:

Let (I, \leq) : a directed set, i.e.:

(1) \leq : partial order on I

(2) $(\forall a, b \in I)(\exists c \in I)(a \leq c \wedge b \leq c)$

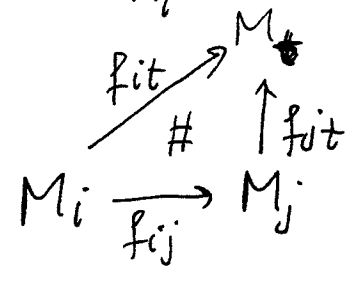
Example J : a set $\mapsto ([J]^{<\omega}, \leq)$: directed set.

Directed system of structures:

$$\mathcal{M} = (M_i, f_{ij})_{i \leq j \in I}$$

connecting functions $f_{ij}: M_i \rightarrow M_j$, $f_{ii} = id_{M_i}$. such that

$$(\forall i \leq j \leq t \in I) f_{it} = f_{jt} \circ f_{ij} \text{ (compatibility)}$$



System \mathcal{M} is elementary if all f_{ij} are elementary.

Example Elementary chain $(M_\alpha)_{\alpha < \mu}$

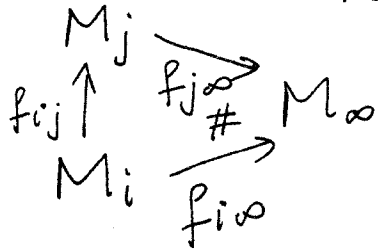
MT1/10

$\mathcal{M} = (M_\alpha, f_{\alpha\beta})_{\alpha \leq \beta < \mu}$ $f_{\alpha\beta} = id_{M_\alpha} : M_\alpha \xrightarrow{\cong} M_\beta$
elementary directed system of structures

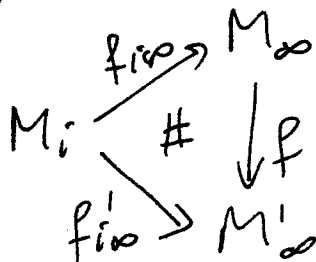
Direct limit of a directed system $\mathcal{M} : M_\infty = \varinjlim \mathcal{M}$

$(M_\infty, f_{i\infty})_{i \in I}$, where $f_{i\infty} : M_i \rightarrow M_\infty$ such that

(1) $\forall i \leq j \in I$ $f_{i\infty} = f_{j\infty} \circ f_{ij}$ [compatible with connecting functions]



(2) $(\forall (M'_\infty, f'_{i\infty})_{i \in I})$ satisfying (1) $\exists ! f : M_\infty \rightarrow M'_\infty$
(universality) $(\forall i \in I) f'_{i\infty} = f \circ f_{i\infty}$



Fact M_∞ exists (and is unique up to \cong).

If \mathcal{M} is elementary, then $f_{i,\infty} : M_i \xrightarrow{\cong} M_\infty$.

Proof 1. Construction of M_∞ :

$S := \dot{\bigcup}_{i \in I} |M_i|$: formally disjoint union.

\sim on S : an equivalence relation

$$M_i \quad M_j \quad \text{MTI/II}$$

$$\downarrow \quad \downarrow \quad \text{def}$$

$$x \sim y \Leftrightarrow f_{it}(x) = f_{jt}(y) \text{ for some } (= \text{every})$$

$$t \geq i, j$$

exercise: \sim is transitive.

$$|M_\infty| := S/\sim$$

- $\sim \upharpoonright |M_i|$: the equality (because f_{ij} : 1-1 (monomorphism))
- $f_{i\infty}(x) = x/\sim$, $f_{i\infty}: |M_i| \xrightarrow{1-1} |M_\infty|$.

L-structure on $|M_\infty|$:

- $c^{M_\infty} = c^{M_i}/\sim$
- $P^{M_\infty}(a_{i_1}/\sim, \dots, a_{i_m}/\sim) \Leftrightarrow M_t \models P(f_{i_1 t}(a_{i_1}), \dots, f_{i_m t}(a_{i_m}))$
 $a_{ij} \in M_{ij}$ for $t \geq i_1, \dots, i_m$
- f^{M_∞} : similarly

the rest is an exercise.

How to extend elementary mappings?

MT2/1

~~Def.~~ \mathbf{BA}_{lg} : Category of Boolean algebras

\mathbf{Comp}_0 : \mathbb{T} of compact Hausdorff 0-dimensional spaces

$$F: \mathbf{BA}_{lg} \rightarrow \mathbf{Comp}_0$$

$$G: \mathbf{Comp}_0 \rightarrow \mathbf{BA}_{lg}$$

$$F(A) = S(A)$$

$$G(X) = C(\text{open}(X))$$

F, G : contravariant functors "inverse" to each other

~~(F, G) is a duality of categories. (look it up)~~
Categories \mathbf{BA}_{lg} and \mathbf{Comp}_0 are dually equivalent.

A, B : Boolean algebras

$$f: A \rightarrow B \text{ homomorphism} \Rightarrow F(f): S(B) \rightarrow S(A)$$

$$F(f)(p) = f^{-1}[p]$$

continuous.

$$\text{Assume } f: A \xrightarrow{\cong} B$$

$$\begin{matrix} \cap & & \cap \\ T \neq M & , & N \neq T \end{matrix}$$

$$\text{Then } \hat{f}: L_n(A) \rightarrow L_n(B)$$

$$\hat{f}(\varphi(\bar{x}, \bar{a})) = \varphi(\bar{x}, f(\bar{a}))$$

homomorphism

of Boolean algebras.

even: monomorphism.

We skip $\hat{}$ in \hat{f} , so:

$$f: L_n(A) \rightarrow L_n(B) \text{ monomorphism}$$

$$f^*: S_n(B) \rightarrow S_n(A) \text{ epimorphism in } \mathbf{Comp}_0$$

i.e.: continuous onto

Lemma (on extensions of elementary mappings) MT2/2

Assume $M, N \models T$, $A \subseteq M$, $B \subseteq N$, $f: A \xrightarrow{\equiv} B$ "onto".

Assume $\overset{\psi}{\underset{a}{\#}}, \overset{\psi}{\underset{b}{\#}}$, $p = \text{tp}(a/A)$, $q = \text{tp}(b/B)$.

Then $f \cup \{Ka, b\}$ is elementary $\Leftrightarrow f^*(q) = p$.

[here $f^*: S(B) \xrightarrow{\cong} S(A)$,
homeomorphism]

Proof exercise.

Def. M is $(< \kappa_0)$ -universal $\Leftrightarrow \forall n \forall p \in S_n(\emptyset) p(M) \neq \emptyset$.

Remark $M: \kappa$ -universal $\Rightarrow M: (< \kappa_0)$ -universal.

Proof Let $p \in S_n(\emptyset)$.

Choose a countable $N \models T$ with $p(N) \neq \emptyset$.

$M: \kappa$ -universal $\Rightarrow \exists f: N \xrightarrow{\equiv} M$
 $\overset{\psi}{\underset{a}{\#}} \neq p \xrightarrow{f} \overset{\psi}{\underset{f(a)}{\#}} \neq p$.

Thm. (1) $M: \kappa$ -saturated $\Rightarrow M: \kappa$ -homogeneous
and κ -universal.

(2) $M: \kappa$ -~~universal~~ ^{homogeneous} and $(< \kappa_0)$ -universal \Rightarrow
 $M: \kappa$ -saturated.

Proof. (1) κ -homogeneity of M :

Assume $f: A \xrightarrow{\equiv} M$, $A \subseteq M$, $|A| < \kappa$, $a \in M$.

We seek $b \in M$ s.t. $g = f \cup \{ \langle a, b \rangle \}$ elementary

MT2/3

\Updownarrow Lemma

$$f^*(tp(b/B)) = tp(a/A).$$

⊗ Let $p = tp(a/A)$, $q = (f^*)^{-1}(p) \in S_1(B)$

\uparrow
 $S_1(A)$

Let $b \in M$ (exists by κ -saturation)
 \uparrow
good. of M

• κ -universality of M :

Assume $N \equiv M$, $\|N\| \leq \kappa$.

We seek $f: N \xrightarrow{\equiv} M$.

Let $\{a_\alpha : \alpha < \mu\}$: an enumeration of N , $\mu = \|N\|$.

We define $f(a_\alpha)$ by induction on $\alpha < \mu$:

• Suppose $f(a_\beta)$ defined for all $\beta < \alpha$ so that

$$f: \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} M$$

Want to find $f(a_\alpha)$ so that

$$f: \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} M.$$

~~By the Lemma it is enough that~~

Let $p = tp(a_\alpha / \{a_\beta : \beta < \alpha\})$.

By the lemma it is enough to find $f(a_\alpha) \in M$

so that $f^*(tp(f(a_\alpha) / \{f(a_\beta) : \beta < \alpha\})) = p$.

So let $q_f = (f^*)^{-1}(p) \in S_{\kappa}(\underbrace{\{f(a_\beta) : \beta < \alpha\}}_{\text{power} < \kappa})$

(MT2/4)

M κ -saturated $\Rightarrow q_f$ realized in M .

Let $f(a_\alpha) \in M$ s.t. $f(a_\alpha) \neq q_f$.

(2) Assume M is κ -homogeneous & $(< \aleph_0)$ -~~saturated~~ ^{universal}.

Want: M : κ -saturated.

So: Let $A \subseteq M$, $|A| < \kappa$, $p \in S_{\kappa}(A)$. Show: $p(M) \neq \emptyset$.

Induction on $|A|$.

Case (a): $|A| < \aleph_0$.

N

$\exists N \supseteq M$ $p(N) \neq \emptyset$. So let $b \in p$.

Let $A^* = A \cup \{b\}$
 $\quad \quad \quad \cup \{a_1, \dots, a_k\}$

Let $q_f = t_p^N(a_1, \dots, a_k, b) \in S_{\kappa+1}(\emptyset)$

q_f is realized in M ($(< \aleph_0)$ -universality),

by $\langle \underbrace{a'_1, \dots, a'_k}_{A'}, b' \rangle$

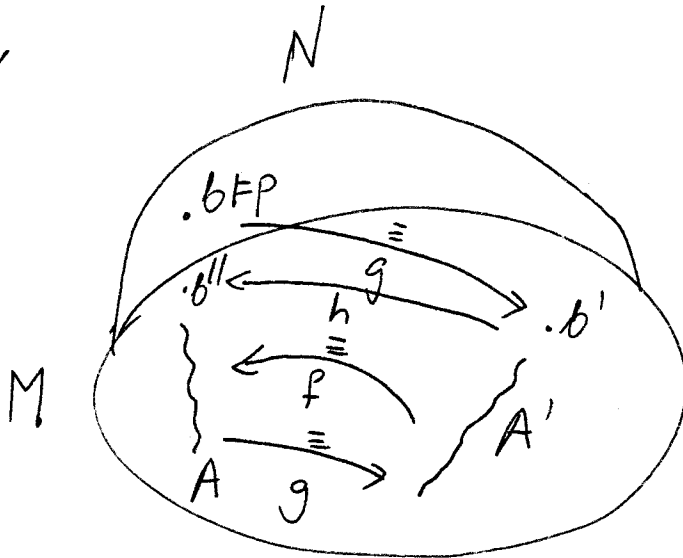
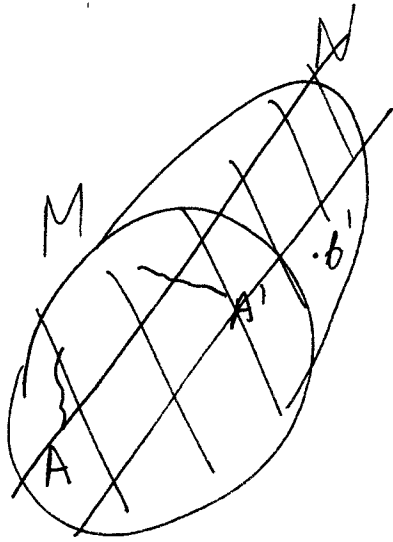
Let $g: A \cup \{b\} \rightarrow A' \cup \{b'\}$, $g(a_i) = a'_i$, $g(b) = b'$.

g : elementary.

$$\Rightarrow g \uparrow_A : A \xrightarrow{\cong} A'$$

$$\Downarrow f := (g \uparrow_A)^{-1} : A' \xrightarrow{\cong} A$$

M: κ -homogeneous $\Rightarrow \exists h : A' \cup \{b''\} \xrightarrow{\cong} A \cup \{b''\}$
for some $b'' \in M$.



$$\begin{array}{ccc} A \cup b & \xrightarrow{\cong} & A \cup b'' \\ g \downarrow \cong & & \cong \uparrow h \\ A' \cup b' & & \end{array}$$

Let $s = h \circ g$

$$s \uparrow_A = \underbrace{(h \uparrow_{A'})}_{\cong} \circ (g \uparrow_A) = \text{id}_A$$

$$s^*(\cancel{tp(b''/A)}) = \cancel{tp(b''/A)}$$

$$s \uparrow_A = \text{id}_A \Rightarrow s^* : S(A) \xrightarrow{\cong} S(A)$$

\cong
 $\text{id}_{S(A)}$

$$\text{hence: } p = tp(b/A) \underset{\uparrow \text{Lemma}}{=} s^*(tp(b''/A)) \underset{\uparrow}{=} tp(b''/A)$$

and $b'' \neq p$ $s^* = \text{id}_{S(A)}$

Case (b) $|A| = \mu$, $x_0 \leq \mu < \kappa$.

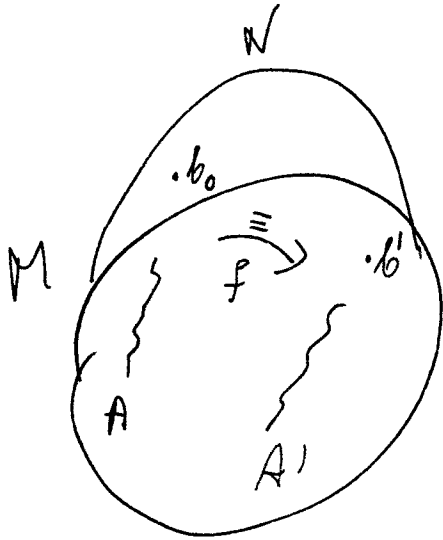
$$A = \{a_\alpha : \alpha < \mu\}, p \in S_1(A).$$

$$p \upharpoonright \emptyset \in S_1(\emptyset) \implies \exists b' \in M \quad b' \neq p \upharpoonright \emptyset,$$

$M: \langle x_0 \rangle$ -universal

$$\exists N \supset M \quad \exists b_0 \in N$$

$\begin{matrix} \pi \\ \downarrow \\ p \end{matrix}$



Will find $A' = \{a'_\alpha : \alpha < \mu\} \subseteq M$

s.t. $f: A b_0 \longrightarrow A' b'$
 given by $f(a_\alpha) = a'_\alpha$
 $f(b_0) = b'$

is elementary!

We find $a'_\alpha, \alpha < \mu$ by induction on $\alpha < \mu$.

So suppose $\alpha < \mu$ and a'_β already defined for all $\beta < \alpha$

so that $\boxed{p \upharpoonright \{a_\beta : \beta < \alpha \cup b\}} = \{a_\beta : \beta < \alpha \cup b_0\} \equiv \{a'_\beta : \beta < \alpha \cup b'\}$

$\equiv: f_0$

We look for a'_α .

Let $q = \text{tp}(a_\alpha / \{a_\beta : \beta < \alpha \cup \{b_0\}\})$

then $(f_0^*)^{-1}(q) \in S(\underbrace{\{a'_\beta : \beta < \alpha \cup \{b'\}\}}_{\text{power} < \mu \leq |A|})$

power $< \mu \leq |A|$

By the lemma it is enough that $a'_\alpha \neq (f_0^*)^{-1}(q)$.

But $M: \kappa$ -

M .

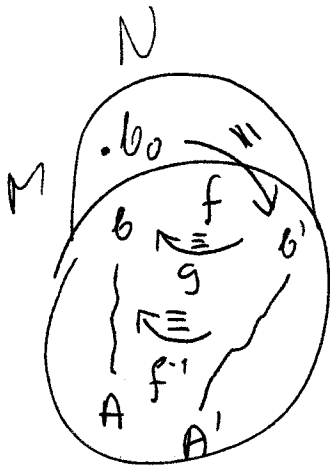
~~Let~~ By the inductive assumption on A :

MT2/7

$(f_0^*)^{-1}(q)$ is realized in M , so we are done with constructing A' .

Now: $f^{-1}: A' \xrightarrow{\cong} A$ in $M \leftarrow \kappa$ -homogeneous, so

$\exists \tilde{g}: A'b' \xrightarrow{\cong} Ab$ for some $b \in M$.



~~Let~~ Let $s = g \circ f$

$s: Ab_0 \xrightarrow{\cong} Ab$

$s \uparrow_A = (g \uparrow_{A'}) \circ (f \uparrow_A) = id_A$,

\parallel
 $(f \uparrow_A)^{-1}$

so $tp(b_0^{\#}/A) = tp(b/A)$
 $p =$

and p is realized in M .

Corollary

M is κ -saturated $\Leftrightarrow M$ is κ -homogeneous and κ -universal and κ -universal.

Proof \Rightarrow by Thm (1).

\Leftarrow κ -homogeneous + κ -universal \Rightarrow

κ -homogeneous + $(\langle \cdot \rangle_0)$ -universal \Rightarrow κ -saturated
Thm (2)

Properties of saturated models.

Thm. Assume $M, N \models T$ saturated models of the same power. Then $M \cong N$.

Proof $M = \{m_\alpha : \alpha < \kappa\}$, $N = \{n_\alpha : \alpha < \kappa\}$,
 $\kappa = \|M\| = \|N\|$. We find $f: M \xrightarrow{\cong} N$

back-and-forth method:

$$f = \bigcup_{\alpha < \kappa} f_\alpha \quad f_\alpha: M \xrightarrow{\cong} N \quad \text{s.t.}$$

partial, elementary

(1) $m_\alpha \in \text{Dom } f_{\alpha+1}$

$n_\alpha \in \text{Rng } f_{\alpha+1}$, $|f_\alpha| \leq 2 \cdot |\alpha|$

(2) $f_0 = \emptyset$

(3) For $\delta \in \text{Lim}$, $f_\delta = \bigcup_{\alpha < \delta} f_\alpha$.

(4) $f_{\alpha+1} = f_\alpha \cup \{ \langle \underset{\substack{\uparrow \\ N}}{m_\alpha}, m \rangle, \langle m, \underset{\substack{\uparrow \\ M}}{n_\alpha} \rangle \}$

Inductive step:

Suppose we have f_α . Want: $f_{\alpha+1}$.

Let $A_\alpha = \text{Dom } f_\alpha \subseteq M$, $B_\alpha = \text{Rng } f_\alpha \subseteq N$.

$$f_\alpha: A_\alpha \xrightarrow{\cong} B_\alpha$$

$$\downarrow$$

$$f_\alpha^*: S(B_\alpha) \xrightarrow{\cong} S(A_\alpha).$$

"forth": Find $n \in N$ st. $f_\alpha \cup \{ \langle m_\alpha, n \rangle \}$ elementary MT2/9

$$\begin{array}{c} \Downarrow \\ (f_\alpha^*)^{-1}(tp(m_\alpha/A_\alpha)) = tp(n/B_\alpha). \end{array}$$

Let $p = tp(m_\alpha/A_\alpha)$.

So $(f_\alpha^*)^{-1}(p) \in S(B_\alpha)$ is realized in N by some n .

"~~back~~": similarly.
back

Thm Assume $M, N \models T$ are homogeneous, of the same power and $\forall n < \omega \forall p \in S_n(\emptyset) (p(M) \neq \emptyset \Leftrightarrow p(N) \neq \emptyset)$.
Then $M \cong N$.

Lemma Under the assumptions of the Thm,

$$\forall A \subseteq M \exists f: A \xrightarrow{\cong} N.$$

Proof. Induction on $|A|$.

Case (a) $|A| < \aleph_0$. $A = \{a_1, \dots, a_n\}$.

Let $p = tp(\langle a_1, \dots, a_n \rangle) \in S_n(\emptyset)$. realized in M
 \Downarrow
 realized in N

by some $\langle b_1, \dots, b_n \rangle \in N$.
 $f(a_i) = b_i$ is good.

Case (b) $|A| = \mu \geq \aleph_0$, $A = \{a_\alpha : \alpha < \mu\}$

We find $f(a_\alpha)$ by induction on $\alpha < \mu$.

Inductive step.

Suppose $\alpha < \mu$ and for every $\beta < \alpha$ we have $f(a_\beta)$

$$\text{s.t. } f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} N.$$

We shall find $f(a_\alpha) \in N$ s.t. $f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} N.$

Let $a_{<\alpha} := \{a_\beta : \beta < \alpha\}$. Likewise $a_{\leq \alpha}$.

$$|a_{\leq \alpha}| < \mu = |A|$$

By inductive assumption: $\exists g : a_{\leq \alpha} \xrightarrow{\equiv} N.$

$$\text{Then } f \circ g^{-1} : \underbrace{g(a_{<\alpha})}_N \xrightarrow{\equiv} \underbrace{f(a_{<\alpha})}_N$$

By homogeneity of N : $\exists f(a_\alpha) \in N$ s.t.

$$f \circ g^{-1} : \underbrace{g(a_{<\alpha}) \cup g(a_\alpha)}_{g(a_{\leq \alpha})} \xrightarrow{\equiv} f(a_{<\alpha}) \cup f(a_\alpha) = f(a_{\leq \alpha})$$

$$\text{Then } f = (f \circ g^{-1}) \circ g : a_{\leq \alpha} \xrightarrow{\equiv} N.$$

Proof of the theorem

$$\kappa := \|M\| = \|N\|$$

$f : M \xrightarrow{\cong} N$ constructed by back-and-forth method

$$f = \bigcup_{\alpha < \kappa} f_\alpha, \quad f_\alpha : M \xrightarrow{\cong} N \text{ (partial elementary), } \alpha < \kappa$$

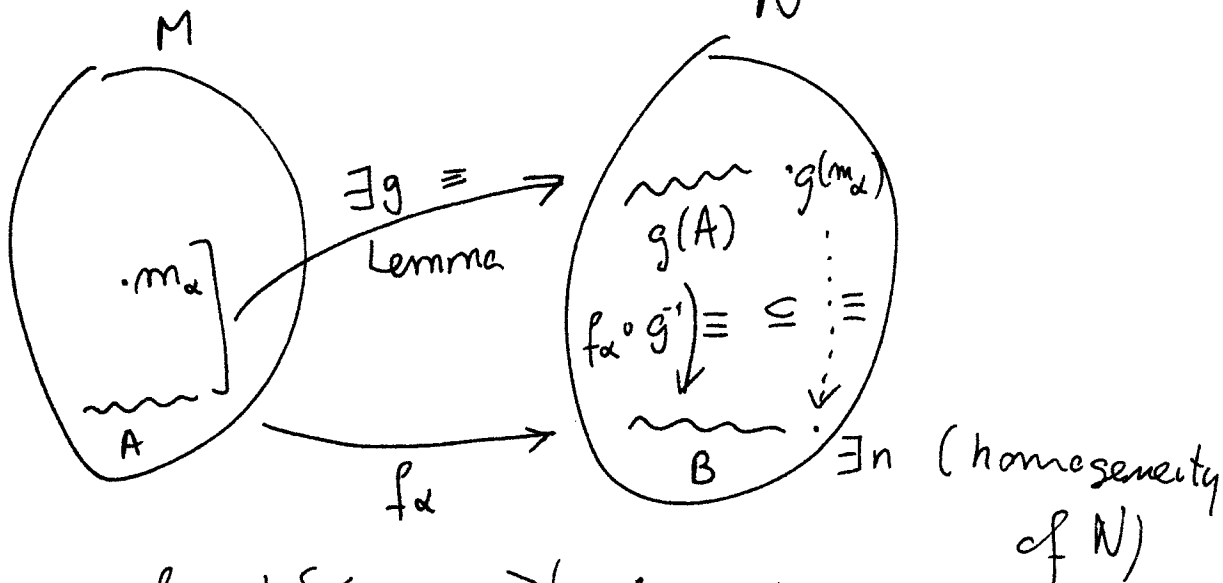
$|f_\alpha| \leq 2 \cdot |\alpha|$ + the same conditions as in the previous thm.

inductive step $f_\alpha \mapsto f_{\alpha+1}$

$A = \text{Dom } f_\alpha$

$B = \text{Rng } f_\alpha$

|"forth";



~~$f_\alpha \cup \{ \langle m_\alpha, n \rangle \}$ elementary~~

$h = (f_\alpha \circ g^{-1}) \upharpoonright_{g(A)} \cup \{ \langle g(m_\alpha), n \rangle \}$ elementary

$h \circ g : A \cup m_\alpha \xrightarrow{\equiv} B \cup n \subseteq N$

\cup
 f_α

"back": similarly.

Constructions of models:

MT3/1

- Saturated \implies • (strongly) homogeneous $\bar{\equiv}$

Thm. $\kappa = 2^{<\kappa}$, $\kappa \in \text{Reg}$, $\kappa > \aleph_0 \implies \exists M \models T$
 saturated, of power κ .

\parallel
 $\kappa^{<\kappa} = \kappa$

Proof

(*) $|S_i(A)| \leq 2^{|A| + \aleph_0}$, because: $|L_i(A)| = |A| + \aleph_0$

Here: $|A| < \kappa \implies |S_1(A)| \leq \kappa$.

Lemma $N \models T$, $\|N\| \leq \kappa \implies X_N := \bigcup \{S_i(A) : A \subseteq N \text{ \& } |A| < \kappa\}$
 the set has power $\leq \kappa$.

Pf $|\{A \subseteq N : |A| < \kappa\}| \leq \kappa^{<\kappa} = \kappa$.

- $|S_i(A)| \leq \kappa$ for such A .

Proof of the thm.

M_α , $\alpha < \kappa$: elementary chain of models of T of power κ .

- M_0 : whatever

• $M_\delta = \bigcup_{\alpha < \delta} M_\alpha$, when $\delta < \kappa$ limit.

- $M_{\alpha+1} \supset M_\alpha$ such that $\forall p \in X_{M_\alpha} p(M_{\alpha+1}) \neq \emptyset$:

$T' = \text{Th}(M_\alpha, m)_{m \in M_\alpha} \cup \bigcup_{\beta < \kappa} \{ \varphi(c_\beta) : \varphi(x) \in p_\beta \}$,

where $X_{M_\alpha} = \{ p_\beta : \beta < \kappa \}$

\uparrow
new constant symbols,

and T' in language $L(M_\alpha) \cup \{ c_\beta : \beta < \kappa \}$.

T^1 : consistent, has model of power κ : $M_{\alpha+1}$
such that $M_\alpha < M_{\alpha+1}$.

$M = \bigcup_{\alpha < \kappa} M_\alpha$: of power κ , saturated:

Let $A \subseteq M$, $|A| < \kappa$ and $p \in S_1^M(A)$
 $\kappa \in \text{Reg} \Rightarrow A \subseteq M_\alpha$ for some $\alpha < \kappa$. CW
↓

proof: $A = \{a_\beta : \beta < \mu\}$ for some $\mu < \kappa$.

$\forall \beta < \mu \exists \alpha_\beta < \kappa \ a_\beta \in M_{\alpha_\beta}$

$\{\alpha_\beta : \beta < \mu\} \subseteq \kappa$, $\mu < \text{cf}(\kappa) = \kappa$

$\Rightarrow \exists \alpha < \kappa \ \forall \beta < \mu \ \alpha_\beta < \alpha$
↑
 $A \subseteq M_\alpha$.

~~Let~~ $M_\alpha < M \Rightarrow p \in S_1^{M_\alpha}(A) = S_1^M(A)$

p realized in $M_{\alpha+1}$ by some $a \in M_{\alpha+1}$

$a \models p$ in $M_{\alpha+1} \Rightarrow a \models p$ in M .

$M_{\alpha+1} < M$

Monster model:

Let $\bar{\kappa}$: a large cardinal number.

"Ideal model" $M \models T$: saturated of power $\bar{\kappa}$

because: $\forall M \models T$ ($\|M\| < \bar{\kappa} \Rightarrow \exists M' < M \ M \cong M'$).

Advantages of saturated model M :

MT 3/3

(i) universality

(ii) strong homogeneity

~~More~~ ^{More} Weakly (a bit):

(1) κ -universality

(2) strong κ -homogeneity

$\text{Aut}(M)$: the group of automorphisms of M

$\text{Aut}(M/A) = \{ f \in \text{Aut}(M) : f|_A = \text{id}_A \}$: automorphisms of M over A
 $A \subseteq M$

Lemma Assume M is strongly κ -homogeneous, κ -saturated, $A \subseteq M$, $|A| < \kappa$. Then:

(1) For $a, b \in M$ ($\text{tp}(a/A) \stackrel{!}{=} \text{tp}(b/A) \Leftrightarrow a, b$ are in the same orbit of $\text{Aut}(M/A)$ on M).

(2) [orbits $\text{Aut}(M/A)$ on M^n] $\xleftrightarrow[\text{onto}]{1:1}$ $S_n(A)$

Proof (1) \Leftarrow : $f \in \text{Aut}(M/A)$, $f(a) = b$

$$\Downarrow \text{tp}(a/A) = \text{tp}(b/A)$$

\Rightarrow : $\text{tp}(a/A) = \text{tp}(b/A) \Rightarrow f: Aa \xrightarrow{\equiv} Ab$

strong κ -homogeneity $f|_A = \text{id}_A, f(a) = b$

$|A| < \kappa \Rightarrow f \in g \in \text{Aut}(M), g \in \text{Aut}(M/A)$
 $g(a) = b$: a, b in the same orbit of $\text{Aut}(M/A)$

$$(2) M^n \supseteq \mathcal{O} \xrightarrow[\varphi]{(1)} p_{\mathcal{O}} \in S_n(A)$$

\uparrow
 orbit of
 $\text{Aut}(M/A)$

\parallel
 common
 type $tp(a/A)$
 for $a \in \mathcal{O}$.

$$\mathcal{O} : \{ \text{orbits of } \text{Aut}(M/A) \text{ on } M^n \}$$

$\downarrow \varphi$

$$S_n(A)$$

$$\mathcal{O}_1 \neq \mathcal{O}_2 \xrightarrow{(1)} p_{\mathcal{O}_1} \neq p_{\mathcal{O}_2} \quad \boxed{\text{so } \varphi: 1-1}$$

[if $p_{\mathcal{O}_1} = p_{\mathcal{O}_2}$ then let $a \in \mathcal{O}_1, b \in \mathcal{O}_2 \Rightarrow \exists g \in \text{Aut}(M/A)$

$M: \kappa$ -saturated $\Rightarrow \varphi$: "onto".

$$g(a) = b \quad \checkmark.]$$

Def Let $\bar{\kappa}$: a (large) cardinal number,

$M \models T$ monster model, if $M: \bar{\kappa}$ -saturated,
 (w.r. to $\bar{\kappa}$) strongly $\bar{\kappa}$ -homogeneous

Thm. Assume $\aleph_0 \leq \kappa \in \mathcal{C.N.}$ Then

$\exists M: \kappa$ -saturated ~~is~~ strongly $\bar{\kappa}$ -saturated.

Proof $M = \bigcup_{\alpha < \kappa^+} M_\alpha$: union of elementary chain
 s.t.:

(1) $M_0 \models T$ any

(2) $M_\delta = \bigcup_{\alpha < \delta} M_\alpha$ if $\delta \in \text{Lim}$,

(3) $M_{\alpha+1} \supset M_\alpha$ s.t.:

(a) $\forall p \in S_1(M_\alpha)$ p realized in $M_{\alpha+1}$

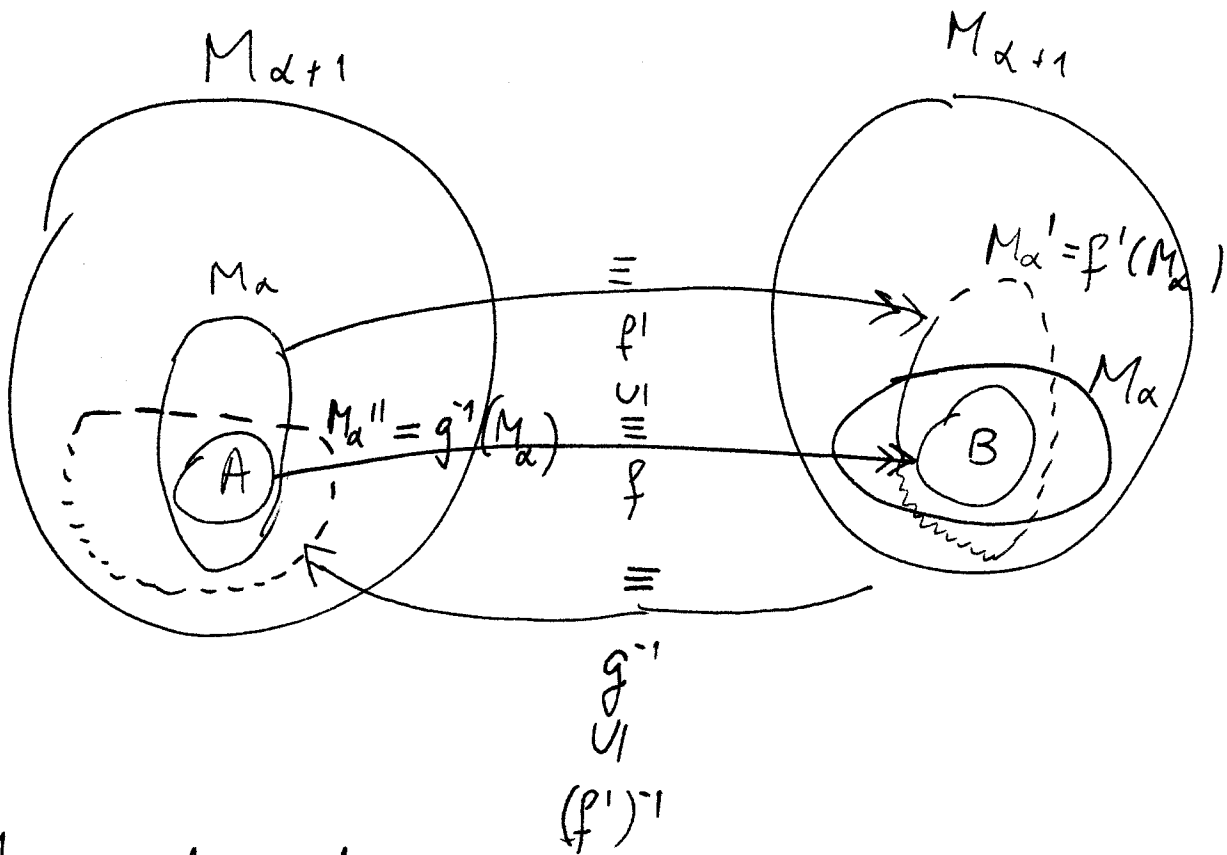
$$(b) \left(\forall f: A \xrightarrow{\equiv} B \right) \left(\exists g \geq f \right) \left(g: A' \xrightarrow{\equiv} B' \text{ in } M_{\alpha+1} \right)$$

$\begin{matrix} \cap & \cap & \cup & \cup \\ M_\alpha & M_\alpha & M_\alpha & M_\alpha \end{matrix}$

MT3/5.

It is enough ~~to~~ that $M_{\alpha+1}$ is $\|M_\alpha\|^+$ -saturated,
 To satisfy (a), (b). $M_{\alpha+1} \succ M_\alpha$

Proof of (b) for such $M_{\alpha+1}$:



I. M : κ -saturated: clear

II. M strongly κ -homogeneous:

Assume $A \subset M$, $|A| < \kappa$. Then $A \subseteq M_\alpha$, $B \subseteq M_\alpha$
 $f: A \xrightarrow{\equiv} M$ for some $\alpha < \kappa$.
 $B = f[A]$

• we construct

a sequence f_β , $\alpha \leq \beta < \kappa^+$:-

• increasing $f_\beta : M \xrightarrow{\equiv} M$

• ~~f_α~~ $f \subseteq f_\alpha$ partial elementary

(*) $M_\beta \subseteq \text{dom } f_\beta \cap \text{rng } f_\beta$.

• ~~f_α~~ f_α constructed according to 3)(b)

$$f_\alpha : M_{\alpha+1} \xrightarrow{\equiv} M_{\alpha+1}$$

• $f_\beta : M_{\beta+1} \xrightarrow{\equiv} M_{\beta+1}$, as in 3.(b)

when β : successor.

• $f_\delta = \bigcup_{\beta < \delta} f_\beta$ when δ limit, still $f_\delta : M_\delta \xrightarrow{\equiv} M_\delta$.

$$f_\infty = \bigcup_{\alpha \leq \beta < \kappa^+} f_\beta, \quad f_\beta \in \text{Aut}(M), \quad f \subseteq f_\beta.$$

Assumptions let $\bar{\kappa}$: a cardinal number large enough
so that:

(1) We consider only small models of T

||
of power $< \bar{\kappa}$, or even $\ll \bar{\kappa}$

(2) We work within a monster model $\mathcal{M} \models T$ (w.r. to $\bar{\kappa}$)

(3) We consider only small models $M \prec \mathcal{M}$

||
of power $< \bar{\kappa}$, or even $\ll \bar{\kappa}$,

Consequences:

(1) For $M, N < \mathcal{M}$, $M \subseteq N \Leftrightarrow M < N$

(2) Convention: For $\bar{a} \in \mathcal{M}$
 $\vDash \varphi(\bar{a})$ means $\mathcal{M} \vDash \varphi(\bar{a})$

(3) For $A \subseteq M < \mathcal{M}$:

$$S_m^M(A) = S_m^{\mathcal{M}}(A) =: S_m(A)$$

Notation Assume $p(\bar{x}), q(\bar{x})$ types (small, over \mathcal{M})

• $p(\bar{x}) \vdash q(\bar{x}) \Leftrightarrow p(\mathcal{M}) \subseteq q(\mathcal{M})$
 "p implies q"

• $p(\bar{x}) \equiv q(\bar{x}) \Leftrightarrow p \vdash q \ \& \ q \vdash p$
 ↑
 equivalent

Special case: $p(\bar{x}) = \{ \varphi(\bar{x}) \}$.

$\varphi(\bar{x}) \vdash q(\bar{x})$: "φ isolates q".

Remark: Syntactically:

$$p(\bar{x}) \vdash q(\bar{x}) \Leftrightarrow \forall \varphi(\bar{x}) \in q \ \exists p_0(\bar{x}) \subseteq p(\bar{x}) \text{ finite} \\ \uparrow \quad \uparrow \\ \text{types over } A \quad T(A) \vdash \bigwedge p_0(x) \rightarrow \varphi(x)$$

Remark (exercise)

$$p(\bar{x}) \vdash q(\bar{x}) \Leftrightarrow \forall M \models T \text{ IA-saturated } p(M) \subseteq q(M).$$

Def. (reminder)

Let $p(\bar{x})$: a type over A .

p is isolated over $A \iff \exists \varphi(\bar{x}) \in L(A)$ $\varphi \vdash p$.
consistent (with T)

Thm (omitting types, Ehrenfeucht)

Assume $p_n(\bar{x}_n), n < \omega$: ~~non~~ a family of non-isolated types over \emptyset . Then $\exists M \models T \forall n \underbrace{p_n(M) = \emptyset}$,
 M omits p_n .

Lemma

Assume A is stable, $p_n(\bar{x}_n), n < \omega$: a family of non-isolated types over A , $\varphi(\bar{x}) \in L_1(A)$, $\varphi(M) \neq \emptyset$.

Then $\exists c \in \varphi(M) \forall n$ p_n non-isolated over $A \cup \{c\}$.
i.e. φ : consistent

Proof [Lemma \Rightarrow Thm]

By the lemma: $\exists \underbrace{\{a_n : n < \omega\}}_A \subseteq M$ s.t.

(1) A satisfies the TV-test A

(2) $p_n, n < \omega$: non-isolated over A .

Construction of $a_n, n < \omega$: recursion on n .

Let $\{\varphi_n(x, \bar{y}) : n < \omega\}$: all formulas of L of this form.

Suppose $n < \omega$ and $\{a_i : i < n\} = a_{<n}$ already
 so that all $p_k, k < \omega$ still non-isolated over $a_{<n}$.

MT3/9

We consider a consistent formula $\varphi_n(x) \in L_1(a_{<n})$

By the Lemma we find $c \in \varphi_n(\mathcal{M})$ so that
 \downarrow
 a_n

all $p_k, k < \omega$, still non-isolated over $a_{\leq n}$.

• the formulas $\varphi_n(x), n < \omega$ may be chosen so that
 after ω steps:

$$\forall \varphi(x) \in L(A) \exists n \varphi = \varphi_n.$$

consistent

Then $A = \{a_n : n < \omega\}$ satisfies TV-test

$A = M \prec \mathcal{M}$, every p_k still non-isolated
 over A .

$$p_k(M) = \emptyset \text{ [if not,}$$

some $\bar{m} \models p_k$. then $\bar{m} \in \bar{a}_0$

$$(\bar{x}_k = \bar{m}) \vdash p_k(\bar{x}_k) \downarrow)$$

Proof of the Lemma.

Let $p(\bar{x})$: one of the types $p_n(\bar{x}_n)$.

Let $h(\bar{x}, y, \bar{a}) \in L(A)$

$$h(\bar{x}, c, \bar{a}) \vdash p(\bar{x}) \Leftrightarrow h(\mathcal{M}, c, \bar{a}) \subseteq p(\mathcal{M}).$$

$$\Leftrightarrow \forall \psi(\bar{x}) \in p(\bar{x}) \quad h(\mathcal{M}, c, \bar{a}) \subseteq \psi(\mathcal{M})$$

$$\Leftrightarrow \forall \psi \in p \quad \mathcal{M} \models \forall \bar{x} (h(\bar{x}, c, \bar{a}) \rightarrow \psi(\bar{x}))$$

$$\Leftrightarrow \forall \psi \in p \quad \psi_h(y) \in t_p(c/A)(y)$$

$$\text{where } \psi_h(y) = \forall \bar{x} (h(\bar{x}, y, \bar{a}) \rightarrow \psi(\bar{x}))$$

hence:

$$t_p(c/A) = t_p(c'/A) \Rightarrow [h(\bar{x}, c, \bar{a}) \vdash p \Leftrightarrow h(\bar{x}, c', \bar{a}) \vdash p]$$

$$h(\bar{x}, c, \bar{a}) \text{ consistent} \Leftrightarrow (\exists \bar{x} h(\bar{x}, y, \bar{a})) \in t_p(c/A)(y).$$

$$\text{Let } X_{h,p} = \{q \in S_1(A) : \text{For } c \models q, h(\bar{x}, c, \bar{a}) \vdash p(\bar{x}) \text{ and } h(\bar{x}, c, \bar{a}) \text{ consistent.}\}$$

"bad types"

$$\text{Let } q \in S_1(A) \text{ then} \quad \text{For } c \models q, h(\bar{x}, c, \bar{a}) \text{ consistent}$$

$$q \in X_{h,p} \Leftrightarrow q(y) \in S_1(A) \cap [\exists \bar{x} h(\bar{x}, y, \bar{a})] \cap \bigcap_{\psi \in p} [\psi_h(y)]$$

$$\text{For } c \models q, h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$$

(*) $X_{h,p}$: nowhere dense in $S_1(A)$.

Proof of (*): (a.a.)

Suppose $\theta(y) \in L_1(A)$ and $\emptyset \neq S_1(A) \cap [\theta] \subseteq X_{h,p}$.

$$\text{Let } \alpha(\bar{x}) = \exists y (h(\bar{x}, y, \bar{a}) \wedge \theta(y))$$

\uparrow
 $L(A)$

• $\alpha(\bar{x})$: consistent :

MT3 / M

Let $c \in \Theta(\mathcal{M})$

$$\Downarrow [\theta] \subseteq X_{n,p}$$

$$\mathcal{M} \models \exists \bar{x} h(\bar{x}, c, \bar{a}).$$

Let $\bar{d} \in \mathcal{M}$ s.t. $\mathcal{M} \models h(\bar{d}, c, \bar{a}).$

$$\bar{d} \text{ satisfies in } \mathcal{M} : \underbrace{\exists y h(\bar{x}, y, \bar{a})}_{\alpha(\bar{x})}$$

• $\alpha(\bar{x}) \vdash p(\bar{x})$, i.e. $\alpha(\mathcal{M}) \subseteq p(\mathcal{M}).$

Let $\bar{d} \in \alpha(\mathcal{M})$. So there is $c \in \mathcal{M}$ s.t.

$$\models h(\bar{d}, c, \bar{a}) \wedge \theta(c)$$

$$\Downarrow [\theta] \subseteq X_{n,d}$$

$$\forall \psi \in p \models \psi_n(c)$$

$$\forall \psi \in p \quad \underbrace{h(\bar{x}, c, \bar{a})}_{\Downarrow} \vdash \psi(\bar{x}) \Rightarrow h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$$

$$\neq \frac{\vdash}{\bar{d}} \Rightarrow \frac{\vdash}{\bar{d}} \quad \textcircled{y}$$

as: $p(\bar{x})$ non-isolated

$$\text{Let } X = \bigcup_{h,p_n} X_{h,p_n} \subseteq S_1(A).$$

meager. Let $q_i \in S_1(A) \cap [\varphi] \setminus X$

$c \models q_i$ good.

\textcircled{pf} (a.e) Suppose $p = p_n$ isolated over $A \cup \Sigma c \mathcal{L}$.

$$\exists h(\bar{x}, c, \bar{a}) \quad \underbrace{h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})}_{\text{consistent}} \Rightarrow \underbrace{q_i \in X_{h,p_n}}_{\substack{\neq \\ \vdash(\bar{c}/A)}} \quad \Downarrow.$$

14.03.2022

Def T is quantifier eliminable if $\forall \varphi \in L \exists \psi \in L$

$$T \vdash \varphi \leftrightarrow \psi$$

ψ open
 \equiv q.f.

Def. For $p(\bar{x}) \in S_n(\emptyset)$ let $p_o(\bar{x}) = \exists \varphi(\bar{x}) \in p(\bar{x})$.

Remark T is q.e. $\Leftrightarrow \forall n \forall p \in S_n(\emptyset) p_o \vdash p$ φ open.

Proof " \Rightarrow " Obvious. " \Leftarrow " Let $\varphi(\bar{x}) \in L$.

$$\bullet \forall p \in [\varphi] \cap S_n(\emptyset) \exists \psi \in p \quad p \in [\psi] \subseteq [\varphi]$$

Why? $p_o \vdash p$
 $p_o \vdash \varphi$

by compactness

\exists finite $p'_o \subseteq p_o$ s.t.
 $p'_o \vdash \varphi$, i.e.

$$p'_o(\mathcal{M}) \subseteq \varphi(\mathcal{M})$$

\Downarrow

$$\varphi(\mathcal{M}) = \left(\bigwedge_{\psi' \in p'_o} \psi' \right) (\mathcal{M}) \subseteq \varphi(\mathcal{M}) \rightsquigarrow \mathcal{M} \models \psi(\bar{x}) \rightarrow \varphi(\bar{x})$$

$$\varphi \vdash \varphi \Rightarrow [\varphi] \cap S_n(\emptyset) \subseteq [\varphi] \cap S_n(\emptyset)$$



Application $L = \{+, \cdot, 0, 1\}$: the language of rings.

ACF_p : the theory of algebraically closed fields of char p , in L .

Axioms:

1) field axioms

2) char $= p \neq 0$: $\underbrace{1 + \dots + 1}_p = 0$

2') $p = 0$: $\underbrace{1 + \dots + 1}_n = 0$ for $n \geq 1$

3) Every polynomial of deg n has a root:
 $0 < n$
 $\forall y_{n-1}, y_{n-2}, \dots, y_0 \exists x \quad x^n + y_{n-1}x^{n-1} + \dots + y_0 = 0.$

Fact ACF_p is complete.

Proof Let $M, N \models ACF_p$. Enough to show that $M \equiv N$. Let $\varepsilon > \|M\|, \|N\|$ and

let $M' \succ M, N' \succ N$.

power ε .

M', N' : uncountable acl fields of the same power and char

$$\begin{array}{c}
 \Downarrow \text{algebra} \\
 M' \cong N' \Rightarrow M' \equiv N' \\
 \quad \quad \quad \Downarrow \\
 \quad \quad \quad M \equiv N
 \end{array}$$

Fact ACF_p is q.e. (Chevalley, Tarski)

Proof (in \mathcal{M}) We will show that $\forall p \in S_n(\emptyset)$

$p_0 + p \Leftrightarrow p_0(\mathcal{M}) \subseteq p(\mathcal{M})$. Let $\bar{a} \models p_0$,

$\bar{b} \models p$, $\bar{a}, \bar{b} \in \mathcal{M}$. It's enough to prove

that $\exists f \in \text{Aut}(\mathcal{M}) f(\bar{a}) = \bar{b}$.

$$\begin{array}{l}
 \bar{a} = (a_1, \dots, a_n) \\
 \bar{b} = (b_1, \dots, b_n)
 \end{array}$$

Let $\langle \bar{a} \rangle, \langle \bar{b} \rangle$: the subrings with \mathbb{A}

of \mathcal{M} generated by \bar{a}, \bar{b} . $\bar{a}, \bar{b} \models p_0 \Rightarrow \langle \bar{a} \rangle \cong \langle \bar{b} \rangle$

$$\langle \bar{a} \rangle \cong \langle \bar{b} \rangle$$

\Downarrow unique \Downarrow

Some algebraic magic.

$$\mathcal{M} \cong \langle \bar{a} \rangle_0 \cong \langle \bar{b} \rangle_0 \subseteq \mathcal{M}$$

(fraction field)

$$\mathcal{M} \cong \underbrace{\langle \bar{a} \rangle_0^{\text{alg}}}_{F_a} \cong \underbrace{\langle \bar{b} \rangle_0^{\text{alg}}}_{F_b} \subseteq \mathcal{M}$$

(not unique)

$$\text{trdeg}(\mathcal{M}/\mathbb{F}_a) = \|\mathcal{M}\| = \text{trdeg}(\mathcal{M}/\mathbb{F}_b)$$

$$\begin{array}{ccc} \mathcal{M} \cong \mathbb{F}_a(X_\alpha, \alpha < \lambda)^{\text{alg}} & \Downarrow & \\ f \cong \downarrow & \curvearrowright & \cong \\ \mathcal{M} \cong \mathbb{F}_b(X_\beta, \beta < \lambda)^{\text{alg}} & & \end{array}$$

$$f \in \text{Aut}(\mathcal{M}), f(\bar{a}) = \bar{b}.$$



Types in $T = \text{ACF}_p$

Let $\mathcal{M} \models T$: a monster model.

subfield K , ^{UI} We will describe $S_n(K)$.
(small)

Let $\bar{a} \subseteq \mathcal{M}$, $|\bar{a}| = n$.

$$K[\bar{x}] \triangleright I(\bar{a}/K) = \{ f \in K[\bar{x}] : f(\bar{a}) = 0 \}$$

Remark 1) $\text{tp}(\bar{a}/K) = \text{tp}(\bar{a}'/K) \Leftrightarrow I(\bar{a}/K) = I(\bar{a}'/K)$

2) $\forall I \triangleleft K[\bar{x}]$ _{prime} $\exists \bar{a} \subseteq \mathcal{M} \ I(\bar{a}/K) = I$.

Proof 1) " \Rightarrow " $\text{tp}(\bar{a}/K) = \text{tp}(\bar{a}'/K) \Rightarrow \exists f \in \text{Aut}(\mathcal{M}/K)$
 $I(\bar{a}/K) = I(\bar{a}'/K) \stackrel{f}{=} f(\bar{a}) = \bar{a}'$

[alternatively: $f \in I(\bar{a}/K) \Leftrightarrow "f(\bar{x}) = 0" \in \text{tp}(\bar{a}/K)$]

" \Leftarrow " Assume $I(\bar{a}/K) = I(\bar{a}'/K) = I$.

$$K[\bar{a}] \cong_K K[\bar{x}] / I \cong K[\bar{a}']$$

$$\Downarrow$$

$$\exists f \in \text{Aut}(K[\bar{x}]/K) \quad f(\bar{a}) = \bar{a}'$$

$$\Downarrow$$

$$\text{tp}(\bar{a}/K) = \text{tp}(\bar{a}'/K).$$

2) $K \subseteq K[\bar{x}] / I = K[\bar{a}] \cong \bar{a} = \bar{x} / I$ and $I(\bar{a}/K) = I$.

•
•
•

□

$$S_1(K) = \{ \text{tp}(a/K) : a \in \mathcal{M} \} : a \text{ top-space.}$$

$$\Downarrow$$

$$p(x) = \text{tp}(a/K), \quad I_p = I(a/K) \triangleleft K[x]$$

a) $I_p \neq \{0\}$, i.e. a is algebraic / K , so

$0 \neq f$ \Downarrow " $f(x) = 0$ " $\in p(x)$. In fact,

irreducible over K " $f(x) = 0$ " $\vdash p(x)$ (isolates)

Let $a' \in \mathcal{M}$ s.t. $f(a') = 0 \Rightarrow I(a'/K) \ni f \ni$

$$p = \text{tp}(a/K) = \text{tp}(a'/K) \Leftarrow I(a/K) = I(a'/K) \Leftarrow \begin{matrix} f \text{ generates} \\ I(a'/K) \end{matrix}$$

$p(x)$ here is called algebraic.

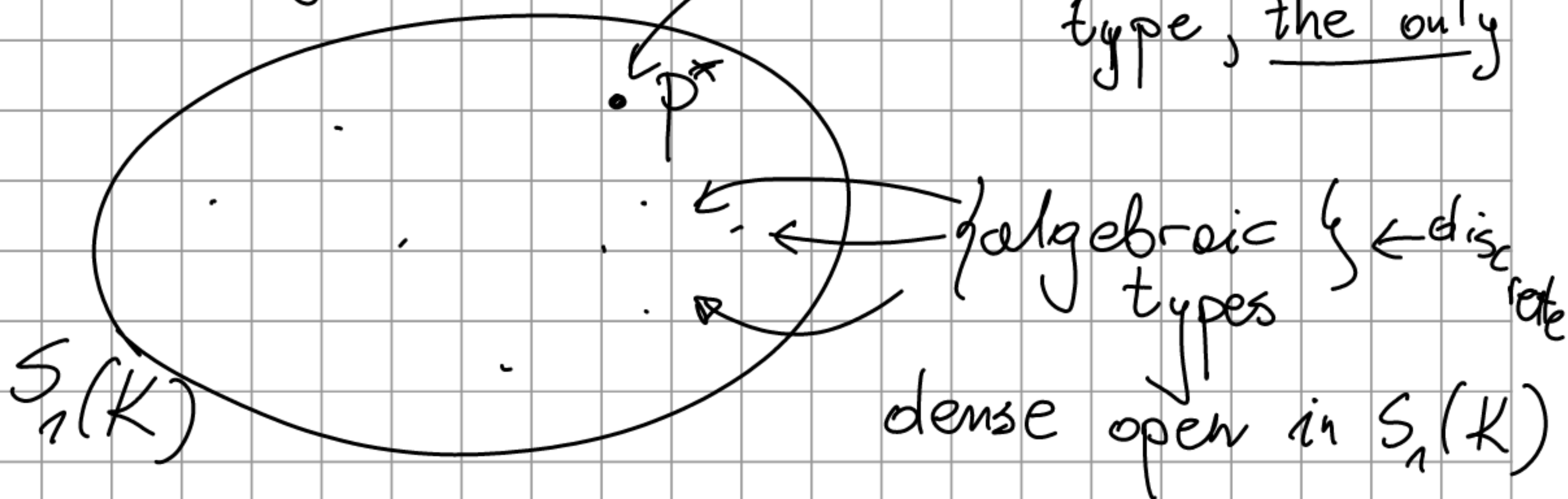
More generally (T : arbitrary)

Def $\varphi(\bar{x}) \in L(A)$ is called algebraic, if $0 < |\varphi(\mathcal{M})| < \aleph_0$, similarly for q : a type.

b) $I_p = \{0\}$: $p = \text{tp}(a/K)$ s.t. $a \in \mathcal{M}$ transcendental over K

↑
the transcendental type over K .

transcendental type, the only



If K cble then $S_n(K)$ cble

Corollary ACF_p is \aleph_0 -stable [recall: T is κ -stable $\Leftrightarrow \forall A \subseteq \mathcal{M}, |A| \leq \kappa$
 $|S_n(A)| \leq \kappa$]

Proof Let $A \subseteq \mathcal{M}$.
 A is abelian

$$A \subseteq K \subseteq \mathcal{M} \quad |S_n(A)| \leq |S_n(K)| = \aleph_0$$

\uparrow
 abelian subfield

Remark T is totally transcendental $\Leftrightarrow T: \aleph_0$ -stable

Proof " \Rightarrow " from def " \Leftarrow ": (A.a.) Let $\kappa > \aleph_0$.

Suppose $|A| \leq \kappa < |S_n(A)|$ for some $A \subseteq \mathcal{M}$.

Shall find $A_0 \subseteq A$ with $|S_n(A_0)| \geq 2^{\aleph_0}$.
 A_0 is abelian

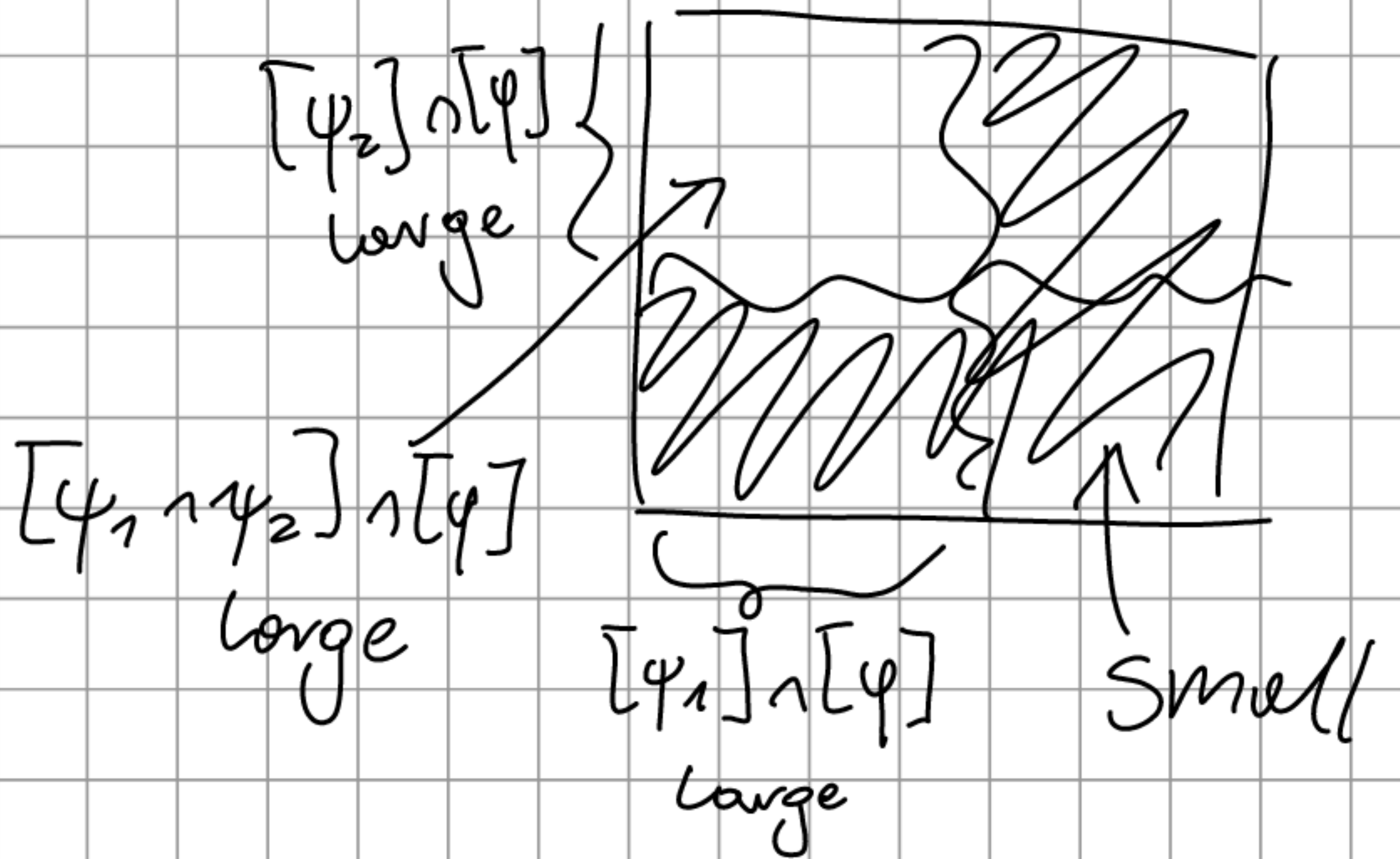
(def) $\varphi(x) \in L(A)$ is large iff $|S_n(A) \cap [\varphi]| > \kappa$
 otherwise: $\varphi(x)$ small.

(a) " $x=x$ " is large

(b) if $\varphi(x)$ is large, then $\exists \psi_1, \psi_2 \in L(A)$ large
 s.t. $\varphi(\mathcal{M}) = \psi_1(\mathcal{M}) \cup \psi_2(\mathcal{M})$

Pf. (b) if not then $\exists \psi \in L_n(A)$: $\psi \wedge \varphi$ is large
 is a complete type is $S_n(A) \cap [\psi]$

1° \mathcal{P}^* : consistent: $\psi_1, \psi_2 \in \mathcal{P}^* \Rightarrow \psi_1 \wedge \psi_2 \in \mathcal{P}^*$



2° \mathcal{P}^* : complete OK

$\mathcal{P}^* \in S_n(A) \cap [\phi]$: the only large type.

$$S_n(A) \cap [\phi] = \underbrace{\mathcal{P}^*}_{\text{large}} \cup \underbrace{\bigcup_{\substack{\psi \in L_n(A) \\ \psi \vdash \phi}}}_{\text{small}} (S_n(A) \cap [\phi])$$

$\leq \aleph$ $\leq \aleph$

$> \aleph$

$\leq \aleph$

↙

c) a tree of large formulas in $L_1(A)$ $\varphi_\eta(x)$,
 $\eta \in 2^{<\omega}$ st. $\varphi_\eta(\mathcal{M}) = \varphi_{\eta_0}(\mathcal{M}) \cup \varphi_{\eta_1}(\mathcal{M})$
↑
by (b)

Let $A_0 \subseteq A$: the set of all params of $\varphi_\eta, \eta \in 2^{<\omega}$

Then $|A_0| \leq \aleph_0^{\aleph_0}$.

For $\eta = 2^{<\omega}$: $\mathcal{P}_\eta^0 = \{ \varphi_{\eta|n}(x) : x < \omega \}$: a consistent
 \mathcal{L} -type over A_0

When $\nu \neq \eta$
then $\mathcal{P}_\eta \neq \mathcal{P}_\nu$

$\mathcal{P}_\eta \in S_\eta(A_0)$

$$\Downarrow \quad |S_\eta(A_0)| \geq 2^{\aleph_0} > \aleph_0 \quad \Downarrow$$

21.03.2022

CONSTRUCTION OF SPECIAL MODELS: N, M ⊨ T

Def. M is atomic if $\forall \bar{a} \in M$ $\text{tp}(\bar{a}/\emptyset) = \text{tp}(\bar{a})$ is isolated.

(2) M is prime if $\forall N \models T \exists f: M \xrightarrow{\cong} N$

Example $T = \text{ACF}_p$, F_p : prime field of char p

a) F_p : atomic (exercise)

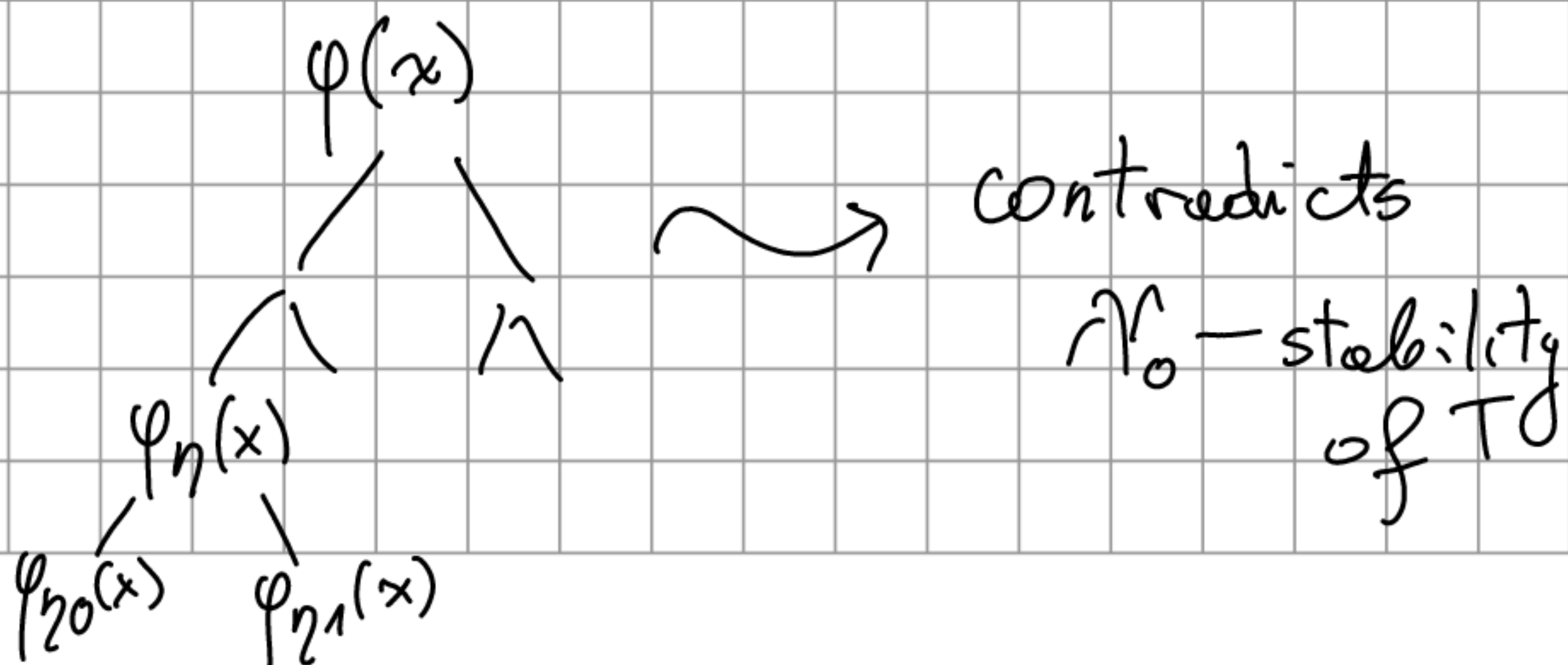
b) F_p : prime (exercise)

Thm. $T: \aleph_0$ -stable $\Rightarrow T$ has a prime model.

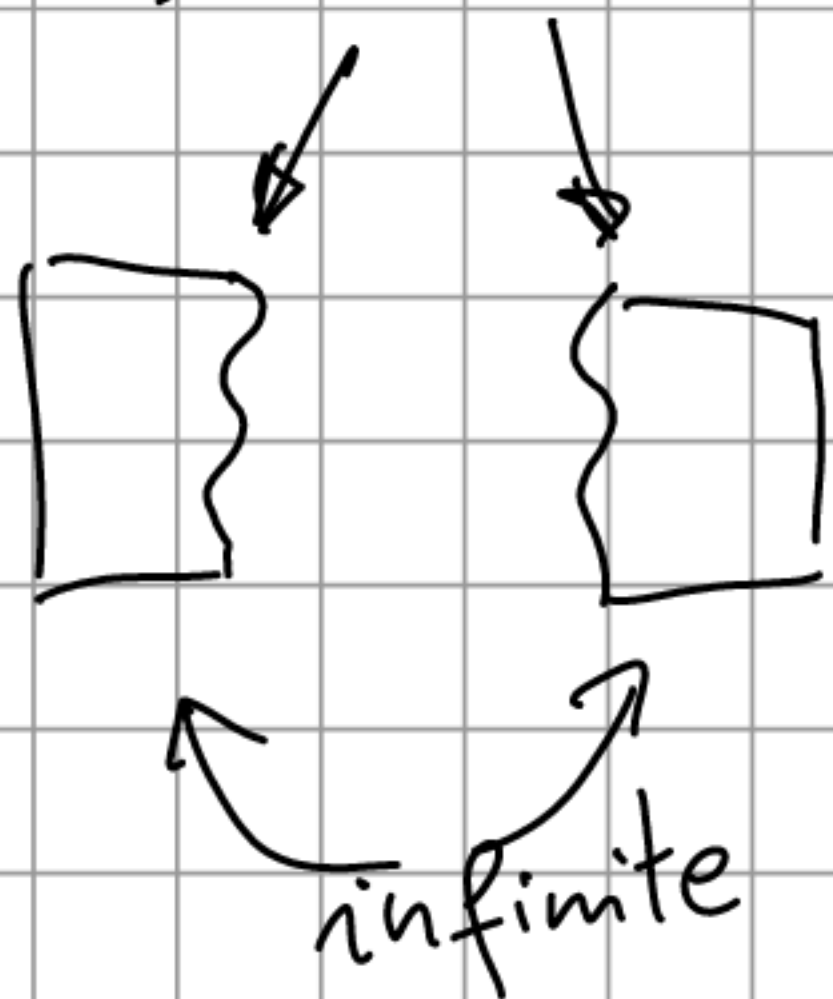
Lemma 1 $T: \aleph_0$ -stable $\Rightarrow \forall A \subseteq \mathcal{M}$ $\{ \text{isolated types} \} \subseteq S_n(A)$ dense

Pf. Suppose $\varphi(x) \in L_n(A)$ s.t. in $S_n(A) \cap [\varphi]$ (consistent with T)

there is no isolated types \Rightarrow a tree of formulas $\varphi_\eta(x) \in L_n(A)$, $\eta \in 2^{<\omega}$



clopen $\{ \} \subseteq S_1(A) \cap [\varphi]$



□

Lemma 2 $(a, b \in \mathcal{M})$ $tp(a)$ isolated and $tp(b/a)$ isolated

$\Leftrightarrow tp(ab)$ isolated

Pf " \Rightarrow ": $\varphi(x) \vdash tp(a), \psi(a, y) \vdash tp(b/a)(y)$

$p_a(y) \subseteq S_y(a)$

Then: $\varphi(x) \wedge \psi(x, y) \vdash tp(ab)$

Let $a', b' \in \mathcal{M}$ satisfy $\varphi(x) \wedge \psi(x, y)$

$\models \varphi(a') \Rightarrow tp(a') = tp(a) \Rightarrow \psi(a', y) \vdash p_{a'}(y)$

$\supseteq S_y(a')$

$\models \psi(a', b') \Rightarrow \models p_{a'}(b')$

\Downarrow

$ab \equiv a'b'$

and $tp(ab) = tp(a'b')$

" \Leftarrow ": $\Theta(x, y) \vdash \text{tp}(a, b)$.

(a) " $\exists y \Theta(x, y)$ " $\vdash \text{tp}(a)$.

because Let $a' \in \mathcal{M}$ satisfy " $\exists y \Theta(x, y)$ ".

So there is b' s.t. $\models \Theta(a', b')$

$\Rightarrow \text{tp}(ab) = \text{tp}(a', b') \Rightarrow \text{tp}(a) = \text{tp}(a')$.

(b) $\Theta(a, y) \vdash \text{tp}(b/a)(y)$

because: similar to (a) ▣

Proof of the thm. Construction of a prime model of T :

$A = \{a_n : n < \omega\} \subseteq \mathcal{M}$ so that:

1) A satisfies TV-test

2) $\forall n$ $\text{tp}(a_n/a_{<n})$ is isolated

At step n choose a_n : $[a_{<n} = \{a_k : k < n\}]$

Let $\varphi(x) \in L_n(a_{<n})$ consistent.

Let $a_n \in \varphi(\mathcal{M})$ s.t. $\text{tp}(a_n/a_{<n})$ is isolated (lemma 1)

Suitable choice of φ 's ensures (1).

$M \models T$ is prime. Let $N \models T$. We find

$f(a_n) \in N$ for $n < \omega$ s.t. \exists ^{arbitrary} $f: M \equiv \rightarrow N$.

At step n $f[a_{<n}] \subseteq N$ with $f: a_{<n} \equiv \rightarrow N$

Let $p(x) = \text{tp}(a_n/a_{<n})$ (isolated)

$$\Downarrow \\ f^*(p)(x) \in S(f(a_{<n}))$$

isolated

too, hence realised by $f(a_n)$

$$\Downarrow \\ f: a_{\leq n} \equiv \rightarrow N$$



Remark (1) A prime model $M \models T$ is atomic

(2) If $M \models T$ is atomic, then M prime

Corollary \hat{F}_p is atomic.

Proof of remark

(1) Let $p(\bar{x}) \in S_n(\emptyset)$ non-isolated. Will show

$p(M) = \emptyset$. Let $N \models T$ be omitting p .

$\exists f: M \xrightarrow{\equiv} N \Rightarrow p(M) = \emptyset$.

(2) Let $M = \prod_T \{a_n : n < \omega\}$ atomic.

Then $\forall n$ $tp(a_n)$ is isolated

$\forall n$ $tp(a_n/a_{<n})$ is isolated

\Downarrow pf of thm

M prime. □

Corollary A prime model of T is unique (up to \cong)

Proof Let $M, N \models T$ both prime $\stackrel{\text{remark}}{\Rightarrow} M, N$

are stable and atomic, so we have embeddings

in both directions, using back-and-forth

we get the iso.

Let $f \in \text{Aut}(\mathcal{M}) : f(\bar{a}') = \bar{a}$
 $b = f(b')$

Then $\bar{a}'b' \stackrel{\equiv}{\underset{f}{\rightarrow}} \bar{a}b \Rightarrow \bar{a}b \models p(\bar{x}, y)$

so $\text{tp}(\bar{a}b)$ is isolated $\stackrel{\text{lemma 2}}{\Rightarrow}$ $\text{tp}(b/\bar{a})$ isolated
 \Downarrow
 $\psi(\bar{a}, y)$

So $q(y) = \text{tp}(b/\bar{a})$

Given $\omega \rightarrow$ we construct a model $M = \{a_n : n < \omega\}$ s.t. $\forall n \text{ tp}(a_n/a_{<n})$ is isolated

\Downarrow lemma 2

M atomic dble $\Rightarrow M$ prime.

Corollary If $\forall n |S_n(\emptyset)| \leq \aleph_0$, then T has a prime model.

Corollary A prime model (of a dble T) is homogeneous (exercise).

The number of countable models of $T: I(T, \aleph_0), n(T)$.

Remark $1 \leq n(T) \leq 2^{\aleph_0}$

$$M \models T \Rightarrow M \cong \underbrace{(N, \dots)}_{\leq 2^{\aleph_0} \text{ L-structures like that}}$$

Recall $n(T) = 1 \Leftrightarrow \forall n |S_n(\emptyset)| < \aleph_0$

($T: \aleph_0$ -categorical)

Vaught conjecture (1961)

$$n(T) > \aleph_0 \Rightarrow n(T) = 2^{\aleph_0}$$

Thm (M. Morley, 1971) $\aleph_0 < n(T) < 2^{\aleph_0} \Rightarrow n(T) = \aleph_1$

Thm (Vaught, 1961) $n(T) \neq 2$

Proof (A.a) suppose $n(T) = 2$.

$$n(T) < 2^{\aleph_0} \Rightarrow T \text{ small (i.e. } \forall n |S_n(\emptyset)| \leq \aleph_0)$$

⋮

25.03.2022

Example (Andrzej Ehrenfeucht) Theory with exactly 3 stable theories.

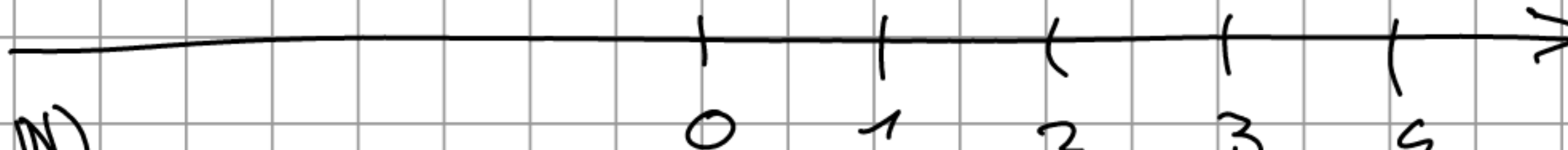
$T_0 = \text{Th}(\mathbb{Q}, \leq)$. Look at $T = T_0(N) = \text{Th}(\mathbb{Q}, \leq, n)_{n \in \mathbb{N}}$.

T_0 is q.e. $\Rightarrow T$ is also q.e.

$S_1^T(\emptyset) = S_1^{T_0}(N)$. The types in $S_1^T(N)$:

realised in (\mathbb{Q}, \leq, N)

isolated



- $p_i(x) \equiv \{x = i\}$, $i \in \mathbb{N}$
- $r_i(x) \equiv \{i-1 \leq x \leq i\}$, $i \in \mathbb{N}$, $-1 \approx -\infty$
- $s(x) \equiv \{x > i : i \in \mathbb{N}\}$

omitted

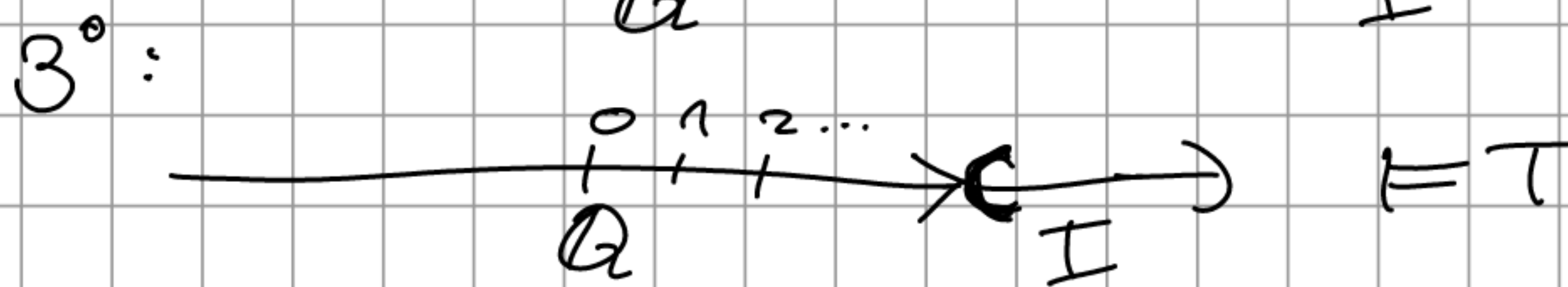
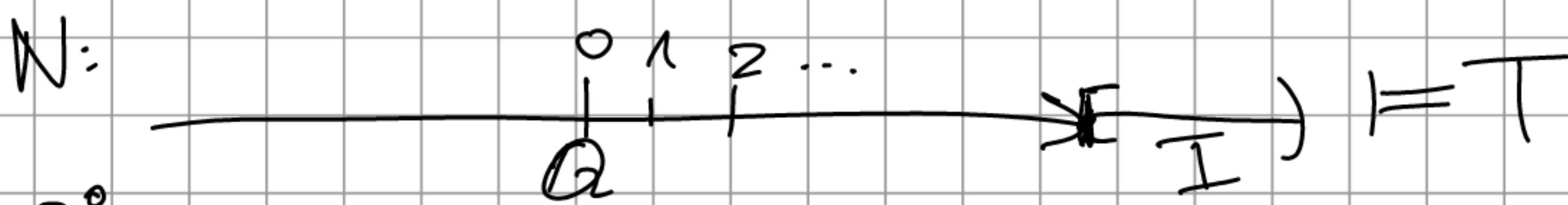
We will point 3 stable models $N \models T$.

1° $N = M$: prime model of T .

2° $s(N)$ has the minimal element.

3° $s(N)$ hasn't minimal element.

2°: $\mathbb{Q} \cup ([0,1) \cap \mathbb{Q})$:



Variants: 3, 4, 5, 6, ...

Problem Does there exist a stable T with $1 < n(T) < \aleph_0$?

Def ($A \subseteq \mathcal{U}$), $M \prec \mathcal{U}$ is prime over A if:

(1) $A \subseteq M$

(2) $\forall N \prec \mathcal{U} \exists f: M \xrightarrow{\cong} N: f|_A = \text{id}_A$

Equivalently M is a prime model of $T(A)$.

• If A is stable \rightarrow full description of prime models of $T(A)$.

• If A is unstable \rightarrow in general not much can be set

Thm \aleph_0 -stable $\implies \forall A \exists M \neq T(A)$
 \mathcal{M} prime

Proof $M = A \cup \{a_\alpha : \alpha < ?\}$

Construction of a_α 's s.t.:

(i) $A \cup \{a_\alpha : \alpha < ?\}$ satisfies the TV-test

(ii) $\forall \alpha < ?$ $\text{tp}(a_\alpha / A a_{<\alpha})$ is isolated.

At some point it has to terminate

(i.e. we cannot add more elements).

Claim Assume $N \not\prec \mathcal{M}$. Then $\forall \alpha \exists f: A \cup a_{<\alpha} \xrightarrow{\cong} N$

s.t. $f|_A = \text{id}_A$.

Proof We define $f(a_\beta)$ for all $\beta < \alpha$

by ind. on β so that $f|_A = \text{id}_A$ and $f: A a_{\leq \beta} \xrightarrow{\cong} N$.

Take $\beta < \alpha$ and suppose $\forall \beta' < \beta$ $f(a_{\beta'}) \downarrow$

so that the condition holds.

$p(x) = \text{tp}(a_\beta / A a_{<\beta})$ is isolated.

$f: A a_{<\beta} \xrightarrow{\cong} f[A a_{<\beta}] \subseteq N$

$f(p)$ is realised by c

$p(x) \in S_1(A a_{<\beta})$ isolated $\xrightarrow{f^*}$ $f(p) \in S_1(f[A a_{<\beta}])$ isolated

Now we put $f(a_\beta) = c$.

Claim ~~1~~

By the claim after some time we cannot get any more elements.

Additional property of the construction:

At the step α we consider a formula $\varphi(x) \in L(A_{\alpha, \alpha})$ with no consistent realisation in $A_{\alpha, \alpha}$, choose a_α s.t. $\models \varphi(a_\alpha)$.

Problems Is a prime model over A unique up to isomorphism over A ?

Answer: not always. However the prime model M over A constructed by the previous construction is unique up to \cong_A and it's called primary over A .

Thm M, N : primary over $A \Rightarrow M \cong_A N$.

Proof $M = A \cup \{a_\alpha : \alpha < \gamma\}$: an "isolated construction" of M over A , i.e. $\text{tp}(a_\alpha / A a_{\beta < \alpha})$ is isolated by a formula $\varphi_\alpha(x)$ over $A a_{\beta < \alpha}$ s.t. $C_\alpha \subseteq \gamma$

Def. $X \subseteq \gamma$ is closed if $\forall \alpha \in X \ C_\alpha \subseteq X$

Remark (1) $\alpha \in \gamma \Rightarrow \exists$ minimal $X \subseteq \alpha$ s.t. X is finite

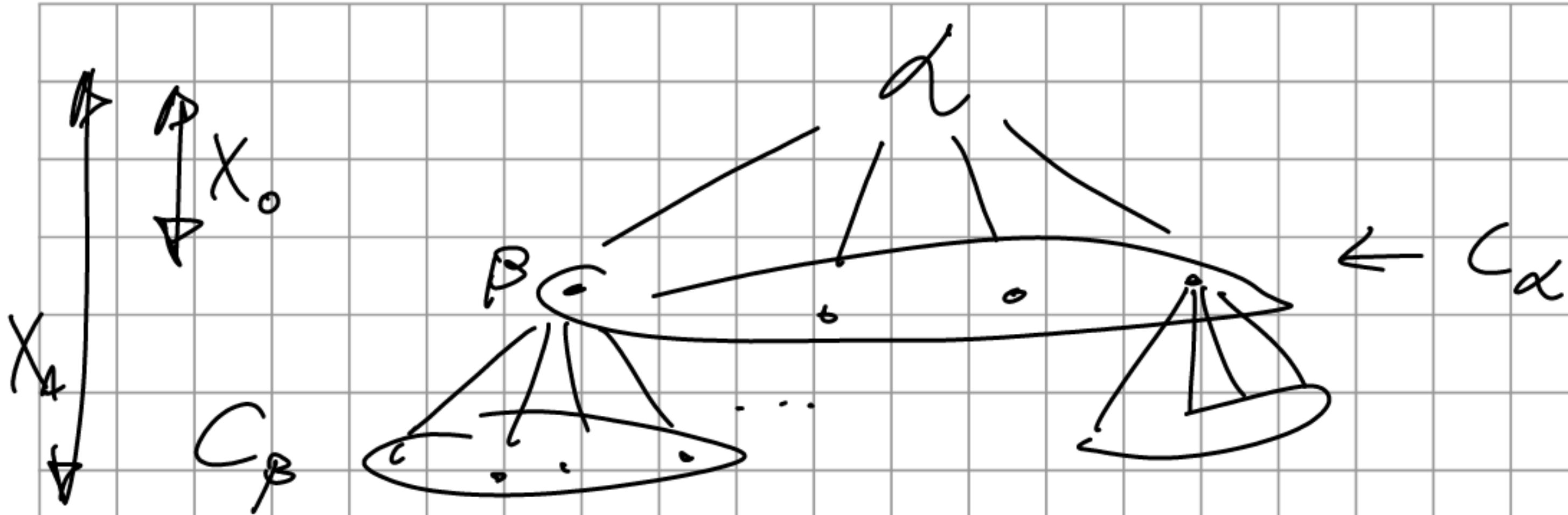
$\underbrace{X \cup \alpha}_\alpha$ is closed.

(2) A union of family of closed subsets of γ is closed.

Proof (remark) (2) is obvious.

(1): Take $X_0 = C_\alpha$, then $X_n = X_{n+1} \cup \bigcup_{\beta \in X_n} C_\beta$
 \uparrow
 finite because C_β finite

Then $X = \bigcup_n X_n$. Then $X \cup \alpha$ is finite and closed!



This tree has no infinite branch
 (because there are no infinite
 decreasing sequence of ordinals).

Remark II

Remark Assume X : closed. Then

$$A \cup \langle a_\alpha : \alpha \in X \rangle \overset{\text{concatenation}}{\uparrow} \langle a_\alpha : \alpha \in \gamma \setminus X \rangle$$

is an isolated construction over A .

Pf. (remark) as $A \cup \langle a_\alpha : \alpha \in X \rangle$ is an i -construction
 over A
 (by the fact that X is closed)

(2) Suppose that $\alpha < \gamma$ and $\alpha \notin X$. Will
 show that $\text{tp}(a_\alpha / A a_\chi a_{\chi \cap (\alpha)})$ is isolated.

$$\varphi_\alpha(A) \vdash \text{tp}(a_\alpha / A a_{<\alpha}) \vdash \text{tp}(a_\alpha / A a_{\alpha} a_{\alpha}^c \text{ } \dots)$$

↑ will show

Let $X_0 \subseteq X$ s.t. $X_0 \cap a \neq \emptyset$.
finite

Enough to show that $\text{tp}(a_\alpha / A a_{<\alpha}) \vdash \text{tp}(a_\alpha / A a_{<\alpha} a_{X_0})$

Wlog By the remark $X_0 \cup a$ is closed.

So: $A \cup \langle a_\beta : \beta < \alpha \rangle \wedge \langle a_\beta : \beta \in X_0 \rangle$ is

an i -construction over A , but for $\beta \in X_0$

$$\text{tp}(a_\beta / A a_{<\alpha} a_{(\beta) \cap X_0}) \vdash \text{tp}(a_\beta / A a_{<\beta})$$

Because $\varphi_\beta(x) \vdash$ and $\varphi_\beta \in$.

So it implies also $\text{tp}(a_\beta / A a_{<\alpha} a_{(\beta) \cap X_0})$.

$$\Rightarrow \text{tp}(a_\alpha / A a_{<\alpha} a_{(\beta) \cap X_0}) \vdash \text{tp}(a_\alpha / A a_{<\alpha} a_{(\beta) \cap X})$$

the
proof
of e123

(we switch a_α with a_β)

We just continue with induction on β

(start with $\beta = \min X_0$).

□

Claim M: Primary / A \Rightarrow atomic / A

Pf. Let $\bar{m} \subseteq M = A \cup \{a_\alpha : \alpha < \gamma\}$.

$\bar{m} \subseteq A \cup a_\chi, \chi \in \gamma$
finite closed

$A \cup a_\chi$: a partial i -construction.

\Downarrow
 $\text{tp}(a_\chi / A)$ is isolated

\Downarrow
 $\text{tp}(\bar{m} / A) \text{ --- } \parallel \text{ ---}$

claim \square

Pf (of thm) $M = A \cup \{a_\alpha : \alpha < \gamma\}$,

$N = A \cup \{b_\alpha : \alpha < \delta\}$: i -constructions / A.

We construct $f: M \xrightarrow{\cong} N$, $f = \bigcup_{\alpha} f_\alpha$: elementary.

(i) $\text{Dom } f_\alpha \supseteq A$, $\text{Rng } f_\alpha \supseteq A$, $f_\alpha \upharpoonright_A = \text{id}_A$

(ii) $|\text{Dom } f_\alpha \setminus A|, |\text{Rng } f_\alpha \setminus A| \leq |\alpha| \cdot \aleph_0$

(iii) $\beta \in \text{Lim } f_\beta = \bigcup_{\alpha < \beta} f_\alpha$

(iv) $a_\alpha \in \text{Dom } f_{\alpha+1}$, $b_{\alpha+1} \in \text{Rng } f_{\alpha+1}$.

(v) $\text{Dom } f_\alpha \setminus A = a_\chi$, $\text{Rng } f_\alpha \setminus A = b_\chi$,

where $X \subseteq \mathcal{I}$, $Y \subseteq \mathcal{J}$ are closed.

The recursive step from f_α to $f_{\alpha+1}$.

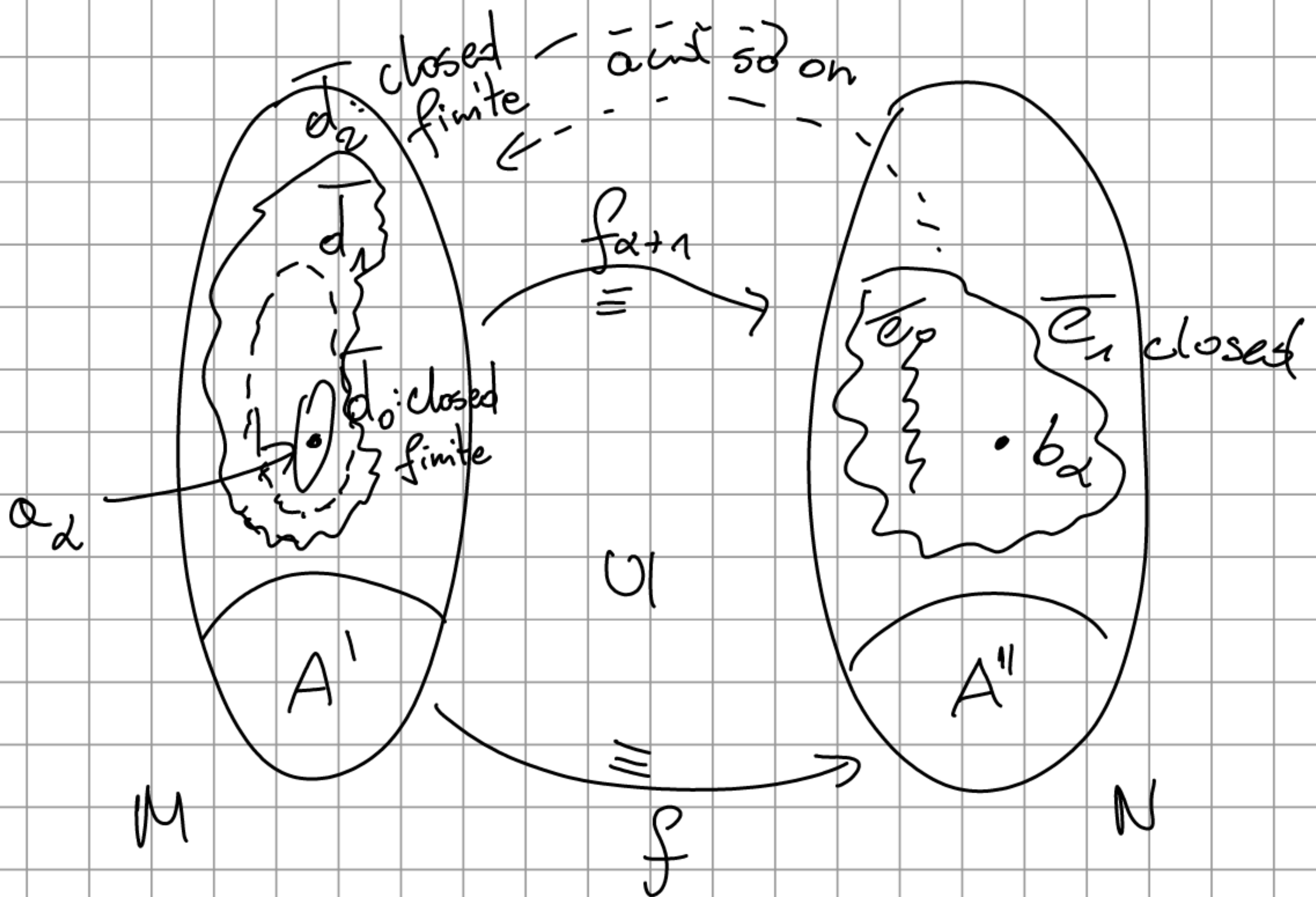
Let $A' = A \cup \text{Dom } f_\alpha$: an \mathcal{I} -construction over A ,

likewise $A'' = A \cup \text{Rng } f_\alpha$: $\text{---} \cup \text{---}$

and M is primary over A' (by the remark)

and N is primary over A'' .

$\wp(x) = \text{tp}(a_\alpha / A')$ is isolated, so $f_\alpha(p)$ is isolated too, therefore $\exists b \in N$ st. $f_\alpha(p) = \text{tp}(b / A'')$.



4.04.22

Def. (1) $\varphi(\bar{x}, \bar{a}) \in L_n(\bar{a})$ is algebraic if

$$0 < |\varphi(\mathcal{M})| < \aleph_0$$

(2) a type $p(\bar{x})$ (over \mathcal{M}) is algebraic if

$$0 < |p(\mathcal{M})| < \aleph_0$$

(3) $a \in \text{acl}(A)$ if $\text{tp}(a/A)$ is algebraic
algebraic closure

(4) $a \in \text{dcl}(A)$ if $a \in \mathcal{M}$ is the only
definable closure
realisation of $\text{tp}(a/A)$

Remark (1) $p(\bar{x})$: an algebraic type $\Leftrightarrow p(\bar{x}) \vdash \varphi(\bar{x})$

for some algebraic formula $\varphi(\bar{x})$

$$(2) |p(\mathcal{M})| = 1 \Leftrightarrow \exists \varphi (p \vdash \varphi \text{ and } |\varphi(\mathcal{M})| = 1)$$

Proof. 1 (\Leftarrow) $p(\mathcal{M}) \subseteq \varphi(\mathcal{M})$

(\Rightarrow) (a.a.) let $n \in \mathbb{N}$ arbitrary. Will show: $|p(\mathcal{M})| \geq n$.

Let $\bar{x}_1, \dots, \bar{x}_n$: disjoint tuples of variables,

$$|\bar{x}_i| = |\bar{x}|.$$

$\{ \varphi(\bar{x}_i) : \varphi(\bar{x}) \in p, i=1, \dots, n \} \cup \{ \bar{x}_i \neq \bar{x}_j : 1 \leq i < j \leq n \}$:

a consistent type.

\mathcal{J} is realised in \mathcal{M} so it has $\geq n$ realisations. \Downarrow

Fact $\text{acl}(A) = \bigcup \{ \varphi(\mathcal{M}) : \varphi(x) \in L_n(A) \text{ algebraic} \}$
 $\text{dcl}(A) = \bigcup \{ \varphi(\mathcal{M}) : \varphi(x) \in L_n(A) \wedge |\varphi(\mathcal{M})| = 1 \}$

Remark Let $\varphi(\bar{x}) \in L_n(\mathcal{M})$. Then $\varphi(\bar{x})$ algebraic
 $\Leftrightarrow 0 < |\varphi(\mathcal{M})| < \aleph_0$

Proof $\mathcal{M} \models \varphi(\bar{x}) \wedge |\varphi(\mathcal{M})| = k \Leftrightarrow \mathcal{M} \models (\exists!^k \bar{x}) \varphi(\bar{x})$
 $\Leftrightarrow \mathcal{M} \models (\exists!^k \bar{x}) \varphi(\bar{x}) \Leftrightarrow |\varphi(\mathcal{M})| = k$

Remark Let $A \subseteq \mathcal{M}$, then: $\text{tp}(ab/A)$ is algebraic
 $a, b \in \mathcal{M}$

$\Leftrightarrow \text{tp}(a/A)$ is algebraic and $\text{tp}(b/Aa)$ is algebraic

Pf. (\Rightarrow) $p(x, y) = \text{tp}(ab/A)$. Let $q(x) = p \upharpoonright_x$
 $= \text{tp}(a/A)$. Let $f: \mathcal{M}^2 \rightarrow \mathcal{M}$ projection

to the first coord. Then $f: p(\mathcal{M}) \rightarrow q(\mathcal{M})$

Why?

Take $a' \in q$, choose $g \in \text{Aut}(\mathcal{M}/A)$, $g(a) = a'$,

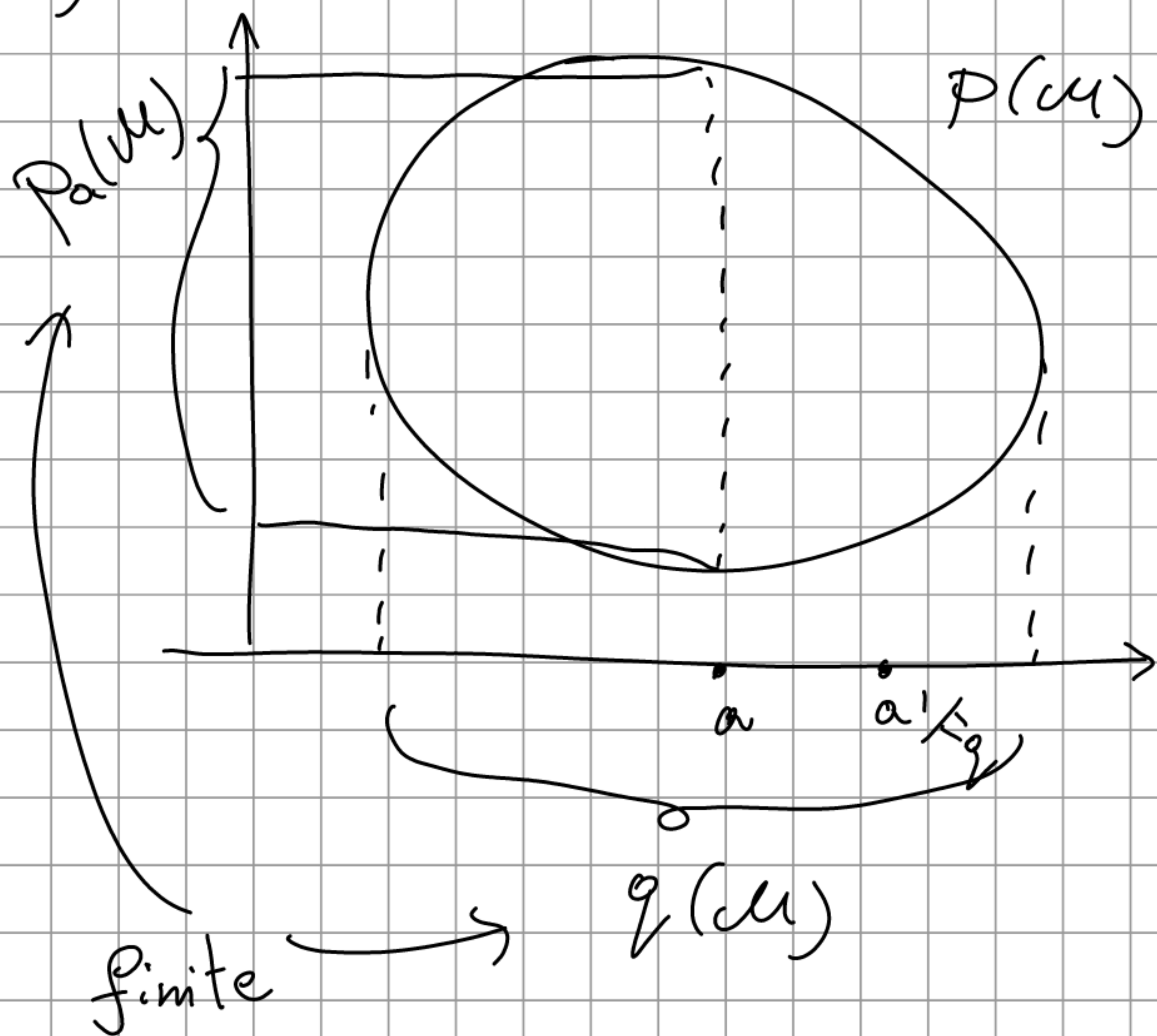
then $b' = g(b) \Rightarrow \text{tp}(ab/A) = \text{tp}(a'b'/A)$.

Then $|p(\mathcal{U})| < \aleph_0 \Rightarrow |p_q(\mathcal{U})| < \aleph_0$ and $(p_a(\mathcal{U}))^{\aleph_0}$

Let $p_a(y) = \text{tp}(b/Aa) = \{ \varphi(a, y) : \varphi(x, y) \in p \}$

Then $p_a(\mathcal{U}) = p(\mathcal{U})_a \leftarrow \text{"verticalization"}$
 \uparrow
 this is finite

(\Leftarrow)



$p(\mathcal{U})_{a'} = q(p(\mathcal{U})_a)$ for any $q \in \text{Aut}(\mathcal{U}/A)$
 with $q(a) = a'$.

$p(\mathcal{U}) = \bigcup_{a' \in q} (\{a'\} \times p_{a'}(\mathcal{U})) : \text{finite} \quad \square$

Remark (1) ⁽ⁱ⁾ $A \subseteq \text{acl}(A)$, ⁽ⁱⁱ⁾ $\text{acl}(\text{acl}(A)) = \text{acl}(A)$,
⁽ⁱⁱⁱ⁾ $A \subseteq B \Rightarrow$ ^(iv) $\text{acl}(A) \subseteq \text{acl}(B)$ (a closure operator)
 $\text{acl}(A) = \bigcup_{\substack{A_0 \subseteq A \\ \text{finite}}} \text{acl}(A_0)$ (finite character)

(2) The same for dcl

Pf. 1 • $a \in \text{acl}(A) \Leftrightarrow a \in \varphi(\mathcal{U})$, $\varphi(x) \in L_1(A)$,
 so $\varphi \in L_1(A_0)$ for some $\substack{A_0 \subseteq A \\ \text{finite}}$, so (iv).

(ii): $a \in \text{acl}(\text{acl}(A)) \Rightarrow a \in \text{acl}(A \cup \bar{b})$,

for some $\bar{b} \subseteq \text{acl}(A) \Rightarrow \text{tp}(\bar{b}/A)$ algebraic

and $\text{tp}(a/\bar{b})$ algebraic $\stackrel{\text{remark}}{\Rightarrow} \text{tp}(a\bar{b}/A)$ is algebraic

$\Rightarrow \text{tp}(a/A)$ algebraic \Rightarrow

Example $T = \text{ACF}_p$, $A \subseteq \mathcal{U} \models T$. $\text{acl}(A) =$

the algebraic closure (in \mathcal{U}) of the field generated by A .



Measuring definable sets and types.

The Cantor-Bendixson rank.

Def. Let X : compact T_2 space.

$$\underbrace{X'}_{\text{CB-derivative}} = X \setminus \underbrace{\{\text{isolated points}\}}_{\text{open in } X} \Rightarrow X' \subseteq X \text{ closed}$$

Iteration: $X^{(\alpha+1)} = (X^{(\alpha)})'$, $X^{(\delta)} = \bigcap_{\alpha < \delta} X^{(\alpha)}$ where $\delta \in \text{Lim}$,
 $X^{(\infty)} = \bigcap_{\alpha \in \text{Ord}} X^{(\alpha)} = X^{(\beta)}$ for some $\beta < |w(X)|^+$
↑ the perfect core of X ↑ minimal cardinality of basis of $\mathcal{C}X$

Def CB: $X \rightarrow \text{Ord} \cup \{\infty\}$

$$\text{CB-rank } p = \begin{cases} \min \{\alpha \in \text{Ord} : p \notin X^{(\alpha+1)}\} & \text{if } p \notin X^{(\infty)} \\ \infty & p \in X^{(\infty)} \end{cases}$$

Now: $X = S(A)$: 0-dimensional (extremely disconnected).

Let: $\text{Clopen}(X) = \{V \subseteq X : V \text{ clopen}\}$.

Def. $CB: \text{Clopen}(X) \rightarrow \text{Ord} \cup \{\infty\}$: the smallest function (value-wise) s.t.: $CB(V) \geq \alpha + 1 \Leftrightarrow \forall n < \omega \exists V_1, \dots, V_n \subseteq V$ $\overset{\text{clopen}}{\text{disjoint}}$ $CB(V_i) \geq \alpha$. (also we define " $\geq \delta$ ")

Then $CB(V) := \min \{ \alpha : \neg CB(V) \geq \alpha + 1 \}$

Properties Let $U, V \subseteq X$ clopen.

(0) $CB(U) = -1 \Leftrightarrow U = \emptyset$

(1) $CB(U) = 0 \Leftrightarrow 0 < |U| < \aleph_0$

(2) $U \subseteq V \Leftrightarrow CB(U) \leq CB(V)$

(3) $CB(U \cup V) = \max \{ CB(U), CB(V) \}$

} Very easy

Pf. 3 Obv. $CB(U \cup V) \geq \max \{ CB(U), CB(V) \}$.

Now assume $CB(U \cup V) \geq \alpha \Rightarrow \max \{ CB(U), CB(V) \} \geq \alpha$.

Pf by ind on α . Base and limit easy.

Successor step $\alpha \rightarrow \alpha + 1$. Assume $CB(U \cup V) \geq \alpha + 1$.

So $\forall n \exists W_1, \dots, W_n \subseteq U \cup V$ $\overset{\text{clopen}}{\text{disjoint}}$ $\bigwedge_{i=1}^n CB(W_i) \geq \alpha$.

By ind hyp.: $\max\{CB(W_i \cap U), CB(W_i \cap V)\} \geq \alpha$.

So $\forall n \left(\exists W_1, \dots, W_n \subseteq U \bigwedge_{i=1}^n CB(W_i) \geq \alpha \right)$
 (or the same for V)

\Downarrow

$CB(U) \geq \alpha+1$ or $CB(V) \geq \alpha+1$.

(4) $CB(V) \geq \alpha+1 \iff (\exists V_n \subseteq V, n < \omega) \bigwedge_n CB(V_n) \geq \alpha$
clopen disjoint

(5) For $p \in X$ $CB(p) = \min\{CB(U) : p \in U \subseteq X\}$
clopen

Notation

In model theory: $X = S(A)$.

$CB(a/A) := CB(\text{tp}(a/A))$

$CB(a/A) = 0 \iff \text{tp}(a/A)$ is isolated

$p \in S(A) \rightsquigarrow CB(p) = CB_A(p)$

$\varphi \in L(A) \rightsquigarrow [\varphi] \subseteq S(A) \rightsquigarrow CB_A(\varphi)$

\parallel
 $CB_A([\varphi] \cap S(A))$

Morley rank "CB on $L(\mathcal{M})$, in $S(\mathcal{M})$ "

Def. RM: $L(\mathcal{M}) \rightarrow \{-1\} \cup \text{Ord} \cup \{\infty\}$:

the minimal function s.t. For $U \subseteq \mathcal{M}^n$
definable
[$\varphi(\bar{x}) \in L_n(\mathcal{M})$ identified with $U = \varphi(\mathcal{M}) \subseteq \mathcal{M}^n$]

(1) $U = \emptyset \Rightarrow \text{RM}(U) = -1$

(2) If $U \neq \emptyset$, then $\text{RM}(U) \geq \alpha + 1$

$\Leftrightarrow \forall n < \omega \exists V_1, \dots, V_n \subseteq U \bigwedge_{i \leq n} \text{RM}(V_i) \geq \alpha$
def. disjoint

Def. For $\varphi(\bar{x}) \in L(\mathcal{M})$, $\text{RM}(\varphi) = \text{RM}(\varphi(\mathcal{M}))$

• For a type $p(\bar{x})$ over \mathcal{M} (not necessarily complete)

$$\text{RM}(p) = \min \{ \text{RM}(\varphi) : p \vdash \varphi \}$$

$$= \min \{ \text{RM}(U) : p(U) \subseteq U \}$$

Remark (1) $\text{RM}(\varphi) = \text{CB}_{\mathcal{M}}(\varphi)$

(2) For $p \in S(\mathcal{M})$ $\text{RM}(p) = \text{CB}_{\mathcal{M}}(p)$

Here \mathcal{M} may be replaced with any

\aleph_0 -saturated $M \prec \mathcal{M}$.

Properties (1) $\varphi \vdash \psi \Rightarrow RM(\varphi) \leq RM(\psi)$

(2) $\rho \vdash \varphi \Rightarrow RM(\rho) \leq RM(\varphi)$

(3) $RM(\varphi \vee \psi) = \max\{RM(\varphi), RM(\psi)\}$

(4) $RM(\varphi) \geq \alpha + 1 \iff \left(\exists \varphi_n \vdash \varphi, n < \omega \right) \wedge_n RM(\varphi_n) \geq \alpha$
pairwise contradictory

(5) $\exists \delta \in Lim$, then $RM(\varphi) \geq \delta \iff (\forall \alpha < \delta) RM(\varphi) \geq \alpha$

Thm T is \aleph_0 -stable $\iff RM("x=x") < \infty$.

Pf. (\Rightarrow) (a.a.) $\exists f$ $tp(\bar{a}) = tp(\bar{b})$, then

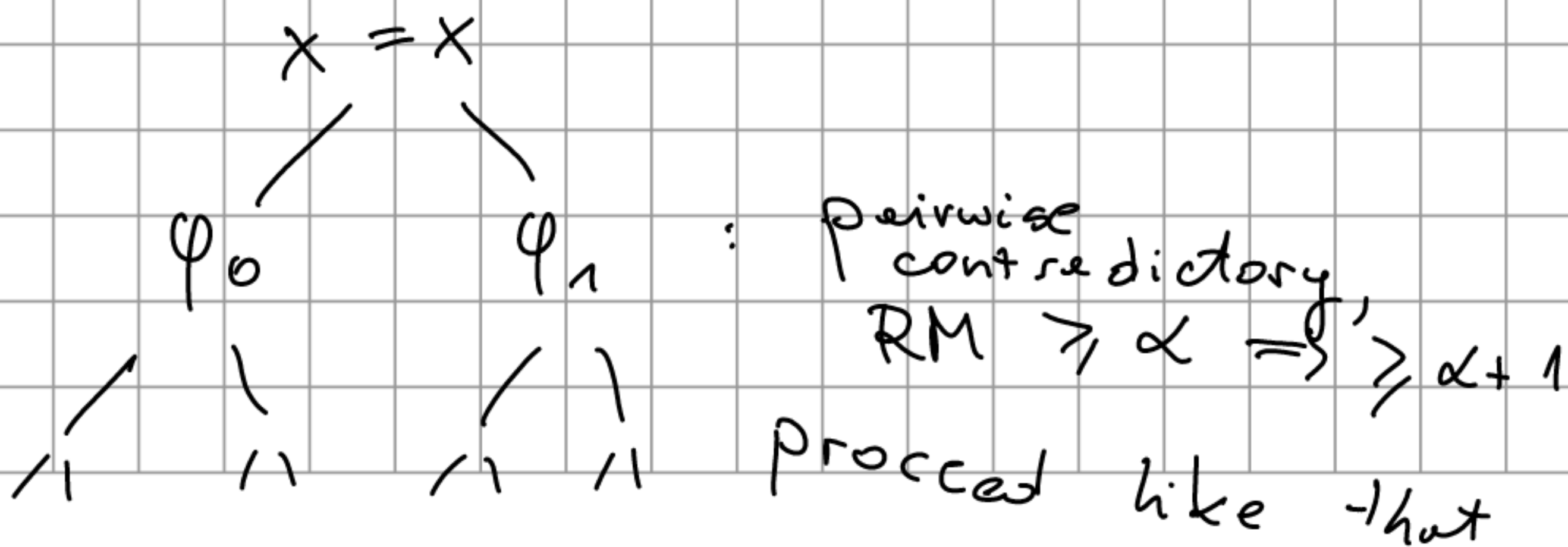
$$RM(\varphi(\bar{x}, \bar{a})) = RM(\varphi(\bar{x}, \bar{b}))$$

$$\exists f \in Aut(\mathcal{M}) \quad f(\bar{a}) = \bar{b} \Rightarrow f(\varphi(\mathcal{M}, \bar{a})) = \varphi(\mathcal{M}, \bar{b})$$

So $|Rng(RM)| \leq 2^{|\mathcal{M}|} \Rightarrow \exists \alpha \in Ord \forall \varphi [RM(\varphi) \geq \alpha$

$\Rightarrow RM(\varphi) = \infty$].

Suppose $RM("x=x") = \infty \Rightarrow \geq \alpha + 1$



We get 2^{n_0} many types over table set A
 $\Rightarrow T$ is not n_0 -stable.

11.04.2022

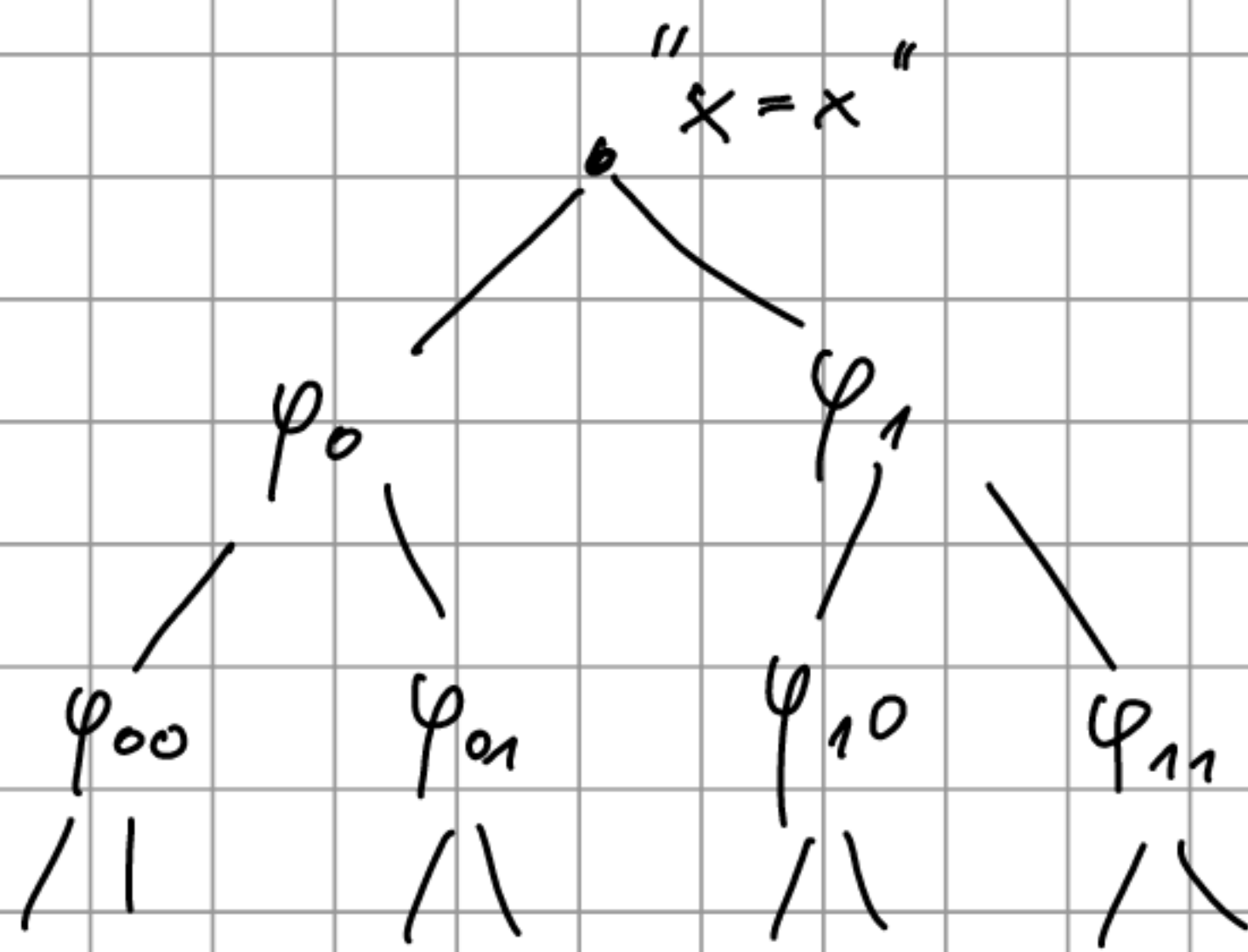
Thm T is \aleph_0 -stable $\iff RM(x=x) \in Ord$

Pf. " \implies " last time. " \impliedby " Suppose T is not \aleph_0 -stable.

$|S(A)| = 2^{\aleph_0}$, $S(A)$: a Polish space

$\implies S(A)^{(\omega)} \neq \emptyset \implies$ get a binary tree of formulas

$\{ \varphi_\eta(x) : \eta \in 2^{<\omega} \}$



Let $\alpha = \min \{ RM(\varphi_\eta) : \eta \in 2^{<\omega} \}$.

If $\eta \subseteq \nu \in 2^{<\omega}$, then $RM(\varphi_\nu) = \alpha$.

$RM(\varphi_\eta) \not\geq \alpha + 1$, so:

(*) $\exists m \in \mathbb{N} \rightarrow \exists \psi_1 \dots \psi_m \vdash \varphi_\eta \wedge_{i=1}^m RM(\psi_i) \geq \alpha$

Take n s.t. $2^n > m$. φ_ν , $\eta \subseteq \nu \in 2^{<\omega}$, $|\nu| = |\eta| + n$
contradicts (*). ~~is~~

Def. Multiplicity of $\varphi(\bar{x}) \in L(\mathcal{M})$ s.t. $\text{RM}(\varphi(\bar{x})) < \infty$:

$\text{Mlt}(\varphi) =$ the largest $m \in \mathbb{N}$ s.t.

$$\exists \psi_1 \dots \psi_m \bigwedge_{i=1}^m \text{RM}(\psi_i) \geq d$$

Properties • $\text{RM}(\varphi_1) = \text{RM}(\varphi_2)$ and $\varphi_1(\mathcal{M}) \cap \varphi_2(\mathcal{M}) = \emptyset$,

then $\text{Mlt}(\varphi_1 \vee \varphi_2) = \text{Mlt}(\varphi_1) + \text{Mlt}(\varphi_2)$

• If $\text{RM}(\varphi_1) < \text{RM}(\varphi_2) < \infty$, then $\text{Mlt}(\varphi_1 \vee \varphi_2) = \text{Mlt}(\varphi_2)$

Example If $\varphi(\bar{x})$ is algebraic, then $\text{Mlt}(\varphi) = |\varphi(\mathcal{M})|$

$$\text{RM}(\varphi) = 0$$

Def. Assume $p(\bar{x})$: a type with $\text{RM}(p(\bar{x})) < \infty$.

$$\text{Mlt}(p(\bar{x})) = \min \{ \text{Mlt}(\varphi(\bar{x})) : p \vdash \varphi \text{ and } \text{RM}(\varphi) = \text{RM}(p) \}$$

Def. $p(\bar{x})$ is stationary, if $\text{Mlt}(p(\bar{x})) = 1$

Remark Assume $p(\bar{x})$: a type over A . Then $\exists q \in S(A)$,

$$p(\bar{x}) \subseteq q(\bar{x}) \text{ s.t. } \text{RM}(p) = \text{RM}(q)$$

Pf. Let $q_0 = \{ \varphi(\bar{x}) \in L(A) : \text{RM}(p \cup \{ \neg \varphi \}) < \text{RM}(p) \}$

• $q_0 \supseteq p$, if $\varphi \in p$, then $RM(p \cup \{\neg\varphi\})$
 $= -1 < RM(p)$

consistent

$RM(p)$

• q_0 : a type: $\varphi_1, \dots, \varphi_n \in q_0$, $RM(p \cup \{\neg\varphi_i\}) < \alpha$

Choose ψ with $p \vdash \psi$ and $RM(\psi) = \alpha$,

ψ_i with $p \vdash \psi_i$ and $RM(\psi_i \wedge \neg\varphi) < \alpha$.

Wlog $\psi = \bigwedge_{i=1}^n \psi_i$

⋮

Let $q_0 \subseteq q \in S(A)$, then $RM(q) = \alpha$. Clearly,
 $RM(q) \leq \alpha$. If $RM(q) < \alpha \Rightarrow q \vdash \varphi$ with

$RM(\varphi) < \alpha$. By compactness \exists finite $q' \subseteq q$, $q' \vdash \varphi$

$\Leftrightarrow \bigwedge q' \vdash \varphi$, $\chi(\bar{x}) \vdash \varphi(\bar{x})$ so $RM(\chi(\bar{x})) < \alpha$,

$\chi(\bar{x}) \in q(\bar{x})$

then $RM(p \cup \{\chi(\bar{x})\}) < \alpha$

\downarrow
 $\neg \chi(\bar{x}) \in q_0(\bar{x}) \subseteq q(\bar{x})$
 $\chi(\bar{x}) \in q(\bar{x})$

\downarrow

Example $\overbrace{RM(p)=0}^{(\Rightarrow) p: \text{algebraic}}$, then $Mlt(p) = |p(\mathcal{M})|$

Example $T = ACF_p$, $K \subseteq \mathcal{M}$, $\varphi \in S_n(K)$, $p = t_p(\bar{a}/K)$
subfield

a) p algebraic $\Leftrightarrow RM(p)=0$,

$(W) = I(\bar{a}/K) \neq \{0\}$, " $W(x)=0$ " $\in p(x)$,

$Mlt(p) = Mlt(W(x)=0) = \# \{ \text{roots of } W \text{ in } \mathcal{M} \}$.

b) p : transcendental, then $I(\bar{a}/K) = \{0\}$,

$RM(p) = 1 = RM(x=x)$

$Mlt(p) = Mlt(x=x) = 1$

So p : stationary.

c) $p = t_p(\bar{a}/K)$, $\bar{a} = (a_1, \dots, a_n) \in \mathcal{M}$. By q.e.

it is determined by $I(\bar{a}/K) \triangleleft K[X_1, \dots, X_n]$

$\{ W(\bar{x}) \in K[\bar{x}] : W(\bar{a}) = 0 \}$.

Let $V_p = V(I(\bar{a}/K)) = \bigcap_{W \in I(\bar{a}/K)} Z(W) =$

$= \bigcap_{i=1}^l Z(W_i) \leftarrow \text{definable on } \mathcal{M} \text{ over } K, (W_1, \dots, W_l)$
 $\underbrace{\quad}_{\{ \bar{b} \in \mathcal{M}^n : W(\bar{b}) = 0 \}}$

$\bar{a} \in V_p \Rightarrow p(\bar{x}) \vdash "x \in V_p"$

$$RM(p) = RM(V_p) = \dim V_p$$

$Mlt(p) = Mlt(V_p) =$ The number of irreducible components of V_p of $\dim = \dim V_p$

$$\text{In } T = Th(ACF_p) : RM(\bar{x} = \bar{x}) < \omega$$

(Order) indiscernible sets:

Let (I, \leq) : a linear ordered set of indices.

Def. $\{\bar{a}_i : i \in I\} \subseteq \mathcal{M}$: order indiscernible over

$A \subseteq \mathcal{M}$, if $\forall k \forall i_1 < \dots < i_k \in I$ $tp(\bar{a}_{i_1} \dots \bar{a}_{i_k} / A) = tp(\bar{a}_{j_1} \dots \bar{a}_{j_k} / A)$
 $j_1 < \dots < j_k \in I$

Recall: 1) Assume $p(\bar{x})$: a non-algebraic type over $A \subseteq \mathcal{M}$. Then $\exists \{\bar{a}_n : n < \omega\} \subseteq p(\mathcal{M})$
infinite order indiscernible

2) (stretching) Assume $\{a_i : i \in I\}$ order indiscernible / A ,

I : infinite, (J, \leq) : a linear ordering. Then

$\exists \{b_j : j \in J\}$: order ind. / A s.t. $\forall k \forall i_1 < \dots < i_k \in I$
 $\forall j_1 < \dots < j_k \in J$

$$tp(\bar{a}_{i_1} \dots \bar{a}_{i_k} / A) = tp(b_{j_1} \dots b_{j_k} / A)$$

Pf. (1) by Ramsey thm.

(2) Let $b_j, j \in J$: new constant symbols.

$$T^* = T(A) \cup \{ \varphi(b_{j_1}, \dots, b_{j_k}) : \varphi(\bar{x}) \in L_k(A), j_1 < \dots < j_k \in J \\ \text{and } \forall i_1 < \dots < i_k \in I \models \varphi(a_{i_1}, \dots, a_{i_k}) \}.$$

T^* : consistent. $b_{j_1}, \dots, b_{j_k} \xrightarrow{\text{interpret as}} a_{i_1}, \dots, a_{i_k}$

has a model M

for any $i_1 < \dots < i_k \in I$

$$M = \left(M, \underbrace{a^M}_{\prod T(A)}, b_j^M \right)_{\substack{a \in A \\ j \in J}} \models T^*$$

$$\Rightarrow \exists f: (M, a^M)_{a \in A} \xrightarrow{\cong} (M, a)_{a \in A} \\ a^M \xrightarrow{f} a$$

Let $b_j = f(b_j^M)$. $\{ b_j : j \in J \} \subseteq M$ is good.

Example (1) $M = (\mathbb{Q}, \leq) \leftarrow \mathbb{Q}$ is order indisc (indexed by itself)

(2) $T = \text{ACF}_p, M \models T, \{ a_i : i \in I \} \subseteq M$, alg-indepen. over $K \subseteq M$ subfields, then it is indisc / K in M

$$\updownarrow \\ \overline{I}(a_{i_1}, \dots, a_{i_k} / K) = \text{deg}$$

25.04.22 Pf. c.d. $\sigma = \tau_1 \circ \tau_2 \circ \dots \circ \tau_l$: product of transpositions of consecutive numbers.

Let $\sigma_t = \tau_1 \circ \dots \circ \tau_t$, $t = 1, \dots, l$, $\sigma_0 = \text{id}$. Then

$\models \varphi(\bar{a}_{i_{\sigma_0}})$ and $\models \varphi(\bar{a}_{i_{\sigma_l}})$. For some

$0 \leq t < l$:

$\models \varphi(\bar{a}_{i_{\sigma_t}}) \wedge \neg \varphi(\bar{a}_{i_{\sigma_{t+1}}})$

Let $\sigma' = \sigma_t$, $\sigma'' = \sigma_{t+1} = \sigma' \circ \tau_{t+1}$

Then $\models \varphi(a_{i_{\sigma'(1)}}, \dots, a_{i_{\sigma'(k)}})$

$\psi(a_{i_1}, \dots, a_{i_k})$ (by renaming)

But $\models \neg \varphi(a_{i_{\sigma'(j(1))}}, \dots, a_{i_{\sigma'(j(k))}})$,

so $\models \neg \psi(a_{i_{\tau(1)}}, \dots, a_{i_{\tau(k)}}) \wedge \psi(a_{i_1}, \dots, a_{i_k})$

($\forall i_1 < i_2 < \dots < i_k \in \bar{I}$)

e.g. $\tau = (3, 4)$ and $k > 4$. Choose $i_1 < i_2 < i_5 < \dots < i_k$.

Let $\chi(x_3, x_4) = \psi(a_{i_1}, a_{i_2}, x_3, x_4, a_{i_5}, \dots, a_{i_k}) \in L(\bar{a}_j) \upharpoonright \bar{I}$
dense \bar{I}

Will show: $|S(\bar{a}_z)| > \kappa$, $|\bar{a}_z| \leq \kappa$.

Namely: let $i < i' \in (i_2, i_5)_I$. Then

$$\text{tp}(a_i / \bar{a}_z) \neq \text{tp}(a_{i'} / \bar{a}_z) \leftarrow \text{enough}$$

$$i_1 < i_2 < i < j < i' < i_5 < \dots < i_k.$$

$$\text{Then } \models \varphi(a_i, a_j), \models \neg \varphi(a_{i'}, a_j)$$

because $i < j$

because $i' > j$

$$\text{Then } \varphi(x, a_j) \in \text{tp}(a_i / \bar{a}_z),$$

$$\neg \varphi(x, a_j) \in \text{tp}(a_{i'} / \bar{a}_z). \quad \downarrow$$

Remark (T : stable, $\varphi(x, \bar{y}) \in L$). There is

$\kappa < \omega$ $\forall I \subseteq \mathcal{M}$ $\forall \bar{a} \subseteq \mathcal{M}$ one of the
indiscernible
infinite

$$\text{sets } I_{\bar{a}}^+ = \{c \in I : \models \varphi(c, \bar{a})\}$$

$$I_{\bar{a}}^- = \{c \in I : \models \neg \varphi(c, \bar{a})\}$$

has $\leq \kappa$ elements.

Pf. (a.c.) If there's no such n , then

$\forall n \exists I_n \exists \bar{a}_n \quad |I_{n, \bar{a}_n}^+|, |I_{n, \bar{a}_n}^-| > n$. Let

$\{c_i, i < \omega\}$: new constant symbols.

\bar{d} : $|\bar{d}| = |\bar{y}|$.

Let $T' = T \cup \underbrace{\{ \{c_i : i < \omega\} \text{ is indiscernible in } T \}}_I$

$\cup \{ \{ \bar{I}_{\bar{d}}^+ = \{c_{2i} : i < \omega\} \} \cup \{ \bar{I}_{\bar{d}}^- = \{c_{2i+1} : i < \omega\} \} \}$.

wlog $|I| = \aleph$

T' is consistent, so it has a model M' .

wlog $M' \upharpoonright_L \prec \mathcal{M}$. So $I \subseteq \mathcal{M}$ indiscernible,

$I_{\bar{d}}^+, I_{\bar{d}}^- \subseteq \mathcal{M} \Rightarrow |S(I)| = 2^\aleph > \aleph$.

Pf.

$|I_{\bar{d}}^+|, |I_{\bar{d}}^-| = \aleph$. For any $I' \subseteq I$ with

$|I'| = |I \setminus I'| = \aleph \quad \exists f \in \text{Aut}(\mathcal{M}) [f[I] = I,$

$f(I') = I_{\bar{d}}^+, f(I \setminus I') = I_{\bar{d}}^-]$. Let $\bar{a}_{I'} = f^{-1}(\bar{d})$.

If $I' \neq I''$, then $\text{tp}(\bar{a}_{I'} / I) \neq \text{tp}(\bar{a}_{I''} / I)$.

\aleph_1 -categorical theories

Examples. 1. ACF_p ,

2. $Th(V, +, k)$ $k \in K \leftarrow$ $\begin{matrix} \text{ctble} \\ \text{field} \end{matrix}$
 \uparrow
inf. vec. space

3. $Th(\mathbb{N}, S)$, $Th(\mathbb{Z}, S)$, $Th(\mathbb{N}, =)$

4. $Th(G, +)$, G : torsion free divisible abelian \uparrow "no structure"

5. $Th(\mathbb{Z}_p^{\aleph_0}, +)$

6. $Th(G)$ where G : an algebraic group

Theorem If $\kappa > \aleph_0$ and T is κ -categorical, then T is \aleph_0 -stable.

Lemma $\forall \kappa \geq \aleph_0 \exists M \models T, \|M\| = \kappa \forall A \subseteq M, |A| < \aleph_0$

$|\{p \in S(A) : p(M) \neq \emptyset\}| \leq \aleph_0$.

Pf. thm (lemma \Rightarrow thm) (A.c.) Suppose

T is not \aleph_0 -stable. $\exists N \models T$ $|S(N)| > \aleph_0$
 \uparrow
ctble

But T : κ -categorical. $N \cong N_1$ s.t. $\|N_1\| = \aleph_1$,

$|\{p \in S(N) : p(N_1) \neq \emptyset\}| = \aleph_1$.

$N_1 \prec N_\kappa \leftarrow$ of power κ . Let M_κ : a model from the lemma. Then $M_\kappa \cong N_\kappa$ \checkmark .

Pf. (lemma) $T \subseteq T^S$: the skolemization in $L^S \supseteq L$.

Let $I = \{a_n : n < \omega\}$: an infinite order indisc.

set in T^S . $I \subseteq J = \{a_\alpha : \alpha < \kappa\}$ (stretching).

Then $J \subseteq N^S \models T^S$. Let $M^S = \mathcal{H}(J) \prec N^S$, i.e.

$M^S = \{t^{N^S}(\vec{j}) : t(\vec{x}) : \text{a term in } L^S, \vec{j} \subseteq J\}$

Will show that M^S satisfies the conditions \uparrow on the

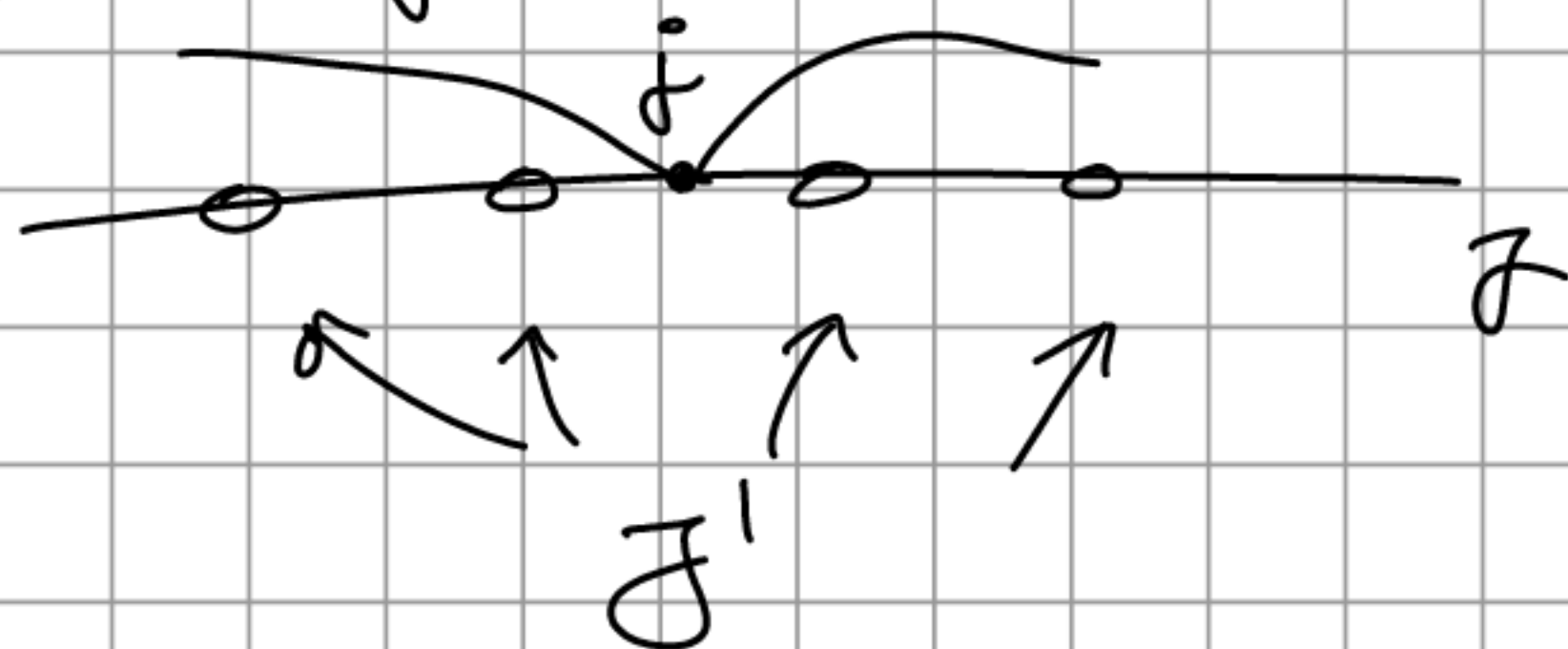
lemma for T^S (then $M := M^S \upharpoonright L$ satisfies conditions for T)

Let $A \subseteq \mathcal{H}(J)$. Wlog $A = \mathcal{H}(J')$ for some $J' \subseteq J$.

$M^S \ni a = t(j_1, \dots, j_k)$ for $\overset{\text{some}}{j_1 < \dots < j_k} \in J$.

\sim eq. rel. on J :

$j \sim j' \stackrel{\text{def.}}{\iff} j$ and j' determine the same cut in J' .



$|J/\sim|$: ctble as $|J'|$ ctble. Now

$$(j_1, \dots, j_k) \sim (j_1', \dots, j_k') \stackrel{\text{def}}{\Leftrightarrow} \bigwedge_{1 \leq i \leq k} j_i \sim j_i'$$

again
ctbly many
classes

Let $a, a' \in M^s$. (*) If $t = t'$ and $(j_1, \dots, j_k) \sim (j_1', \dots, j_k')$
 $t(j_1, \dots, j_k) = t'(j_1', \dots, j_k')$ then $\text{tp}^{M^s}(a/A) = \text{tp}^{M^s}(a'/A)$.

(it obvs. implies the lemma). $\exists j'$

Pf (*): $\varphi(x) \in L(A)$, $b = \bar{t}(j'')$

$$\varphi(x, b), b \in A$$

$$\varphi(x, \bar{y}) \in L$$

$\bar{j} \sim \bar{j}'$ + order
indiscernibility of \bar{j}

$$\models \varphi(a, b) \Leftrightarrow \models \varphi(t(\bar{j}), \bar{t}(\bar{j}'')) \Leftrightarrow \models \varphi(t(\bar{j}'), \bar{t}(\bar{j}''))$$

$$\Leftrightarrow \models \varphi(a', b)$$



Remark

Let $T: \mathcal{M}_0$ -stable, $p \in S(A)$, $\text{RM}(p) < \infty$, $\text{Mit}(p) = 1$.

Then $\forall B \supseteq A \exists! p_B \in S(B)$ $\text{RM}(p_B) = \text{RM}(p)$

Def. $I = \{a_\alpha : \alpha < \beta\} \subseteq \mathcal{M}$ is a Morley sequence in $\text{pES}(A)$ if $\forall \alpha < \beta$ $a_\alpha \text{ F P}_{A_{a_{\alpha < \alpha}}} \in S(A_{a_{\alpha < \alpha}})$

$\subseteq \text{P, Mlt}(1)$
 $\text{pES}(A)$

Remark A Morley sequence in P is indiscernible over A .

Pf. Enough to show order-indiscernibility. Wlog $I = \{a_\alpha : \alpha < \beta\}$, $\beta = \kappa \geq \aleph_0$. Induction on $k < \omega$:

$$\forall \alpha_1 < \dots < \alpha_k < \kappa \quad \text{tp}(a_{\alpha_1} \dots a_{\alpha_k} / A) = \text{tp}(a_{\beta_1} \dots a_{\beta_k} / A)$$

$$\beta_1 < \dots < \beta_k$$

Step $k \rightarrow k+1$, $\alpha_{k+1} > \alpha_k$, $\beta_{k+1} > \beta_k$.

$$\text{p} \subseteq \text{tp}(a_{\alpha_{k+1}} / A_{a_{\alpha_1} \dots a_{\alpha_k}}) \subseteq \text{P}_{A_{a_{\alpha < \alpha_{k+1}}}}$$

the same

RM, Mlt=1

$$\Rightarrow \text{tp}(a_{\alpha_{k+1}} / A_{a_{\alpha_1} \dots a_{\alpha_k}}) = \text{P}_{A_{a_{\alpha_1} \dots a_{\alpha_k}}}$$

Likewise $\text{tp}(a_{\beta_{k+1}} / A_{a_{\beta_1} \dots a_{\beta_k}}) = \text{P}_{A_{a_{\beta_1} \dots a_{\beta_k}}}$

Consider $f: A_{a_{\alpha_1} \dots a_{\alpha_k}} \rightarrow A_{a_{\beta_1} \dots a_{\beta_k}}$ s.t.

$f|_A = \text{id}$, $f(a_{\alpha_i}) = a_{\beta_i}$, $i=1, \dots, k$. Then f is elementary.

and $f(\underbrace{p_{A_{a_{\alpha_1} \dots a_{\alpha_k}}}}_p) = p_{A_{a_{\beta_1} \dots a_{\beta_k}}}$

and has the same RM as $\text{RM}(p)$

$f \cup \langle a_{\alpha_{k+1}}, a_{\beta_{k+1}} \rangle$ elementary

So $\text{tp}(a_{\alpha_1} \dots a_{\alpha_{k+1}} / A) = \text{tp}(a_{\beta_1} \dots a_{\beta_{k+1}} / A)$. \square

Thm (Morley, Shelah) If $T: \aleph_0$ -stable and $\kappa \geq \aleph_0$ then T has a saturated model of power κ .

9.05.2021 Thm (Morley, Shelah) T: \aleph_0 -stable \Rightarrow

T has a saturated model of power κ

Pf. $M = \bigcup_{\alpha < \kappa} M_\alpha \leftarrow$ elementary chain of models of T of power κ
 $\|M_\alpha\| = \kappa$

• $M_\delta = \bigcup_{\alpha < \delta} M_\alpha$ when $\delta \in \text{Lim}$

• $\alpha \rightarrow \alpha+1 : M_{\alpha+1} \succ M_\alpha$ s.t.

(a) $\forall A \subseteq M_\alpha$ finite $\forall p \in S(A)$ $p(M_{\alpha+1}) \neq \emptyset$

(b) $\forall A \subseteq M_\alpha$ finite $\forall p \in S(A)$ stationary $\exists I \subseteq M_{\alpha+1}$ ($|I| = \kappa \wedge$

I is a Morley sequence in p)

Claim M is saturated

• M is \aleph_0 -saturated (easy)

• M is κ -saturated: (a.c.) let $A \subseteq M$,

$|A| < \kappa$, $p \in S(A)$, $p(M) = \emptyset$. Choose

A and p so that $(RM(p), \text{Mlt}(p))$ is

lexicographically minimal. Then $\text{Mlt}(p) = 1$.

pf. let choose $\varphi \in \mathcal{P}$ with $\text{RM}(\mathcal{P}) = \text{RM}(\varphi)$,
 $\text{Mlt}(\mathcal{P}) = \text{Mlt}(\varphi)$.

- If $\text{Mlt}(\varphi) > 1$, then

(*) $\exists \psi(x) \in L(\mathcal{M})$ $\text{RM}(\varphi \wedge \psi) = \text{RM}(\varphi \wedge \neg \psi) = \text{RM}(\varphi)$

So $\psi(x) = \psi(x, \bar{c})$, $\varphi(x) = \varphi(x, \bar{a})$. Choose

$\bar{c}' \subseteq M$ s.t. $\text{tp}(\bar{c}'/\bar{a}) = \text{tp}(\bar{c}/\bar{a}) \in S_k(\bar{a})$, ($k = |\bar{c}'|$)

(we can choose it by (a))

(*) holds for $\psi(x, \bar{c}')$ in place of ψ

Let $A' = A \cup \bar{c}' \subseteq M$, $|A'| < \kappa$

$S(A)$

$S(A')$

\subseteq
 \mathcal{P}



\mathcal{P}_1
 \mathcal{P}_2
 \vdots
 \mathcal{P}_l

s.t. $\text{RM}(\mathcal{P}_i) = \text{RM}(\mathcal{P})$

($1 \leq \text{Mlt}(\varphi)$)

Also (*) \Rightarrow $\text{Mlt}(\varphi) = \text{Mlt}(\varphi \wedge \psi') + \text{Mlt}(\varphi \wedge \neg \psi')$

Look at \mathcal{P}_1 : either $\varphi \wedge \psi \in \mathcal{P}_1$ or $\varphi \wedge \neg \psi \in \mathcal{P}_1$

$(\text{RM}(\mathcal{P}_1), \text{Mlt}(\mathcal{P}_1)) \prec_{\text{lex}} (\text{RM}(\mathcal{P}), \text{Mlt}(\mathcal{P}))$

$\mathcal{P}_1(M) \subseteq \mathcal{P}(M) = \emptyset \Rightarrow \mathcal{P}_1(M) = \emptyset \downarrow$

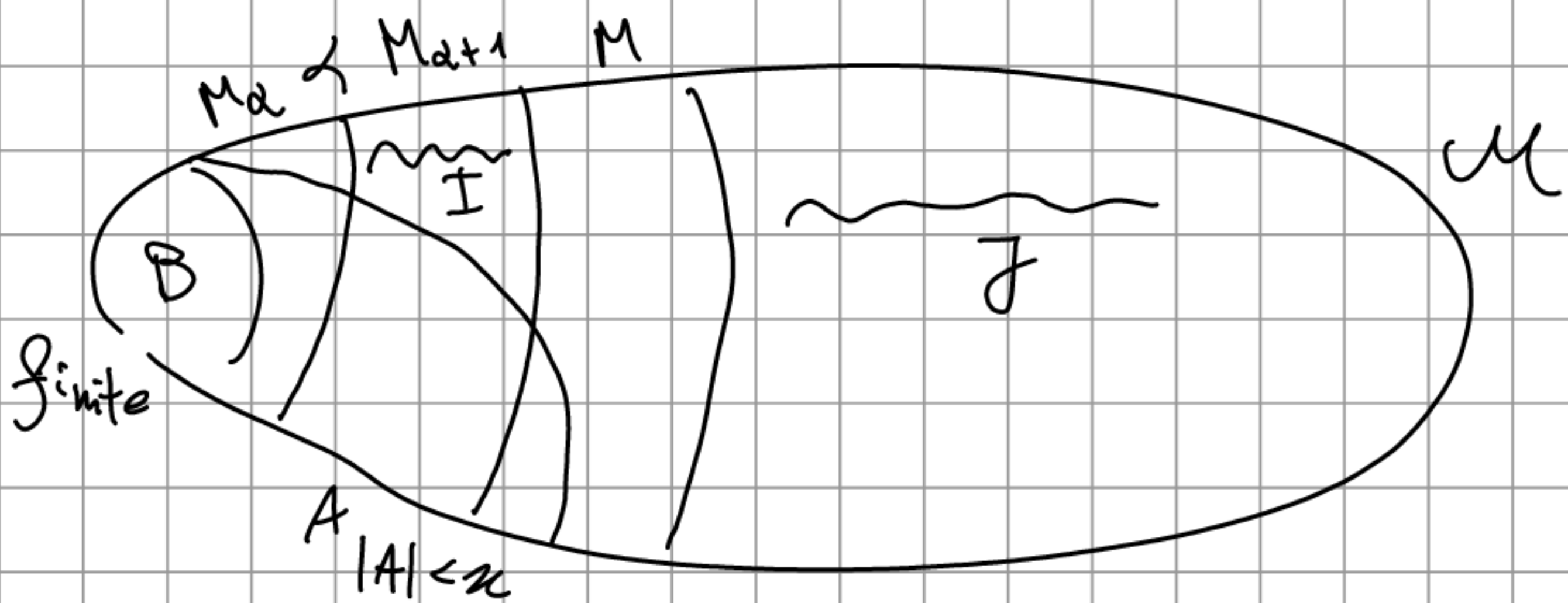
Therefore $\text{Mlt}(\varphi) = 1$. Choose a finite $B \subseteq A$

s.t. $\text{RM}(\varphi) = \text{RM}(\varphi|_B)$, $\text{Mlt}(\varphi) = \text{Mlt}(\varphi|_B)$
 (enough that $\varphi \in L(B)$)

By (b) $\exists I \subseteq M$: a Morley sequence in \mathcal{P}' ,

$|I| = \kappa$. Let $\mathcal{J} = \{a_\alpha : \alpha < \kappa\}$: a M -sequence
 in $\mathcal{P}'_{AI} \in \mathcal{S}(AI)$

Then $I \cup \mathcal{J}$ is a Morley sequence in \mathcal{P}'_{AI} .



Let $\chi(x) \in \mathcal{P} \subseteq \mathcal{P}'_{AI} \Rightarrow \mathcal{J} \subseteq \chi(\mathcal{U})$,
 infinite

However $I \cup \mathcal{J}$ indiscernible $\Rightarrow \{i \in I : \models \chi(i)\}$ is
 also cofinite in I

$I \cup \mathcal{J} = (I \cup \mathcal{J})^+ \cup (I \cup \mathcal{J})^-$
 finite

(by the lemma
 from prev. lecture)

$$\left| \bigcup_{\chi \in \mathcal{P}} I_{\chi}^{-} \right| < \kappa \Rightarrow \left| \bigcap_{\chi \in \mathcal{P}} I_{\chi}^{+} \right| = \kappa \Rightarrow \left| \bigcap_{\chi \in \mathcal{P}} I_{\chi}^{+} \right| \neq \emptyset.$$

($|\mathcal{P}| < \kappa$) Any $c \in \bigcap_{\chi \in \mathcal{P}} I_{\chi}^{+}$ realises p in M ▀

So: if $\kappa > \aleph_0$, $T: \kappa$ -categorical

$\Downarrow \cup$

$T: \aleph_0$ -stable

\Downarrow

$\exists M \models T \quad \|M\| = \kappa$
saturated

S -isolation ($S = \text{"set"}$)

Def. (1) $p \in S(A)$ is S -isolated if $\exists B \subseteq A$
finite

$p|_B \vdash p$.

(2) M is S -atomic over A if

$\forall \bar{a} \subseteq M$ tp (\bar{a}/A) is S -isolated
finite

(3) M is an S -model, if $\forall A \subseteq M$ $\forall p \in S(A)$
finite $p(M) \neq \emptyset$
 \parallel
 \aleph_0 -saturated

(4) M is S -prime over A if M is S -model

and $\forall N \supseteq A$ $\exists f: M \xrightarrow[A]{=} N$ ($f|_A = \text{id}_A$)
 $\hat{M} \uparrow \aleph_0$ -saturated

Remark T : \aleph_0 -stable, $p \in S(B)$, $B \subseteq A$
finite

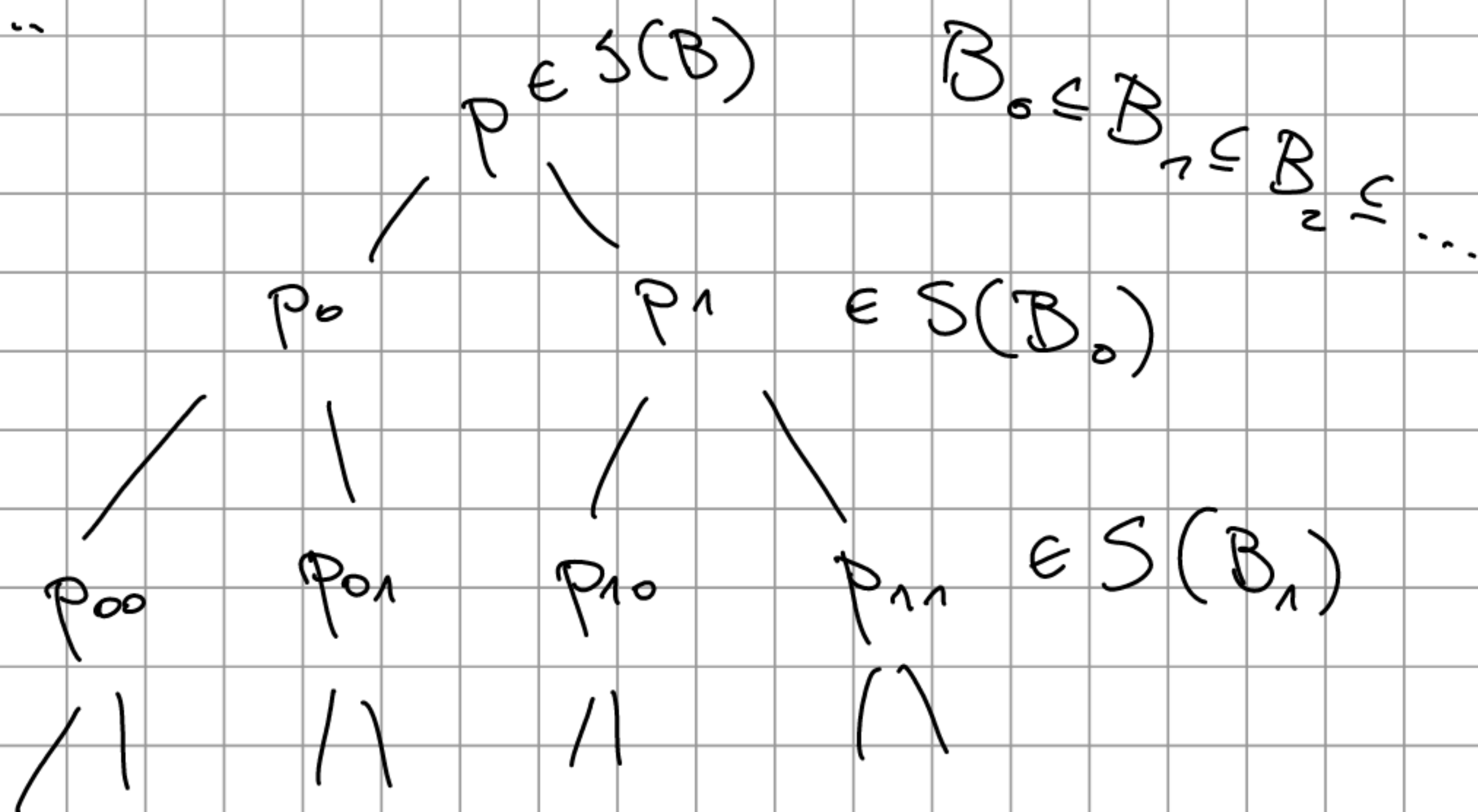
$\Rightarrow \exists q \in S(A)$ q : s-isolated
 $p \subseteq q$

Proof (A.c.) Suppose there's no such q .

(1) $p \not\perp$ a type in $S(A)$. So there's $B_0 \subseteq A$
 $B \subseteq B_0$
 and $p_0 \neq p_1 \in S(B_0)$ extending p .

(2) $p_i \not\perp$ a type in $S(A)$. So: there is $B_1 \subseteq A$
 $B_0 \subseteq B_1$
 and $p_{00}, p_{01}, p_{10}, p_{11} \in S(B_1)$
 pairwise distinct

(3)



Let $B_\omega = \bigcup_n B_n \subseteq A$: dble set, but $|S(B_\omega)| \geq \aleph_0$. ▣

Property of s -isolation:

$\text{tp}(ab/A)$ s -isolated \Leftrightarrow $\text{tp}(a/A)$ is s -isolated and $\text{tp}(b/Aa)$ is s -isolated.

(exercise)

Corollary $T: \aleph_0$ -stable $\Rightarrow \forall A \overset{\mu}{\exists} M \supseteq A$
 \uparrow
 s -prime over A

Proof (sketch) $M = A \cup \{a_\alpha : \alpha < \aleph_0\}$ s.t.:

(1) $\text{tp}(a_\alpha/Aa_{<\alpha})$ is s -isolated

(2) M is \aleph_0 -saturated

it terminates at some point

Then M is s -prime

The model M constructed this way is s -constructible/ A and s -primary/ A , it's unique up to $\frac{\aleph_0}{A}$.

Corollary $T: \kappa_0$ -stable, $M: s$ -prime $/ A$

$\Rightarrow M: s$ -atomic

Proof Let $N: s$ -primary $/ A$

\Downarrow property of
 s -isolation

$N: s$ -atomic

$M: s$ -prime $/ A \Rightarrow \exists f: M \xrightarrow[A]{=} N$

\Downarrow

$M: s$ -atomic

Def (M, N) is a Vaughtian pair for T , if

$N \not\equiv M \models T$ and for some $\varphi(x) \in L_1(N)$

non-algebraic
consistent

$\varphi(N) = \varphi(M)$

(here (M, N, φ) : Vaughtian triple)

Lemma 1 Assume $T: \kappa_0$ -stable (enough T : small). If

T has a Vaughtian pair, then there's $V_p(M, N)$

s.t. M, N : cble and saturated.

Proof Let (M_0, N_0, φ) : Vaughtian triple.

$\varphi(x, \bar{a})$
 \uparrow
 N_0

Let $L' = L \cup \{\bar{a}\} \cup \{P(x)\}$
 new constant symbols new predicate symbol

T' complete theory in L' s.t.

(0) $T' \supseteq \text{Th}(N_0, \bar{a}) = \text{Th}(M_0, \bar{a})$ and for

any $M' \models T'$:

(1) $N := P(M') \upharpoonright_L \prec M := M' \upharpoonright_L$

(2) $\bar{a}^{M'} \in P(M') : \bigwedge_i P(a_i) \in T'$

(3) (M, N, φ) : a V. triple:

$\varphi(x, \bar{a}^{M'})$

- $[\forall x (\varphi(x) \rightarrow P(x))] \in T'$

- $[\exists x \neg P(x)] \in T'$

Ad (1): Let $\varphi(\bar{x}, y) \in L$.

$T' \ni [\forall \bar{x} [\bigwedge_i P(x_i) \wedge \exists y \varphi(\bar{x}, y) \rightarrow \exists y (P(y) \wedge \varphi(\bar{x}, y))]]$

Fact $\exists M' \models T'$ ($M := M' \upharpoonright_L$ and $N := P(M') \upharpoonright_L$)

over both stable and saturated

Pf (fact) $M' = \bigcup_{n < \omega} M'_n$: models of T'
 elem. chain

• M_0' : arbitrary

• $n \rightsquigarrow n+1$: $M_{n+1}' \supseteq M_n'$ such that:

(i) $\forall A \subseteq M_n'$ $\forall p \in S^L(A)$ $p(M_{n+1}') \neq \emptyset$
finite

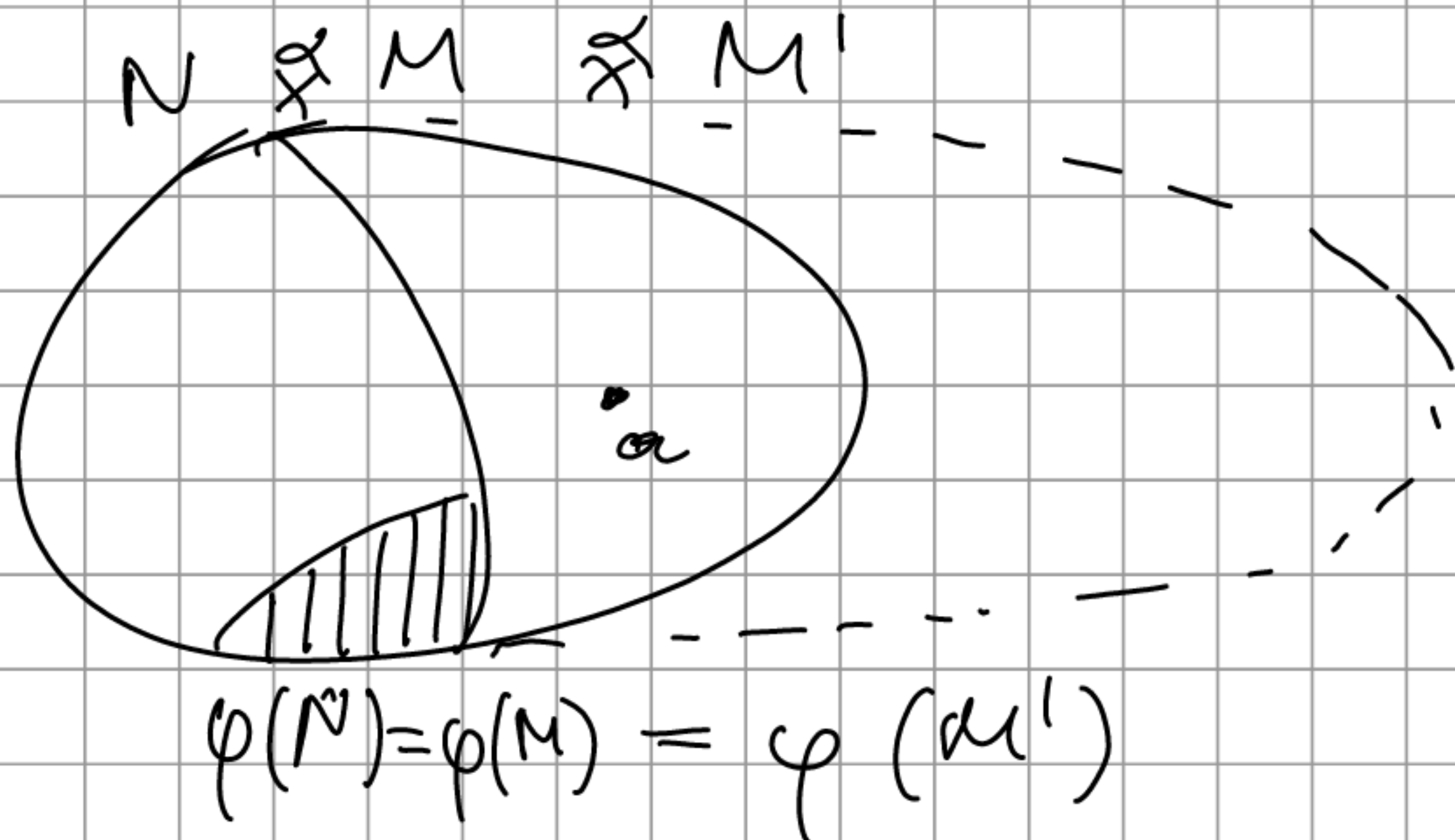
(ii) $\forall A \subseteq P(M_n')$ $\forall p \in S^L(A)$ $p(x) \cup \{p(x)\}$ is
finite realised in M_{n+1}' .

Fact, Lemma 1 ~~is~~

Lemma 2 (stretching Vaughtian pair)

Let (M, N, φ) : V. triple, M, N : \mathcal{N}_0' -saturated,

T : \mathcal{N}_0' -stable. Then $\exists M' \not\cong M$ (M', N, φ) is
 \mathcal{N}_0' -saturated a V. triple



Proof Let $a \in M \setminus N$, $p = \text{tp}(a/N)$, $\text{RM}(p) = 1$.

So $p \subseteq q \in S(M)$, $\text{RM}(q) = \text{RM}(p)$
unique

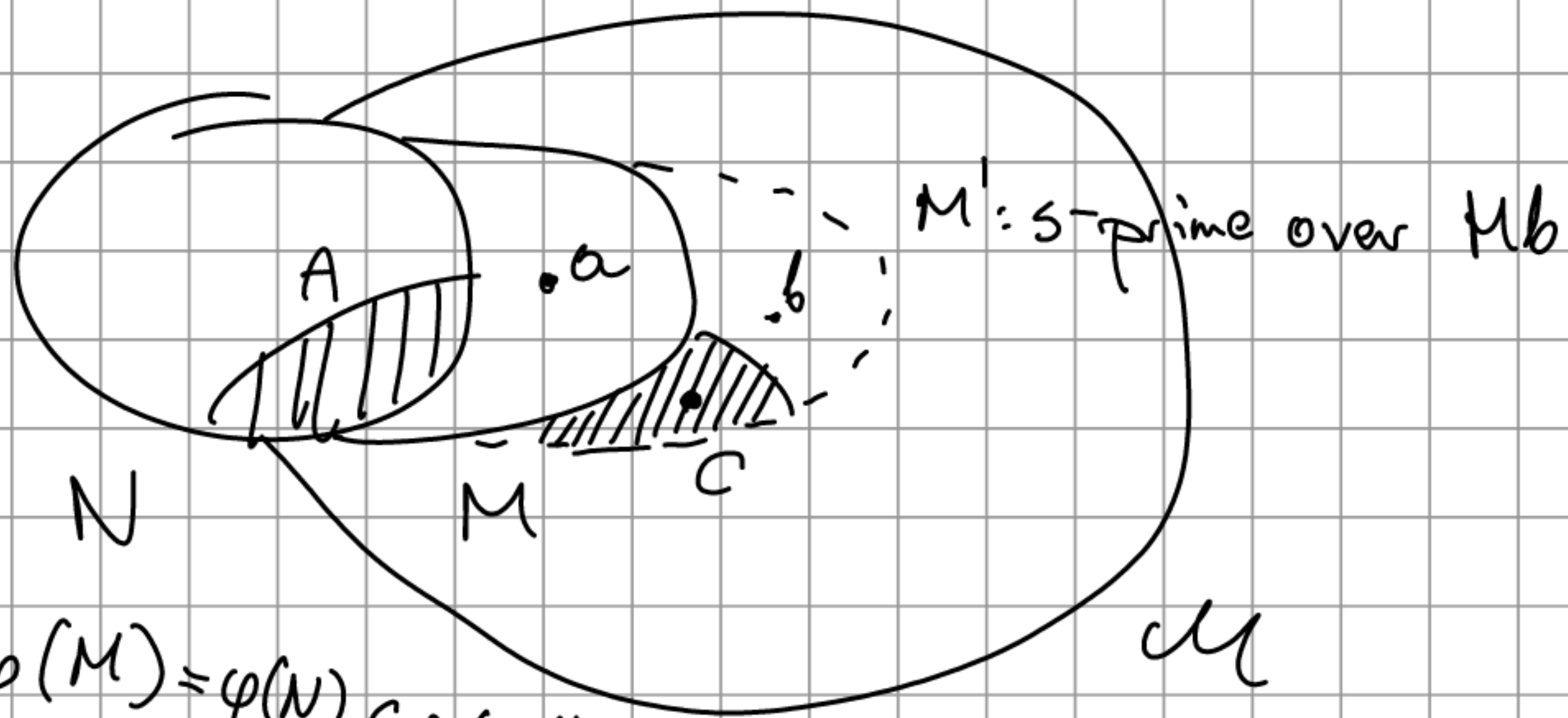
16.05.2022

Lemma 2 (Stretching Voughtian pair)

$T: \mathcal{A}_0$ -stable, $(M, N, \varphi): \text{Voughtian triple}$

$\Rightarrow \exists M' \xrightarrow{\mathcal{A}_0\text{-saturated}} M$ $(M', N, \varphi): \text{Vt.}$

Proof



$A = \varphi(M) = \varphi(N) \subseteq \varphi(M')$

$tp(b/M) \geq tp(a/M), RM(b/A) = RM(b/M) = RM(a/N)$

Claim: $\varphi(M) = \varphi(M')$

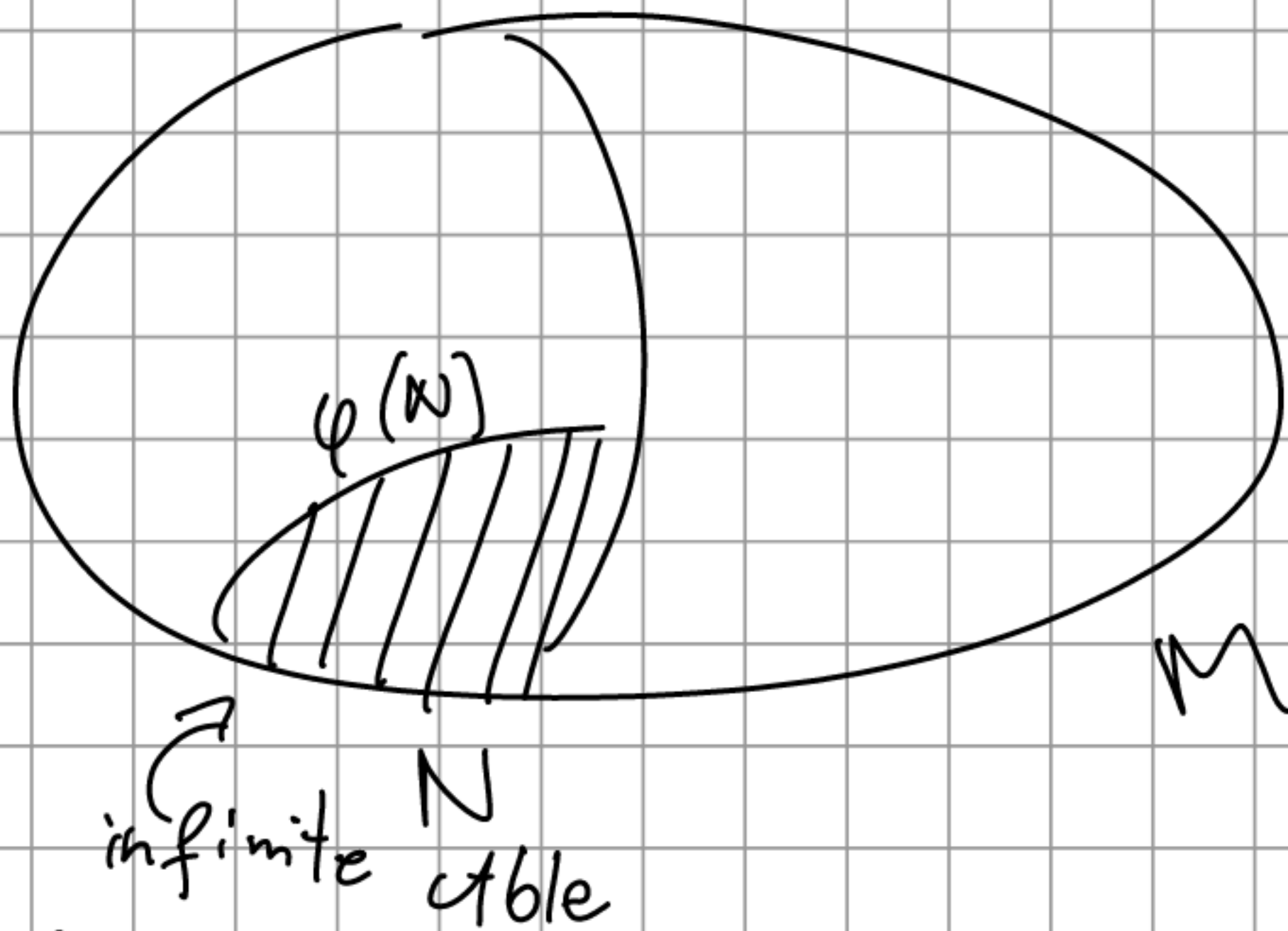
Corollary Suppose $\kappa > \mathcal{A}_0$, $T: \kappa$ -categorical. □

T has no Voughtian pair and $T: \mathcal{A}_0$ -stable.

Proof $T: \mathcal{A}_0$ -stable: proved earlier.

No Voughtian pair: (A.c.) suppose $(M, N, \varphi): \text{Vt.}$

By lemma 1 wlog M, N are \aleph_0 -saturated, N cble
 by lemma 2 wlog $\|M\| = \kappa$, so M is not
 saturated



$$\varphi(x) = \{\varphi(x)\} \cup \{x \neq a\}_{a \in \varphi(N)} \Rightarrow \text{not realised in } M.$$

But $T: \aleph_0$ -stable $\rightsquigarrow \exists M' \|M'\| = \kappa$
 saturated

$$M \not\cong M' \quad \Downarrow$$

Theorem If $T: \aleph_0$ -stable without a Vaughtian pair.

Then $\forall \kappa > \aleph_0$ $T: \kappa$ -categorical.

Corollary (Morley thm, 1964) Let $\kappa > \aleph_0$, \mathcal{D}

(1) $T: \kappa$ -categorical

(2) $T: \aleph_1$ -categorical

Lemma 3 ($T: \mathcal{A}_0$ -stable with no Veughtian pair).

\exists a strongly minimal formula $\varphi(x, \bar{c})$, $\bar{c} \subseteq M \neq T$
prime

$$RM(\varphi) = \text{Mlt}(\varphi) = 1$$

Proof Problem from list 6 ~~□~~

Lemma 4 ($T: \mathcal{A}_0$ -stable, no v. pair) Assume

$M \neq T$, $\varphi(x) = \varphi(x, \bar{a})$, then:
s.m. \uparrow \downarrow \uparrow \downarrow
 M

(a) M is prime and minimal / $\varphi(M) \cup \bar{a}$

(b) If $\|M\| > \mathcal{A}_0$, then $\dim(\varphi(M) / \bar{a}) = \|M\|$

Explanation 1. $A \subseteq M \not\subseteq \mathcal{U}$, M : minimal / A

$$\Leftrightarrow \neg \exists M' \not\subseteq M$$

2. acl -dimension: Assume $\mathcal{U} = \varphi(M)$, where

$\varphi(x, \bar{a})$: s.m., $\text{acl}_{\bar{a}}: \mathcal{P}(\mathcal{U}) \rightarrow \mathcal{P}(\mathcal{U})$,

$$\text{acl}_{\bar{a}}(A) = \text{acl}(A\bar{a}) \cap \mathcal{U}$$

3. $(\mathcal{U}, \text{acl}_{\bar{a}})$ is a pregeometry, i.e.

$$\bullet \text{cl}(\text{cl}(A)) = \text{cl}(A) \supseteq A$$

$$\bullet A \subseteq B \Rightarrow \mathcal{d}(A) \subseteq \mathcal{d}(B)$$

$$\bullet \mathcal{d}(A) = \bigcup_{\substack{A_0 \subseteq A \\ \text{finite}}} \mathcal{d}(A_0)$$

$$\bullet a \in \mathcal{d}(Ab) \setminus \mathcal{d}(A) \Rightarrow b \in \mathcal{d}(Aa)$$

\leadsto Basis $B \subseteq \mathcal{U}$ \mathcal{d} -independent set, $\dim(\mathcal{U}) = |B|$.

In the case of $\text{ad}_{\bar{a}}$ on $\varphi(\mathcal{U})$ additionally:

$$\text{Let } p(x) \in S(\bar{a}) \cap [\varphi(x)] \text{ i.e. } \text{RM}(p) = \text{MH}(p) = \underline{1}$$

s.m.

Then an $\text{ad}_{\bar{a}}$ -independent set $I \subseteq \varphi(\mathcal{U})$

is a Marley sequence in $\mathcal{P} \Rightarrow$ indiscernible over \bar{a} .

Pf (lemma 4)

(a) Suppose M is ^{prime and} not minimal / $\varphi(\mathcal{U}) \cup \bar{a}$.

$$\text{So } \varphi(\mathcal{U}) \cup \bar{a} \subseteq N \not\subseteq M$$

\uparrow
prime / $\varphi(\mathcal{U}) \cup \bar{a}$

But then $(M, N, \varphi) =$ a V. triple \downarrow

(b) Let $I \subseteq \varphi(\mathcal{U})$. Then $\bar{a} \cup \varphi(\mathcal{U}) \subseteq \text{ad}_{\bar{a}}(I) \cup \bar{a} \subseteq M$
an $\text{ad}_{\bar{a}}$ -basis of $\varphi(\mathcal{U})$

$$\|M\| \stackrel{(\bar{a})}{=} |\bar{a} \cup \varphi(M)| \stackrel{\substack{\uparrow \\ \varphi(M) \\ \text{uncountable}}}{=} |I| = \dim(\varphi(M)/\bar{a})$$

Proof of Morley Thm \mathcal{M}_r

Let $\kappa > \aleph_0$, $M, N \models T$ of power κ .

M_0 : prime

$\bar{a} \quad \varphi(x, \bar{a})$: s.m. (lemma 3)

By lemma 4: M : prime & minimal / $\varphi(M, \bar{a}) \cup \bar{a}$

N : ——— || ——— / $\varphi(N, \bar{a}) \cup \bar{a}$

Let $I \subseteq \varphi(M)$ bases.
 $J \subseteq \varphi(N)$

Let $p \in S(\bar{a}) \cap [\varphi(x)]$: a s.m. type

I, J : Morley sequences in p . By L4b

$|I| = |J| = \kappa$. $f: I\bar{a} \rightarrow J\bar{a}$ s.t.

$f = g \upharpoonright I\bar{a} = \text{id}_{I\bar{a}}$

f is elementary

$f: I \xrightarrow[\text{onto}]{1-1} J$

\Downarrow

$f: \text{ad}(I\bar{a}) \xrightarrow{\cong} \text{ad}(J\bar{a})$

\cup
 $\varphi(M)\bar{a}$

\cup
 $\varphi(N)\bar{a}$

$$f \uparrow \varphi(M)\bar{a} : \varphi(M)\bar{a} \xrightarrow{\cong} \varphi(N)\bar{a}$$

$$\uparrow \cong \quad \uparrow \cong$$

$$M \quad N$$

\cap

$$f' : M \xrightarrow{\cong} N$$

(lemme 4a)



Thm (Baldwin, Locken, 1971) Assume $T: \mathcal{A}_1$ -categorical

not \mathcal{A}_0 -categorical. Then:

(1) T has \mathcal{A}_0 many stable models

(2) Every model of T is homogeneous.

\mathcal{N}_1 -categorical theories:

- \mathcal{N}_0 -stable
- have prime model
- no V. pairs
- s.m. sets
- M. sequences, "bases"

Stable formulas

T: fixed complete theory with infinite models

$L \ni \varphi = \varphi(\bar{x}, \bar{y})$
stable

distinguished variables parameters variables

Def Let $\delta(x, y) \in L$.

- (1) $\delta(x, c)$, $c \in \mathcal{M}$: an instance of δ
- (2) $\varphi(x)$ is positive δ -formula if
 $\models \varphi \leftrightarrow$ [positive] boolean combination of instances of δ

(3) $(A \subseteq \mathcal{M})$ δ -type = a type consisting of δ -formulas,

$$L_\delta(A) = \{ \delta\text{-formulas over } A \}$$

(4) $S_\delta(A) = \{ \text{complete } \delta\text{-types over } A \}$
(ultrafilters in $L_\delta(A)$)

Remark Let $\varphi \in L(A)$, $A \subseteq M$. Then

φ is a δ -formula $\Leftrightarrow \models \varphi \Leftrightarrow$ boolean comb. of instances of δ over M

Pf (\Rightarrow) $\mathcal{M} \models \varphi(\bar{x}) \Leftrightarrow \bigwedge_i \bigvee_j \delta(x_i, c_{ij})$ $c_{ij} \in \mathcal{M}$

$$\mathcal{M} \models (\exists y_{ij}) (\varphi(\bar{x}) \Leftrightarrow \bigwedge_i \bigvee_j \delta(x_i, y_{ij}))$$

$$M \models \text{---} \quad \parallel \quad \text{---}$$

So c_{ij} may be taken from M .

Example T : the theory of a single equivalence relation E with two infinite classes. Let $\delta(x, y) = \bar{E}(x, y)$.

$$\models x = x \leftrightarrow E(x, a) \vee E(x, b)$$

is a δ -formula / \emptyset , where $a, b \in \mathcal{M} \neq T, \models \neg E(a, b)$

Def (1) $\delta(x, y)$ has order property, if
 $\exists_{i < \omega} a_i, b_i \in \mathcal{M} \quad \forall i, j < \omega \models \delta(a_i, b_j) \Leftrightarrow i \leq j$

(2) $\delta(x, y)$ is stable if it does not have the order property

Lemma 1 (1) $\varphi(x, y), \psi(x, z)$: stable

$\Rightarrow \neg \varphi(x, y), (\varphi \vee \psi)(x, yz), (\varphi \wedge \psi)(x, yz)$
are stable

(2) Let $\psi(y, x) = \varphi(x, y)$, then
 φ stable $\Leftrightarrow \psi$ stable

(3) φ : stable $\Leftrightarrow \exists n < \omega \neg \exists a_i, b_i (i \leq n)$

$$\bigwedge_{i, j} \models \varphi(a_i, b_j) \Leftrightarrow i \leq j$$

Pf. exercise

Def Let $p \in S(M)$ or $p \in S_\delta(M)$.

A δ -definition of p : a formula $\psi(y) \in L(M)$

s.t. $\forall c \in M (\delta(x, c) \in p \iff \models \psi(c))$

(i.e. $\{c \in M : \delta(x, c) \in p\} = \psi(M)$)

Lemma 2 Assume $\delta(x, y)$ is stable, $p \in S(M)$

or $p \in S_\delta(M)$. Then:

(1) $p(x)$ has a δ -definition $\psi(y)$ that

is a positive δ^* -formula, where $\delta^*(y, x) = \delta(x, y)$

(2) $A \subseteq M$ and M is $|A|^+$ -saturated, then

$\exists c_1, c_2, \dots \in M \quad c_i \models p \upharpoonright_{A \setminus c_i}$

and the δ -definition of p is equivalent

to a positive boolean combination of

formulas $\delta(c_i, y)$, $i < \omega$.

23.05.2022

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Lemma 2 Assume δ is stable, $\varphi \in S(M)$ or $\varphi \in S_{\sigma}(M)$. Then

(1) $\varphi(x)$ has a δ -definition $\psi(y)$, that is a positive δ -formula.

(2) If $A \subseteq M \leftarrow |A|^+$ -saturated, then $\exists c_1, c_2, \dots \in M \bigwedge_i c_i \models \varphi \upharpoonright_{A \setminus \{c_i\}}$
and ψ from (1) is a positive Boolean combination of formulas $\delta(c_i, x), i < \omega$

Proof Let $N < \omega$ s.t.

(i) $\neg (\exists c_i, a_i, i \leq N) (\bigwedge_{i < j} \models \delta(c_i, a_j) \Leftrightarrow i < j)$

(ii) $\neg (\exists c_i, b_i, i \leq N) (\bigwedge_{i < j} \models \neg \delta(c_i, b_j) \Leftrightarrow i < j)$

Let $c^* \models \varphi$. So: $\delta(x, a) \in \varphi \Leftrightarrow \delta(c^*, a)$ for $a \in M$.

We shall define $c_i \in M, A_i, B_i \subseteq M, i < \omega$ as follows:

• $c_0 \in M$ arbitrary ($c_0 \models \varphi \upharpoonright_A$)

• Suppose we have $c_0, \dots, c_n, A_0, \dots, A_n, B_0, \dots, B_n$. We shall find $c_{n+1}, A_{n+1}, B_{n+1}$.

• $A_{n+1} \subseteq M$ s.t. $\forall W \subseteq \{0, \dots, n\}$ if $(\exists a \in M) (\bigwedge_{j \in W} \models \delta(c_j, a) \text{ and } \models \neg \delta(c^*, a))$ i.e. $\delta(x, a) \notin \varphi$

then \exists such an $a_W \in A_{n+1}$

• $B_{n+1} \subseteq M$ s.t. $\forall W \subseteq \{0, \dots, n\}$ if $(\exists b \in M) (\bigwedge_{j \in W} \models \neg \delta(c_j, b) \text{ and } \models \delta(c^*, b))$

then \exists such an $b_W \in B_{n+1}$.

• for $b \in B_{\leq n+1} \models \delta(c^*, b)$, for $a \in A_{\leq n+1} \models \neg \delta(c^*, a)$
we may assume

• $c_{n+1} \in M$ s.t. $(\forall a \in A_{\leq n+1}) (\forall b \in B_{\leq n+1}) (\models \delta(c_{n+1}, b) \text{ and } \models \neg \delta(c_{n+1}, a))$

[for (2): $c_{n+1} \models \varphi \upharpoonright_{A_{\leq n+1} \cup B_{\leq n+1}}$ is enough]

• So for $a \in A_{\leq n}, b \in B_{\leq n} \models \neg \delta(c_n, a) \wedge \delta(c_n, b)$

• if $0 \leq i_0 < \dots < i_n < \omega$ and $\exists a \in M$

$\models \delta(c_{i_0}, a) \wedge \delta(c_{i_1}, a) \wedge \dots \wedge \delta(c_{i_n}, a) \wedge \neg \delta(c^*, a)$

then $\exists d_0, \dots, d_n \in M (\forall 0 \leq j, r \leq n) (\models \delta(c_{i_j}, d_r) \Leftrightarrow j \leq r)$

pf Let $W_r = \{i_0, \dots, i_{r-1}\} \subseteq \{i_0, \dots, i_n\}$. Let $d_r = a_{w_r} \in A_{(i_{r-1})+1}^2$ s.t.
 $\models \delta(c_{i_0}, d_r) \wedge \dots \wedge \delta(c_{i_{r-1}}, d_r) \wedge \neg \delta(c^*, d_r), \models \neg \delta(c_j, d_r)$ for all $j > i_{r-1}$

• Symmetrically: if $0 \leq i_0 < \dots < i_n < \omega$ and $\exists b \in M$

$$\models \neg \delta(c_{i_0}, b) \wedge \dots \wedge \neg \delta(c_{i_n}, b) \wedge \delta(c^*, b)$$

then $\exists d_0, \dots, d_n \in M \quad \forall 0 \leq j, r \leq n \quad (\models \neg \delta(c_{i_j}, d_r) \Leftrightarrow j < r)$

• Therefore:

(a) If $a \in M$ and $\exists W \subseteq \{0, \dots, 2N\}$ s.t. $\bigwedge_{i \in W} \models \delta(c_i, a)$, then $\models \delta(c^*, a)$
 $|W| = N+1$

(b) If $a \in M$ and $\exists W \subseteq \{0, \dots, 2N\}$ s.t. $\bigwedge_{i \in W} \models \neg \delta(c_i, a)$, then $\models \neg \delta(c^*, a)$
 $|W| = N+1$

• Let $a \in M$. $\delta(x, a) \in \mathcal{P} \Leftrightarrow \models \delta(c^*, a) \Leftrightarrow \bigvee_{\substack{W \subseteq \{0, \dots, 2N\} \\ |W| = N+1}} \bigwedge_{i \in W} \delta(c_i, a)$

put $y := \bigwedge_{i \in W} \delta(c_i, a)$

• So $\forall a \in M \quad (\delta(x, a) \in \mathcal{P} \Leftrightarrow \models \mathcal{P}(c_{\leq 2N}, a))$: α . delta definition of \mathcal{P} .

Lemma \blacksquare

Corollary Let $\delta \in L$. \square :

(1) $\delta(x, y)$: stable

(2) $\forall M \forall \mathcal{P} \in \mathcal{S}(M)$ \mathcal{P} has a δ -definition

(3) $(\forall \lambda \geq \lambda_0) (\forall A \subseteq M) (|S_\delta(A)| \leq \lambda \Rightarrow \exists \mathcal{P} \in \mathcal{S}(M) \text{ s.t. } \mathcal{P} \text{ has a } \delta\text{-definition})$

(4) $(\exists \lambda \geq \lambda_0) \dots$ " " " " " "

Proof (1) \Rightarrow (2): by lemma 2

• (2) \Rightarrow (3) $A \subseteq M \text{ s.t. } |S_\delta(A)| \leq \lambda \Rightarrow \exists \mathcal{P} \in \mathcal{S}(M) \text{ s.t. } \mathcal{P} \text{ has a } \delta\text{-definition}$
 $\|M\| \leq \lambda$

↑ there are $\leq \|M\|$ δ -definitions $/ M$

• (3) \Rightarrow (4) trivial

(4) \Rightarrow (1) by contradiction: suppose δ has order property. Let $\lambda \geq \aleph_0$.

We shall find A s.t. $|A| \leq \lambda$ and $|S_\delta(A)| > \lambda$

• $(\exists a_i, b_i, i < \omega)(\neq \delta(a_i, b_j) \Leftrightarrow i < j)$, choose $(I, \leq) \neq \text{DLO}$, $|I| > \lambda$

where $\exists J \subseteq I : J$ dense. By compactness $|J| \leq \lambda$

$$(\exists a_i, b_i, i \in I) \bigwedge_{i, j \in I} (\neq \delta(a_i, b_j) \Leftrightarrow i < j)$$

• δ defines an order:

$$0 < \frac{a_j}{b_i} < 0 \leq 0 < \dots$$

$i \in I \geq j$

Let $A = \{b_j : j \in J\}$, then $|A| \leq \lambda$ and $|S_\delta(A)| > \lambda$.

When $i_1 < i_2 \in I$ then $\text{tp}_\delta(a_{i_1}/A) \neq \text{tp}_\delta(a_{i_2}/A)$,
just choose $j \in J$ s.t. $i_1 < j < i_2$.

Corollary (1) T -stable $\Leftrightarrow (\forall \delta \in L)(\delta$ -stable in $T)$

(2) T -stable and $\lambda^{\aleph_0} = \lambda$, then T - λ -stable

Proof (1) \Leftrightarrow T - λ -stable. Then $\forall A \in \mathcal{M}$ $|S_\delta(A)| \leq |S(A)| \leq \lambda$, so by

previous corollary δ -stable.

• (\Leftarrow) later...

(2) $|T| = \aleph_0$, assume $\lambda = \lambda^{\aleph_0}$. Let $|A| \leq \lambda$, ~~$|S(A)| \leq \lambda$~~

$$S(A) \xrightarrow{\lambda^{-1}} \langle p|_\delta : \delta \in L \rangle \in \prod_{\delta \in L} S_\delta(A)$$

$$|\prod_{\delta \in L} S_\delta(A)| \leq \lambda^{\aleph_0} = \lambda \rightsquigarrow |S(A)| \leq \lambda$$

(\Leftarrow) Assume $\forall \delta \in L$ δ -stable in T . Then $|S(A)| \leq |\prod_{\delta \in L} S_\delta(A)| \leq \lambda^{\aleph_0}$

\Downarrow
 T stable in λ if $\lambda^{\aleph_0} = \lambda$

Def (1) $\delta(x, y) \in L$ has strict order property (SOP)

if $(\exists b_i, i < \omega) (\delta(\mathcal{M}, b_0) \neq \delta(\mathcal{M}, b_1) \neq \dots)$

$$[\Leftrightarrow \delta(b_0, \mathcal{M}) \neq \delta(b_1, \mathcal{M}) \neq \dots]$$

↑
alternative definition

T has SOP $\Leftrightarrow (\exists \sigma \in L) (\sigma \text{ has SOP})$

(2) δ has independence property (IP)

if $(\exists b_i, i < \omega) \{ \delta(\mathcal{M}, b_i) : i < \omega \}$ is independent

T has IP $\Leftrightarrow (\exists \sigma \in L) (\sigma \text{ has IP})$

Theorem (Shelah)

T has OP $\Leftrightarrow T$ has SOP or T has IP
(is unstable)

Examples • $DLO_0 = Th(\mathbb{Q}, \leq)$ has SOP : $\varphi(x, y) = x \leq y$
has SOP,

but DLO_0 has NIP

• Random graph has IP and NSOP

TYPES AND DEFINABLE

TYPES & TYPE-DEFINABLE SETS.

$p(\bar{x})$: a type $\leadsto p(\mathcal{M}) = \{ a \in \mathcal{M}^n : a \models p \}$: a type-definable set (over A)
 $|\bar{x}| = n$ (over A)
 $= \bigcap_{\varphi \in p} \varphi(\mathcal{M})$

$$p = \{ \varphi_i : i \in I \} \rightarrow \left(\bigwedge_{i \in I} \varphi_i \right) (\mathcal{M})$$

Let $p(\bar{x}), q(\bar{x})$: types, $(p \wedge q)(\bar{x}) = (p \cup q)(\bar{x})$,

$$(p \vee q)(\bar{x}) = \{ (\bigwedge p_0) \vee (\bigwedge q_0) (\bar{x}) : p_0 \in p, q_0 \in q, \text{ finite} \}$$

Fact (1) $(p \wedge q)(\mathcal{M}) = p(\mathcal{M}) \cap q(\mathcal{M})$

(2) $(p \vee q)(\mathcal{M}) = p(\mathcal{M}) \cup q(\mathcal{M})$

$$(\exists \bar{y} \varphi)(\bar{x}, \bar{y}) = \exists \bar{y} \bigwedge_i \varphi_i(\bar{x}, \bar{y}) \stackrel{\text{def}}{=} \left\{ \exists \bar{y} \bigwedge p_0(\bar{x}, \bar{y}) : p_0 \in P_{\text{finite}} \right\}$$

where $p = \bigwedge \varphi_i : i \in I$

$$(3) (\exists \bar{y} p)(\mathcal{M}) = \pi_{\bar{x}} \left(p(\mathcal{M}) \right)_{\mathcal{M}^{\bar{x}} \times \mathcal{M}^{\bar{y}}}$$

$$(4) (\forall \bar{y} p)(\mathcal{M}) = \mathcal{M}^{\bar{x}} \setminus \pi_{\bar{x}}(\mathcal{M}^{\bar{x}\bar{y}} \setminus p(\mathcal{M}))$$

Remark $\mathcal{M} \setminus p(\mathcal{M})$ is usually not type-definable

Let $\Delta = \{ \varphi_0(x, y) \dots \varphi_n(x, y) \} \in L$
 \downarrow
 Δ -formula, Δ -type

Skolem trick There's formula $\delta(x, \bar{y})$ s.t. "instances of formulas Δ "
 = "instances of δ " over any set A of power ≥ 2 .

Proof $\delta = \delta(x, y, z_0, \dots, z_n) =$

$$\begin{cases} \varphi_0(x, y), & \text{when } z_0 = z_1 \\ \varphi_1(x, y), & \text{when } z_0 \neq z_1 \wedge z_0 = z_2 \\ \varphi_2(x, y), & \text{when } z_0 \neq z_1, z_2 \wedge z_0 = z_3 \\ \vdots \\ \varphi_n(x, y), & \text{when } z_0 \neq z_1, \dots, z_n \end{cases}$$

• $\delta(\mathcal{M}, a, \bar{c}) = \varphi_i(\mathcal{M}, a)$ for some $i \in \{0, \dots, n\}$

• For every $a \in A$, every $i \in \{0, \dots, n\}$ $\exists \bar{c} \in A$ $\varphi_i(\mathcal{M}, a) = \delta(\mathcal{M}, a, \bar{c})$ ■

$\varphi \in S_\delta(A) \ [\varphi \in S_\Delta(A)] \ (\Delta \text{ possibly infinite})$

$\bullet \text{CB}_\delta(\varphi) = \text{CB in } S_\delta(A) \ (\text{CB-delta rank, CB-local rank})$

$\bullet \varphi \in L_\delta(A) \rightsquigarrow \text{CB}_{\delta,A}(\varphi) = \text{CB}(S_\delta(A) \cap [\varphi])$

$\bullet \text{CB}_\delta \rightsquigarrow R_\delta: L(\mathcal{M}) \longrightarrow \{ \neg 1 \} \cup \text{Ord} \cup \{ \infty \}$
local δ -rank
local Δ -rank

The smallest function s.t. $\forall \alpha \in \text{Ord} \cup \{ \neg 1 \}$

$R_\delta(\varphi) \geq \alpha + 1 \iff \forall n < \omega \exists \psi_1, \dots, \psi_n \bigwedge_i R_\delta(\varphi \wedge \psi_i) \geq \alpha$
 δ -formulas
pairwise contr.

[Alternatively]

$R_\delta(\varphi) = \text{CB}_{\delta, \mathcal{M}}(\{ p \in S_\delta(\mathcal{M}) : p \cup \{ \varphi \} \text{ consistent} \})$

$R_\delta(p) = \min \{ R_\delta(\varphi) : p \vdash \varphi \}$

[Alternatively]

$R_\delta(p) = \text{CB}_{\delta, \mathcal{M}}(\{ q(x) \in S_\delta(\mathcal{M}) : p \cup q \text{ consistent} \})$

Remark (a) $R_{\delta(x,y)}: L_x(\mathcal{M}) \xrightarrow{\text{onto}} (\{ \neg 1 \} \cup \text{initial segment of ordinals} \cup \{ \infty \})$
proper
initial segment
of ordinals
possibly

(b) $R_\delta(\varphi) \geq \alpha + 1 \iff \exists \psi_1, \psi_2, \dots \bigwedge_i R(\varphi \wedge \psi_i) \geq \alpha$
pairwise contr.
 δ -formulas

(c) $R_\delta(\varphi \vee \psi) = \max \{ R_\delta(\varphi), R_\delta(\psi) \}$

(d) $p(x): a \text{ type } / A \Rightarrow \exists q(x): a \text{ complete type } / A \text{ with } p \subseteq q \implies R_\delta(p) = R_\delta(q)$

Example

$R_\delta = R_{\{ \delta, \neg \delta \}} \xrightarrow{\Delta} R_{\delta'} \uparrow$
Shelah trick (for some δ')

\rightsquigarrow We assume that " δ is closed under negation":
 i.e. $\forall \bar{a} \exists \bar{b} \models \delta(x, \bar{a}) \iff \neg \delta(x, \bar{b})$

(1) $R_\delta(\varphi(x, \bar{a})) \geq k + 1 \iff \forall m < \omega \exists \psi_1, \dots, \psi_m \bigwedge_i R_\delta(\varphi(x, \bar{a}) \wedge \psi_i) \geq k$
 δ -formulas
pairwise contr.

$$\Leftrightarrow \forall n \exists p_1, \dots, p_n \in S_\delta(\mathcal{M}) \wedge R_\delta(p_i \cup \{\varphi\}) \geq k$$

distinct

$$\left[\begin{array}{l} p_i \neq p_j \text{ for } i \neq j \\ \exists c_{ij} \delta(x, c_{ij}) \wedge \neg \delta(x, c_{ji}) \end{array} \right]$$

only a def. of ψ_i

enough to have this in the condition

$$\Leftrightarrow \forall m \exists \langle c_{ij} : 1 \leq i \neq j \leq n \rangle \left\{ \bigwedge_{\substack{i, j \\ i \neq j}} \delta(x, c_{ij}) \wedge \neg \delta(x, c_{ji}) \in p_i \right\} \text{ and } R_\delta(\varphi(x, \bar{a}) \wedge \psi_i(x, \bar{c})) \geq k$$

$\psi_i(x, \bar{c})$: a δ -formula

[Notice: $\psi_i(x, \bar{c})$, $i=1, \dots, m$ are explicitly pairwise \perp]

Lemma For every $\varphi(x, \bar{y})$, $k < \omega$ there is a type $\Phi_{\varphi, k}(\bar{y})$ s.t.
 $\forall \bar{a} \in \mathcal{M} R_\delta(\varphi(x, \bar{a})) \geq k \Leftrightarrow \models \Phi_{\varphi, k}(\bar{a})$

PP Induction on k .

$k=0$ $R_\delta(\varphi(x, \bar{a})) \geq 0 \Leftrightarrow \models \exists x \varphi(x, \bar{a})$ so $\Phi_{\varphi, 0}(\bar{y}) = \{ \exists x \varphi(x, \bar{y}) \}$

$k \rightarrow k+1$ $R_\delta(\varphi(x, \bar{a})) \geq k+1 \Leftrightarrow \models \exists \bar{c} \bigwedge_{1 \leq i \leq n} \Phi_{\varphi, \psi_i, k}(\bar{a}, \bar{c})$
 $\models \Psi_n(\bar{a})$ for a type $\Psi_n(\bar{y})$

$$\Phi_{\varphi, k+1}(\bar{y}) = \bigcup_n \Psi_n(\bar{y})$$

□

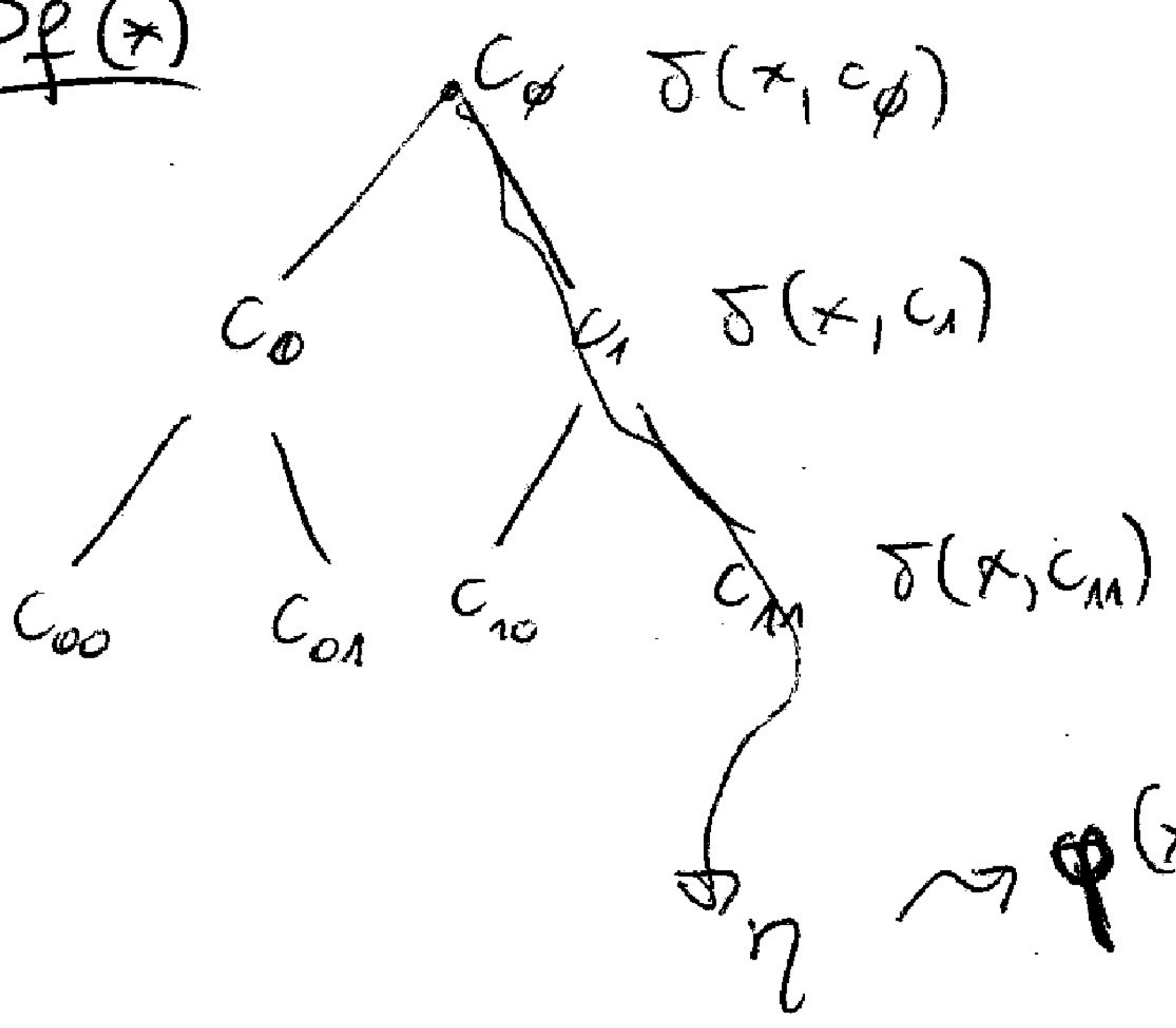
Corollary (1) $R_\delta(\varphi) \geq \omega \Leftrightarrow R_\delta(\varphi) = \infty$

(2) δ is stable $\Leftrightarrow R_\delta(x=x) < \omega$

here $\begin{cases} \chi^0 = x \\ \chi^1 = \neg x \end{cases}$

Proof (1) \Rightarrow (*): $\exists \langle c_\eta, \eta \in 2^{<\omega} \rangle \forall \eta \in 2^{<\omega} p_\eta(x) = \{ \varphi \} \cup \{ \delta(x, c_\eta) \} : n < \omega$ is consistent

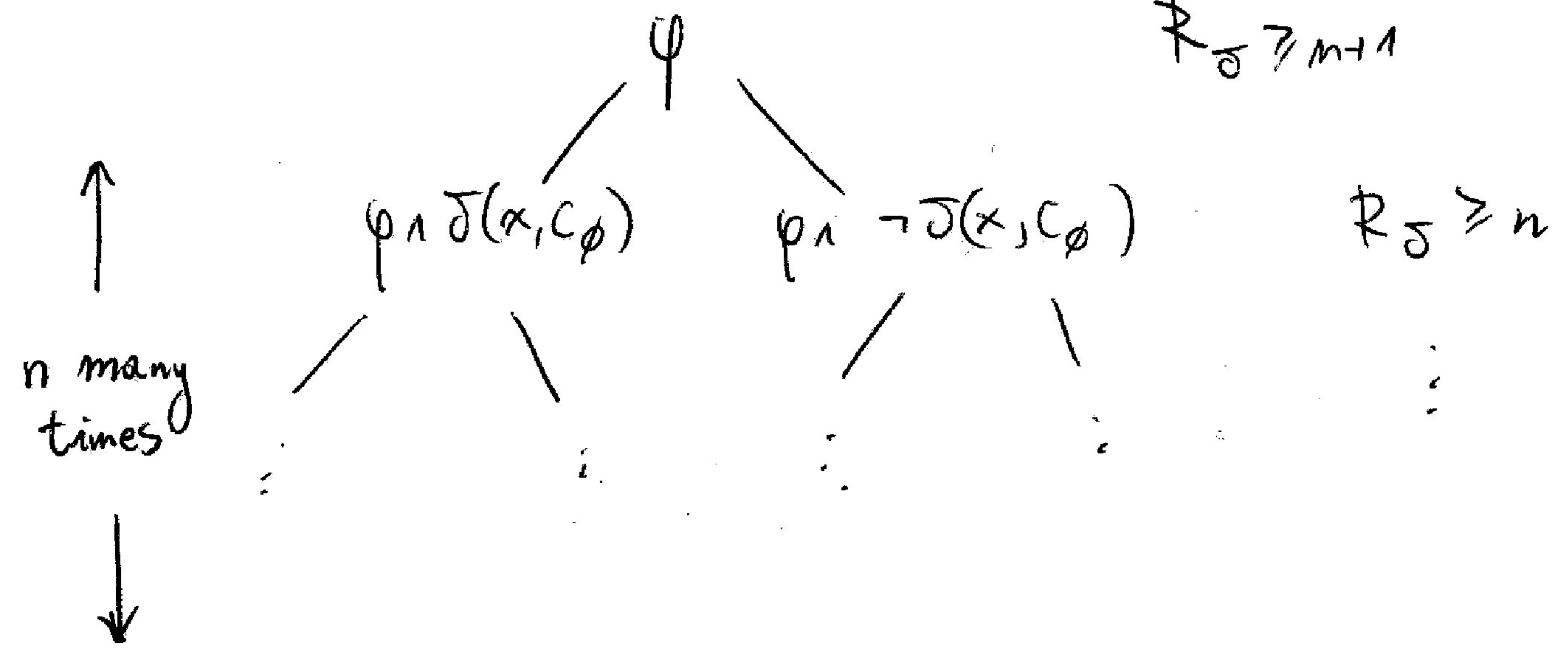
PP (*)



$\varphi(x) \cup \{ \delta(x, c_\varphi), \neg \delta(x, c_1), \neg \delta(x, c_{11}), \dots \}$
 consistent

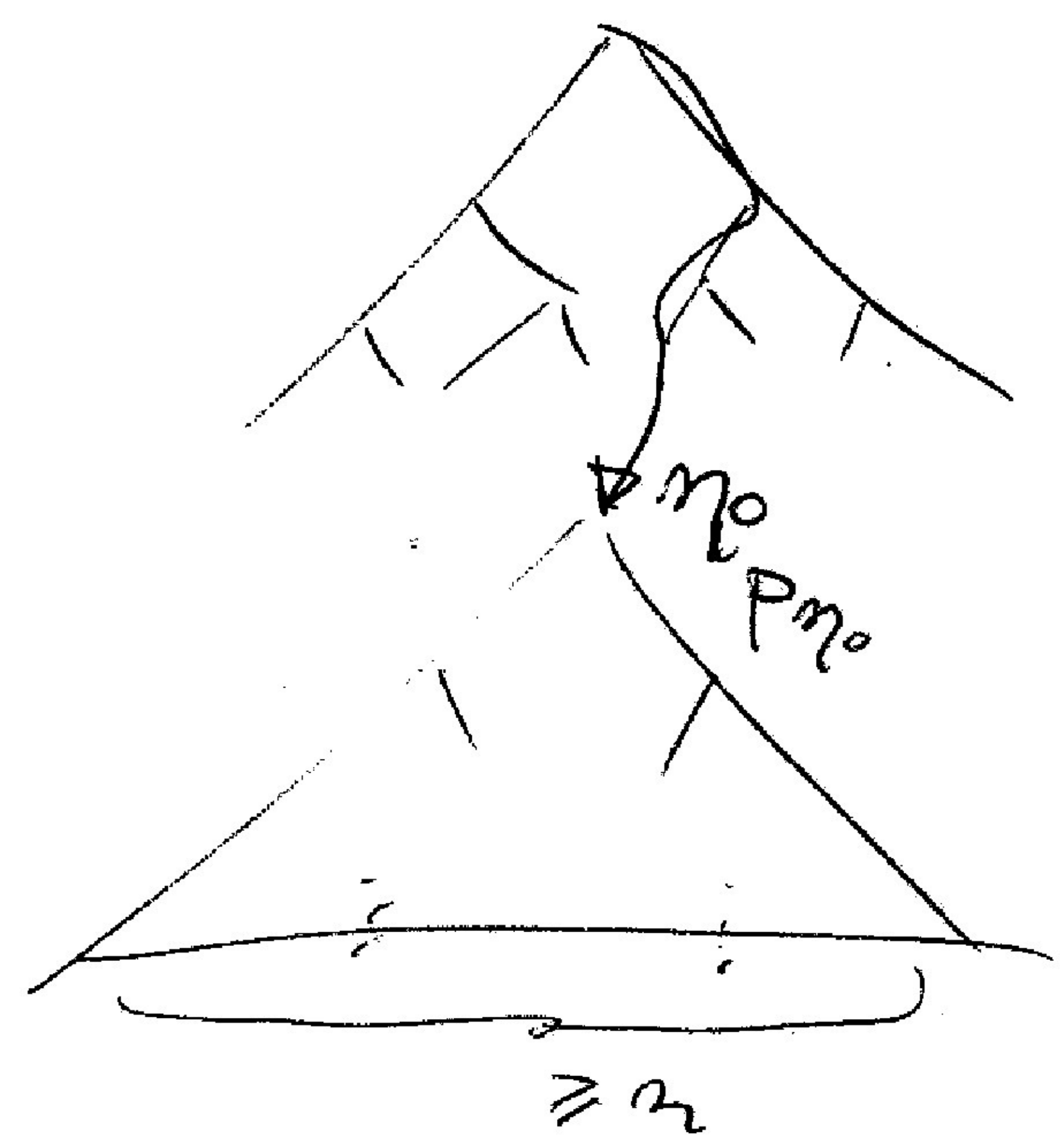
goal

Enough to show $\forall m \exists c_\eta, \eta \in 2^{<m} \forall \eta \in 2^m p_\eta(x)$ consistent. But this follows from $R_\delta(\varphi) \geq m+1$



(*) □

Proof of \Rightarrow in (1): (A.c.) suppose $R_\delta(\varphi) < \infty$. For $\eta \in 2^{<\omega}$ $R_\delta(p_\eta(x)) < \infty$.
 Let $\eta_0 \in 2^{<\omega}$ s.t. $\alpha := R_\delta(p_{\eta_0})$ is minimal.



So $\forall m \exists \varphi_1, \dots, \varphi_m$ $\wedge_i R_\delta(p_{\eta_0} \wedge \varphi_i) \geq \alpha$
 δ -formulas pairwise \downarrow

PF (2) ~~unstable $\Leftrightarrow R_\delta(\Rightarrow)$~~ (A.c.) $R_\delta(x \neq x) \geq \omega \stackrel{(1)}{\Rightarrow} R_\delta(x=x) = \infty$
 the proof of (1) get a tree of instances of $\delta \Rightarrow |S_\delta(\{c_\eta : \eta \in 2^{<\omega}\})| = 2^{\aleph_0}$
 $\Rightarrow \delta$ unstable

(\Leftarrow) $R_\delta(x=x) < \omega$, A : a set, $|A| \leq \kappa$, shall prove $|S_\delta(A)| \leq \kappa$

(a) $p \in S_\delta(A) \rightsquigarrow \exists \varphi \in \mathcal{P} R_\delta(p) = R_\delta(\varphi) \leq \kappa < \omega$ (for some)

There are $\leq \kappa$ -many such $\varphi \in \mathcal{L}(A)$

(b) For $\varphi \in \mathcal{L}_\delta(A)$ $| \{ p \in S_\delta(A) : \varphi \in \mathcal{P} \text{ and } R_\delta(p) = R_\delta(\varphi) \} | < \aleph_0$

□

Corollary $R_{\delta}(p) \geq k+1 \Leftrightarrow \forall m \exists \varphi_1, \dots, \varphi_m \text{ } \delta\text{-formulas pairwise } \perp \text{ } R_{\delta}(p \cup \{\varphi_i\}) \geq k$
 any type

BINARY RANKS: $R_{\delta,2}$

$$R_{\delta,2}(\varphi) \geq \alpha+1 \Leftrightarrow \exists c [R_{\delta,2}(\varphi \wedge \delta(x,c)) \geq \alpha \text{ and } R_{\delta,2}(\varphi \wedge \neg \delta(x,c)) \geq \alpha]$$

Lemma (1) $R_{\delta,2}(\varphi(x,\bar{a})) \geq k \Leftrightarrow \models \Theta_{\varphi,k}(\bar{a})$ for a formula $\Theta_{\varphi,k}(\bar{y})$

(2) $R_{\delta,2}(\varphi(x,\bar{a})) = k \Leftrightarrow \models \Theta_{\varphi,k}(\bar{a}) \wedge \neg \Theta_{\varphi,k+1}(\bar{a})$

(3) $R_{\delta,2}(\varphi) \geq \omega \Leftrightarrow R_{\delta,2}(\varphi) = \infty$

(4) $R_{\delta,2}(x=x) < \infty \Leftrightarrow \delta \text{ is stable}$

Comments (a) Usually $R_{\delta,2}(\varphi \vee \psi) = \max\{R_{\delta,2}(\varphi), R_{\delta,2}(\psi)\}$.

(b) So usually it is not true that $\forall p \forall A \exists q \in S(A) R_{\delta,2}(p) = R_{\delta,2}(q)$

Definability again

Lemma (Sketch) Let $\delta(x,y)$: stable, $p(x) \in S_{\delta}(A)$. Then

$$\exists \chi \in L(A) \forall \bar{a} \in A \models \chi(\bar{a}) \Leftrightarrow \delta(x,\bar{a}) \in p(x)$$

a δ -definition

Proof $R_{\delta,2}(p) = k < \omega$. Choose $\psi \in p$ with $R_{\delta,2}(\psi) = k$. Then for $\bar{a} \in A$

$$\delta(x,\bar{a}) \in p \Leftrightarrow R_{\delta,2}(\psi \wedge \delta(x,\bar{a})) \geq k \Leftrightarrow \underbrace{\Theta_{\psi \wedge \delta,k}(\bar{a})}_{\chi(\bar{a})}$$

Corollary $[\delta: \text{stable}, a \in \mathcal{M}_{A \neq}] \delta(a,A) = \{b \in A : \models \delta(a,b)\} = \{b \in A : \models \chi(b)\}$
 for some $\chi(y) \in L(A)$

Proof $p = \text{tp}_{\delta}(a/A) \rightsquigarrow \chi$ from the lemma

$M \models L, T = Th(M), ER(T) = \{E(\bar{x}, \bar{y}) \in L : M \models "E \text{ is an equivalence relation}"\}$
 $E \rightsquigarrow S_E : \text{new sort symbol.}$

$L \rightsquigarrow L^{eq} : \text{many sorted language.}$

$L^{eq} :$

- the standard sort $S_ =$
- $S_E, E \in ER(T)$ (those are imaginary sort symbols)
- symbols of L refer to $S_ =$ ~~(the standard sort)~~
- new function symbols of $L^{eq} :$
 - $E \in ER(T) \rightsquigarrow F_E(\bar{x})$
 $\begin{matrix} \text{"} E(\bar{x}, \bar{y}) \text{"} & & \text{of sort } S_E & \text{of } S_ = \end{matrix}$

$M \models T \rightsquigarrow M^{eq} = \bigcup_{E \in ER(T)} S_E^{M^{eq}}, \text{ where } S_E^{M^{eq}} = \{ \bar{a} / E : \bar{a} \in M^n \}$
 $S_ =^{M^{eq}} = M$
 the home sort of M^{eq}
 standard

$M_E := S_E^{M^{eq}}$, elements of M_E : imaginary elements, imaginaries where $E \neq x=y$

$F_E^{M^{eq}}(\bar{a}) = \bar{a} / E$. The symbols of L are naturally interpreted in M .

~~$T^{eq} = Th(M^{eq})$~~

Fact (1) For $\varphi(\bar{x}) \in L$ and $\bar{a} \in M^n$ $M \models \varphi(\bar{a}) \Leftrightarrow M^{eq} \models \varphi(\bar{a})$

Def $T^{eq} := \text{Cr}(T \cup \{ "F_E : S_ = \xrightarrow{\text{onto}} S_E" \}_{E \in ER(T)} \cup \{ F_E(\bar{x}) = F_E(\bar{y}) \Leftrightarrow E(\bar{x}, \bar{y}) \}_{E \in ER(T)})$

Fact (1) $M \models T \Leftrightarrow M^{eq} \models T^{eq}$

(2) $\forall M^* \models T^{eq} \exists M \models T (M^* \stackrel{\text{essentially}}{=} M^{eq} \text{ and } M^*_ = M)$
 i.e. $\exists f: M^* \xrightarrow{\cong} M^{eq}$
 $f|_M = \text{id}_M$

(3) T^{eq} : complete

(4) $\exists E(1), \dots, E(k) \in ER(T), \varphi(x_1, \dots, x_k) \in L^{eq}, x_i \in E(i)$

then there's $\psi(y_1, \dots, y_k) \in L$ s.t. $T^{eq} \vdash (\forall y_1, \dots, y_k) (\psi(y_1, \dots, y_k) \leftrightarrow$
of sort $S =$

$$\varphi(F_{E(1)}(y_1), \dots, F_{E(k)}(y_k))$$

Corollary (1) $\forall M \prec \mathcal{M} \quad M^{eq} \prec \mathcal{M}^{eq}$

(2) $\mathcal{M}^{eq} = dcl^{eq}(\mathcal{M}_=), M^{eq} = dcl^{eq}(M_=)$

(3) $A \subseteq \mathcal{M} = \mathcal{M}_=$ is definable [with parameters] in M^{eq}

$\Leftrightarrow A$ is definable [with parameters] in \mathcal{M}

(4) \mathcal{M}^{eq} is κ -saturated and strongly κ -homogeneous

Proof of fact (1), (2): obvious

(3) Let $M^*, N^* \models T^{eq} \stackrel{(2)}{\Rightarrow} M^* = M^{eq}, N^* = N^{eq}$ for some $M, N \models T$

$T^{complete} \Rightarrow M \equiv N \stackrel{(4)}{\Rightarrow} M^{eq} \equiv N^{eq}$

(4) Induction on φ :

- φ : atomic
 - the connectives
- } easy

• $\varphi: \exists x_0 \varphi_0(x_0, x_1, \dots, x_k)$. By the ind. hyp. φ_0 is "equivalent" ^{in T^{eq}} to $\varphi_0(\bar{y}_0, \dots, \bar{y}_k)$
of sort S_{E_0} So: $T^{eq} \vdash \exists x_0 \varphi_0(x_0, F_{E(1)}(\bar{y}_1), \dots, F_{E(k)}(\bar{y}_k))$

||| (by axioms of T^{eq})

$$\exists \bar{y}_0 \varphi_0(F_{E(0)}(\bar{y}_0), \dots, F_{E(k)}(\bar{y}_k))$$

|||

$$\exists \bar{y}_0 \psi(\bar{y}_0, \dots, \bar{y}_k)$$

Proof of corollary (1) Goal: $M^{eq} \prec \mathcal{M}^{eq}$. Take $\bar{a}^{eq} \in M^{eq} \models \varphi(\bar{a}^{eq}), \bar{a} = \bar{b}/E$

by fact (4) $M^{eq} \models \varphi(\bar{a}^{eq}) \Leftrightarrow M \models \psi(\bar{b}) \Leftrightarrow M \models \psi(\bar{b}) \Leftrightarrow \mathcal{M} \models \psi(\bar{b})$

$\Leftrightarrow \mathcal{M}^{eq} \models \psi(\bar{b}) \Leftrightarrow \mathcal{M}^{eq} \models \varphi(\bar{a})$

Fact Assume $E \in ER(T^{eq})$. Then there is $\psi \in ER(T)$ on \mathcal{M}^n and 3

$\exists f: \mathcal{M}^n \xrightarrow{\text{onto}} \text{dom}(E)$ s.t. $\forall a, b \in \mathcal{M} \quad (\mathcal{M}^{eq} \models \psi(\bar{a}, b) \leftrightarrow f(\bar{a}) E f(b))$
 \uparrow
 0 -definable in T^{eq} . So: $\exists g: \mathcal{M} \xrightarrow{1-1 \text{ onto}} \mathcal{M}^{eq}_E$ in $(\mathcal{M}^{eq})^{eq}$ g : 0 -definable

Def Assume $X \subseteq \mathcal{M}^{eq}$ and $c \in \mathcal{M}^{eq}$. c is a name (code) of X if

$\forall f \in \text{Aut}(\mathcal{M}^{eq}) \quad (f[X] = X \Leftrightarrow f(c) = c)$

Remark (1) $X \subseteq \mathcal{M}^{eq} \xrightarrow{\text{def}} X$ has a name $c \in \mathcal{M}^{eq}$

(2) If c : name of X then $X = \psi(\mathcal{M}^{eq}, c)$ for some $\psi \in L^{eq}$

Proof (1) $1^\circ X \subseteq \mathcal{M} \xrightarrow{\text{def}} X = \psi(\mathcal{M}, \bar{a})$. $E_\psi \in ER(T)$,

$E_\psi(\bar{y}_0, \bar{y}_1) = \forall x (\psi(x, \bar{y}_0) \leftrightarrow \psi(x, \bar{y}_1))$

So for $\bar{a}_0, \bar{a}_1 \in \mathcal{M} \quad \models E_\psi(\bar{y}_0, \bar{y}_1) \Leftrightarrow \psi(\mathcal{M}, \bar{a}_0) = \psi(\mathcal{M}, \bar{a}_1)$

$c = \bar{a} / E_\psi = F_{E_\psi}(\bar{a}) \in \mathcal{M}_{E_\psi}$
 \uparrow
 good, name for X

$2^\circ X \subseteq \mathcal{M}^{eq} \xrightarrow{1^\circ}$ a name of X in $(\mathcal{M}^{eq})^{eq} \Rightarrow$ get $c' \in \mathcal{M}^{eq}$ a name of X
 $(\mathcal{M}^{eq})^{eq} = \mathcal{M}^{eq}$

(2) $X \subseteq \mathcal{M}^{eq}$, c : a name of X in \mathcal{M}^{eq} . By def. $\forall f \in \text{Aut}(\mathcal{M}^{eq})$
 $(f[X] = X \Leftrightarrow f(c) = c)$

hence X is invariant under $\text{Aut}(\mathcal{M}^{eq}/c)$

\Downarrow
 X definable over c

Example $X = \psi(\mathcal{M}, \bar{a})$, $c = \bar{a} / E_\psi$ a name of X

$x \in X \Leftrightarrow \exists \bar{y} (\psi(x, \bar{y}) \wedge F_{E_\psi}(\bar{y}) = c)$

$\delta(x, c)$
 defines X

From now on we work in \mathcal{M}^{eq}, T^{eq}
 \mathcal{M}^*

Definition Assume $X \subseteq \mathcal{M}$. X is definable almost over $A \subseteq \mathcal{M}$
definable

\Leftrightarrow the set $\{f[X] : f \in \text{Aut}(\mathcal{M}/A)\}$ is finite

Lemma X is definable almost over $A \Leftrightarrow X$ is definable over $\text{acl}^{eq}(A)$

Proof (\Leftarrow) $X = \varphi(\mathcal{M}, \bar{a})$ $f[X] = \varphi(\mathcal{M}, f(\bar{a}))$, $f \in \text{Aut}(\mathcal{M}/A)$
 $\bar{a} \in \text{acl}^{eq}(A)$ finitely many possibilities

(\Rightarrow) $X = \varphi(\mathcal{M}, \bar{b})$, $|\{f[X] : f \in \text{Aut}(\mathcal{M}/A)\}| < \aleph_0$. A type over A in $\bar{y}_1, \dots, \bar{y}_k$

$\{\varphi(\mathcal{M}, \bar{y}_i) \neq \varphi(\mathcal{M}, \bar{y}_j) \}_{1 \leq i \neq j \leq k+1} \cup \{\bar{y}_i \in \text{tp}(\bar{b}/A)\}_{1 \leq i \leq k+1}$

This type has to be inconsistent. So for some $\Theta(\bar{x}) \in \text{tp}(\bar{b}/A)$

$\Theta \cup \{\bigwedge_{1 \leq i \leq k+1} \Theta(\bar{y}_i)\}$ is inconsistent with T .

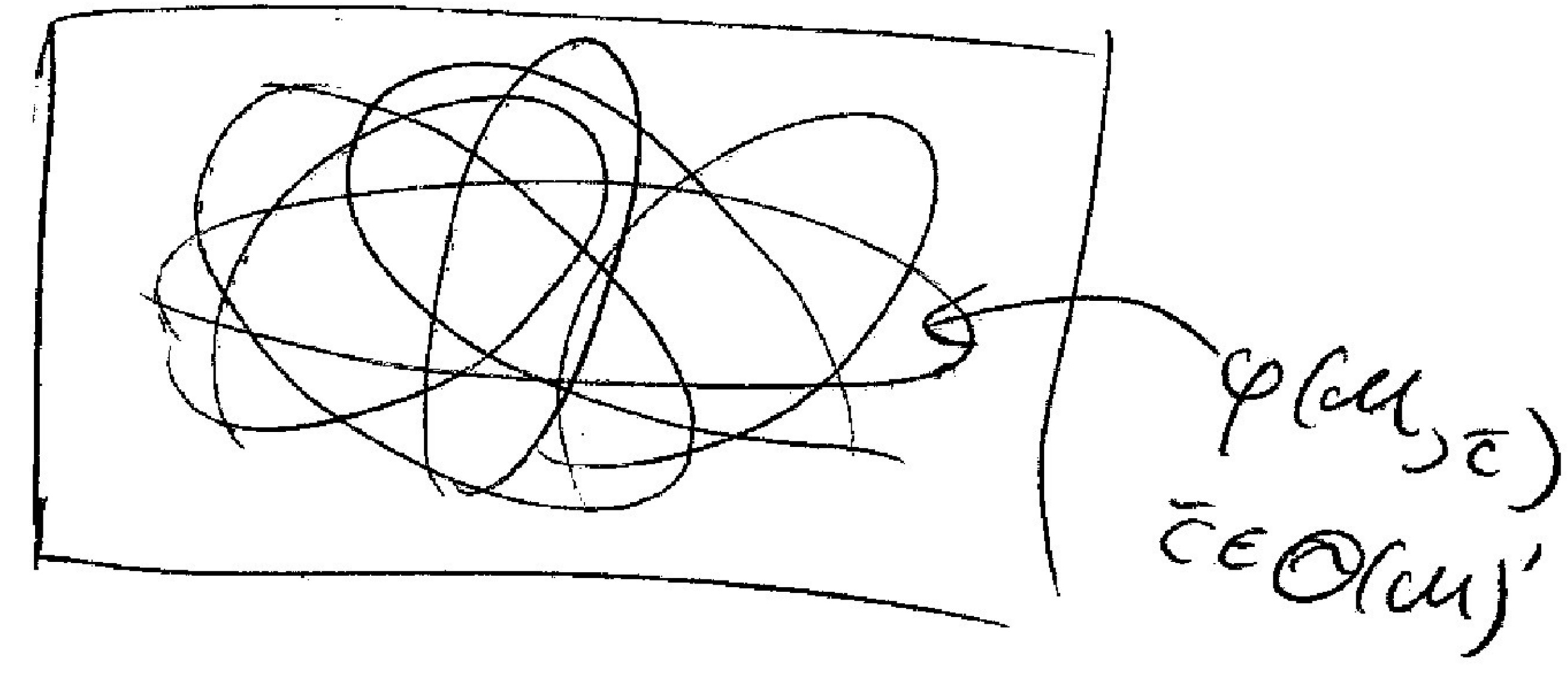
So: $|\{\varphi(\mathcal{M}, \bar{c}) : \bar{c} \in \Theta(\mathcal{M})\}| \leq k$

$E(x, x') : \forall \bar{c} \in \Theta(\mathcal{M}) (\varphi(x, \bar{c}) \leftrightarrow \varphi(x', \bar{c}))$
 $\Theta \in L(A)$

E defines in \mathcal{M}^{eq} an equiv. rel. with finitely many classes.

$E \in \underline{\text{FE}}(A)$

formulas over A defining in \mathcal{M} an equiv. rel. with finitely many classes



E -classes are atoms of the algebra of sets generated by Θ

$X =$ union of some E -classes E_1, \dots, E_n

\uparrow definable in $\mathcal{M} \Rightarrow$ have names $c_1, \dots, c_n \in \mathcal{M}^{eq}$
 $\Rightarrow c_1, \dots, c_n \in \text{acl}^{eq}(A)$

So X is definable over $\text{acl}^{eq}(A)$

FORKING OF TYPES

Example $p \in S(A)$, $A \subset B$, $RM(p) < \infty$

or
 $q \in S(B)$

1° $RM(p) = RM(q)$: " q is a free extension of p "

2° $RM(q) < RM(p)$: " q is a degenerated extension of p "
forking

Def (1) $\{\varphi(x, a_i) : i < \omega\}$ is k -contradictory ($k < \omega$) if every k -element subset is inconsistent (with T)

(2) $\varphi(x, a)$ divides over A if

$$\exists k < \omega \exists \{\varphi(x, a_i) : i < \omega\} : k\text{-contradictory} \wedge \bigwedge_{i < \omega} tp(a_i/A) = tp(a/A)$$

(3) A type $p(x)$ (over M) divides over A if $p \vdash \varphi(x, \bar{a})$ for some $\varphi(x, \bar{a})$ that divides over A

Remark In (2) can request $a_0 = a$ and $\langle a_i, i < \omega \rangle$ is indiscernible order

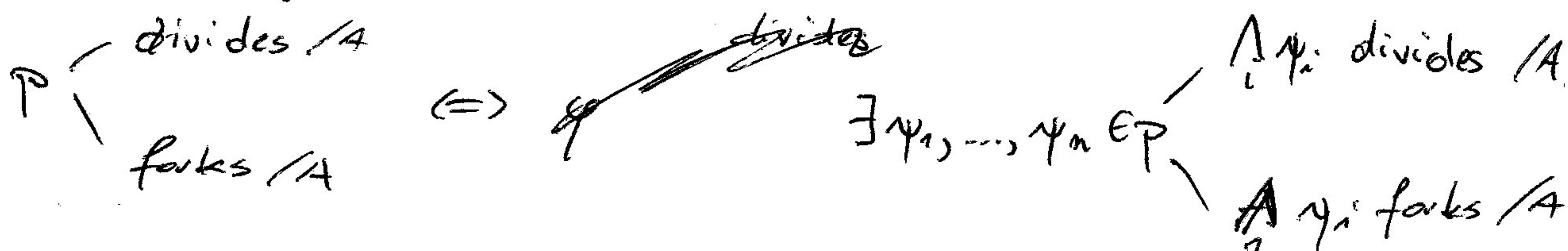
Def (1) $\varphi(x, \bar{a})$ forks over A if $\varphi(x, \bar{a}) \vdash \bigvee_{i \leq n} \varphi_i(x, \bar{b}_i)$
for some $\varphi_i(x, \bar{b}_i)$, $i \leq n$
divides over A

(2) A type $p(x)$ forks / A if $p(x) \vdash \underbrace{\varphi(x, \bar{a})}_{\text{forks / } A}$ for some $\varphi(x, \bar{a})$

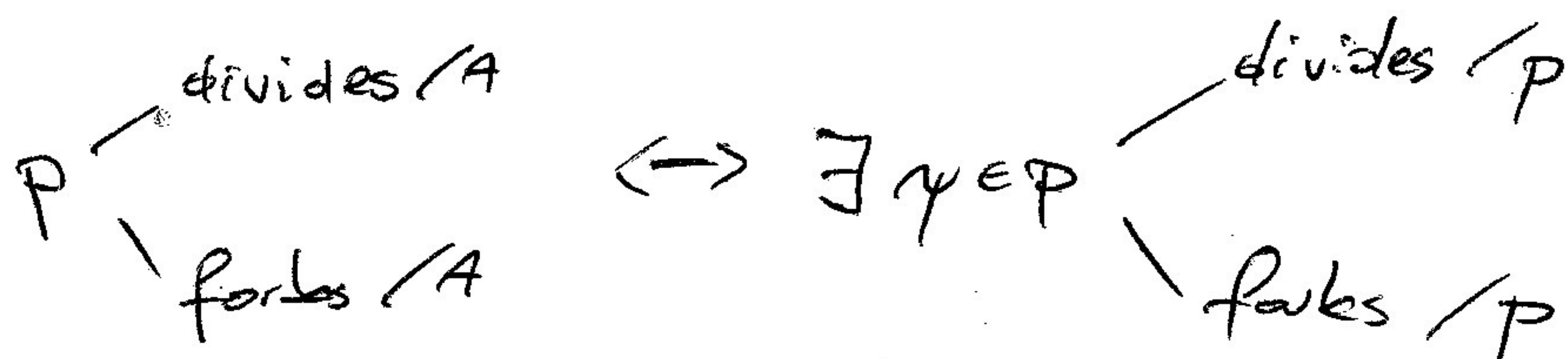
Properties (1) dividing \Rightarrow forking

(2) $\exists \varphi \vdash \psi$, then $\left\{ \begin{array}{l} \psi \text{ divides } /A \Rightarrow \varphi \text{ divides } /A \\ \psi \text{ forks } /A \rightarrow \varphi \text{ forks } /A \end{array} \right.$

(3) Assume p : a type. Then



(4) $p \in S(B)$, then



(5) p : a type / B and p dnf / $A \Rightarrow \exists q \in S(B)$, $q \supseteq p$, q dnf / A
 "does not fork"

Pf (5) Let $\Gamma = \{ \neg \psi(x, \bar{b}) : \psi \in L, \bar{b} \in B \text{ \& } \psi(x, \bar{b}) \text{ forks } /A \}$

$p \cup \Gamma$ is consistent type over B

If not then $p \cup \Gamma_0$ is inconsistent, thus $p \vdash \bigvee_{i=0}^k \psi_i(x, \bar{b}_i)$
 $\{\neg \psi_0, \dots, \neg \psi_m\}$

But each of ψ_i forks / A , thus $\bigvee_{i=0}^k \psi_i(x, \bar{b}_i)$ forks / A ,

thus p forks / A . \downarrow

Let $q(x) \in S(B)$. Then q does not fork / A .
 $p \cup \Gamma \stackrel{q}{\vdash}$

From now on: assume T : stable.

ORKING AND RANKS

Remark Assume $\varphi(x, \bar{b})$ forks / A . Then:

(1) $\exists \Delta_0 \subseteq L \forall \Delta_0 \subseteq \Delta \subseteq L \forall p \in S(A) R_{\Delta}(p \cup \{\varphi(x, \bar{b})\}) < R_{\Delta}(p)$
 finite finite

(2) $RM(p) < \infty$, then $RM(p \cup \{\varphi(x, b)\}) < RM(p)$ (2)

Proof (1,2) $\varphi \vdash \bigvee_{i=0}^k \varphi_i(x, b_i)$. Let $\Delta_0 = \{\varphi_i(x, y_i) : i \leq k\} \leftarrow \text{good}$.
divides A

Let $\Delta_0 \subseteq \Delta \subseteq L$ and assume $R_\Delta(p) < \infty$. Will show that

$R_\Delta(p \cup \{\varphi(x, b)\}) < R_\Delta(p)$. (a.c.) Suppose $R_\Delta(p \cup \{\varphi(x, b)\}) = R_\Delta(p)$.
 Then there's $q \in S(A \cup \{b, b_i : i \leq k\})$ $R_\Delta(q) = R_\Delta(p)$. Then

$q \supseteq \varphi(x, b) \vdash \bigvee_i \varphi_i(x, b_i)$, thus (wlog) $\varphi_0(x, b_0) \in q$,
 but p_0 divides A . By the def. of dividing we
 get $\{\varphi_0(x, b_0^i) : i < \omega\} = L$ -contradictory.

and $tp(b_0^i/A) = tp(b_0/A)$.

$R_\Delta(p \cup \{\varphi_0(x, b_0^i)\}) = R_\Delta(p)$ for $i < \omega$

extend to $q_i \in S(C)$, $R_\Delta(q_i) = R_\Delta(p)$
 where $C \supseteq A \cup \{b, b_0^i : i < \omega\}$

$q_i \upharpoonright \Delta \in S_\Delta(C)$, $\{q_i \upharpoonright \Delta : i < \omega\}$ is infinite (by L-contradictionality)
 and $R_\Delta(p \cup (q_i \upharpoonright \Delta)) = R_\Delta(p)$ for all i ~~of φ~~

(because there can be only finitely many
 such types) □

Corollary If $p \in S(A)$, then p dnf A .

~~Assume $\varphi(x, y)$ is~~

Desell Lemma Assume $\varphi(x, y) \in L$. Then $\exists m_\varphi < \omega \forall I$: infinite indiscernible $\forall a$ ($|p(I, a)| < m_\varphi$ or $\neg \varphi(I, a) < m_\varphi$)

Def. Let $I \subseteq \mathcal{M}$, $A \subseteq \mathcal{M}$.
infinite indiscernible \leftarrow permanent assumption

$A_{av}(I/A) = \{\varphi(x) \in L(A) : \varphi(I) \text{ infinite}\} \in S(A)$
 the average type of I over A cof. infinite in I

Lemma If $\varphi(x): a$ formula over \mathcal{M} , $a \in \text{acl}(A)$ and $\models \varphi(a)$, then $\varphi \text{ dnf } / A$ (3)

Def. I is based on A iff $\forall B \text{ Aut}(I/B) \text{ dnf } / A$.
 $(\Leftrightarrow \text{Aut}(I/\mathcal{M}) \text{ dnf } / A)$

Remark For $\alpha < \beta \in \text{Ord}$ let $a_\alpha \models \text{Aut}(I/A \upharpoonright a_{<\alpha})$. Then:

(1) $I \cup \{a_\alpha : \alpha < \beta\}$ is indiscernible

(2) $\{a_\alpha : \alpha < \beta\}$ is indiscernible $/ A$

Proof Hint: (a) If $f \in \text{Aut}(\mathcal{M}/I)$, $A \subseteq \mathcal{M}$, then $f(\text{Aut}(I/A)) = \text{Aut}(I/f(A))$

(b) If $a \notin I$, then $I \cup \{a\}$ is indiscernible $\Leftrightarrow a \models \text{Aut}(I/I)$

Lemma If $a \notin \text{acl}(A)$, then $\exists I$: indiscernible $/ A$, $a \in I$
based on A

Proof Choose $M \supseteq Aa$, where $\kappa = \max(|A|, \aleph_0)$.
 \uparrow
 κ^+ -saturated

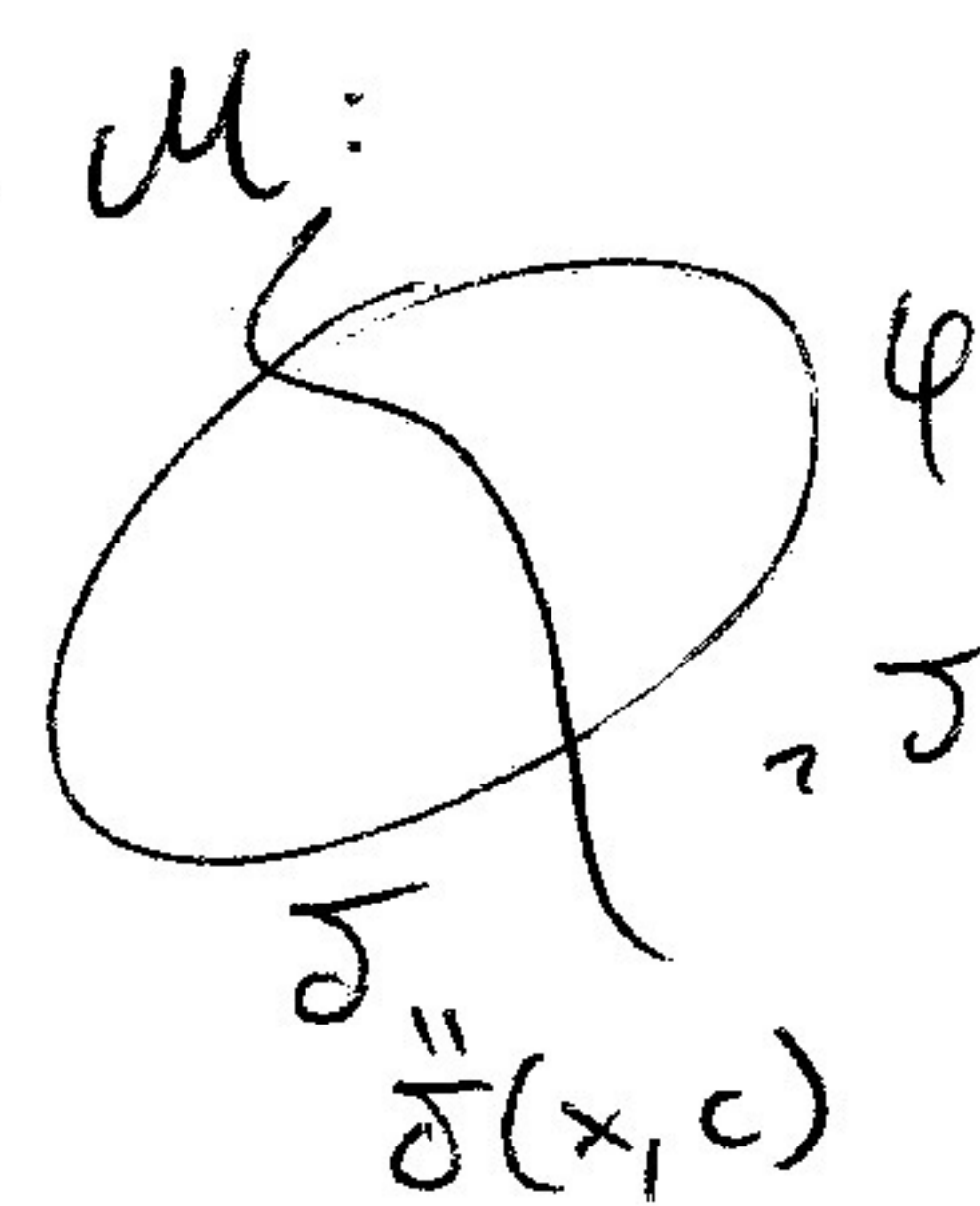
Let $p = \text{tp}(a/A) \subseteq q \in S(M)$. Then $\forall \Delta \subseteq L$ finite $\text{Mlt}_\Delta(q) = \Delta$.

(non forking)
 dnf $/ A$

Then $\exists B \subseteq M$, $|B| \leq \omega$, $|B \setminus A| \leq \omega$

$\forall \Delta \subseteq L$ finite $\left\{ \begin{array}{l} \text{Mlt}_\Delta(q) = \text{Mlt}_\Delta(q|_B) \\ R_\Delta(q) = R_\Delta(q|_B) \end{array} \right.$

(if $\Delta \neq \emptyset$: choose $\varphi(x) \in \Delta$ with $\omega > R_\Delta(q) = R_\Delta(\varphi)$ and $\text{Mlt}_\Delta(q) = \text{Mlt}_\Delta(\varphi)$)



$\delta: \Delta$ -formula
 $R_\Delta(\varphi \wedge \delta) = R_\Delta(\varphi \wedge \neg \delta) = R_\Delta(\varphi)$

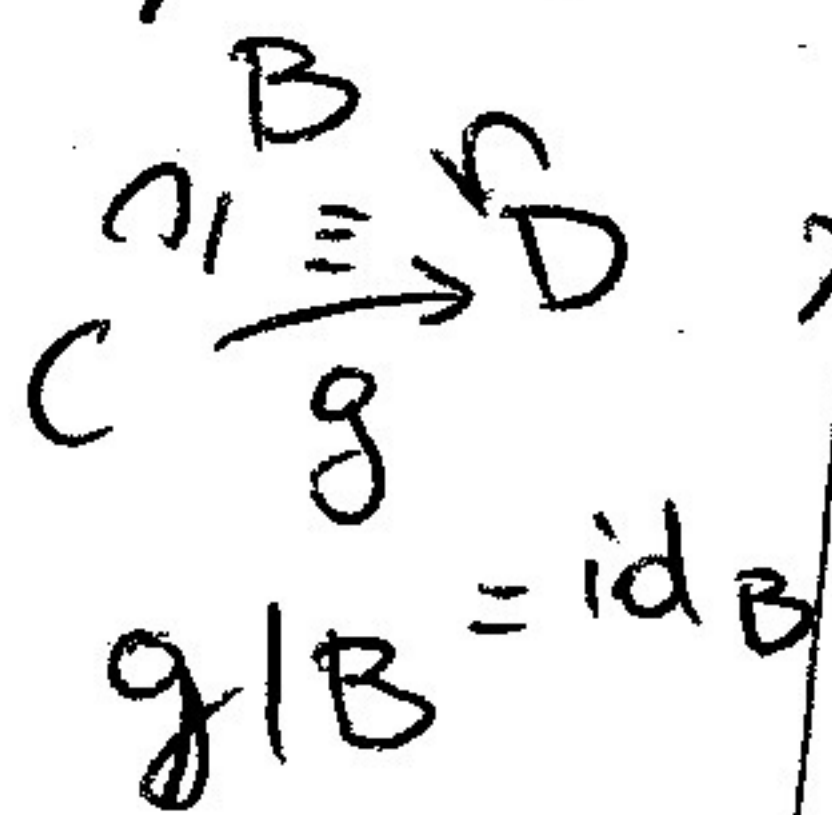
Let $a_n \models q|_B a_n$ for $n=0,1,\dots,\omega$

choose $c' \in M$ with $\text{tp}(c'/a) = \text{tp}(c/a)$

Claim $I = \{a_n : n < \omega\}$ is indiscernible $/ B$

Then $R_\Delta(\varphi \wedge \delta(x,c')) = R_\Delta(\varphi \wedge \neg \delta(x,c')) = R_\Delta(\varphi)$

[hint: as with Morley sequence]
 crucial point: if

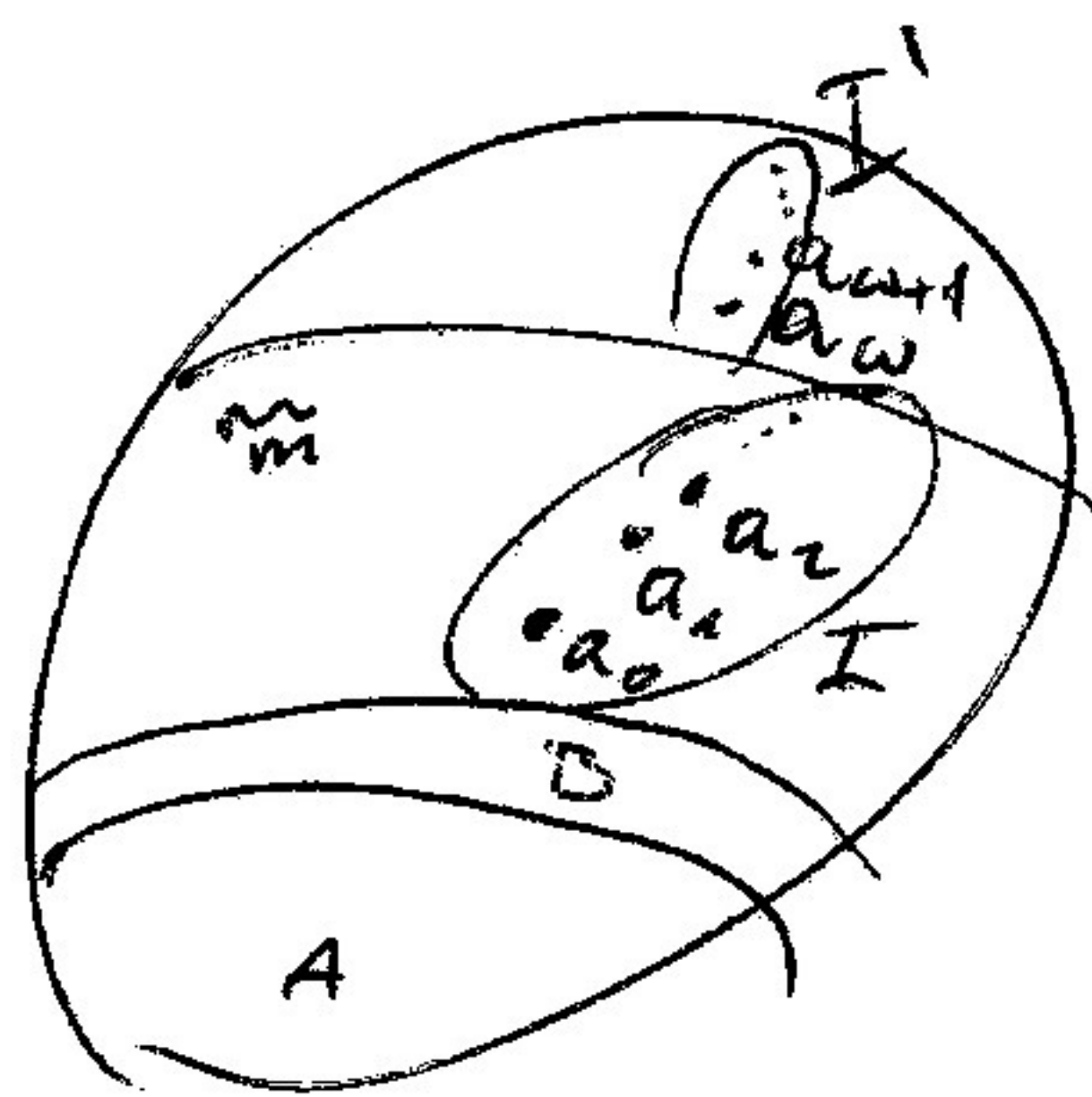


φ (or) φ

[then $g(q|_c) = q|_d$]

Claim $q = \text{Av}(I/M)$. ~~Let $C \in \mathcal{M}$ and $\varphi(x, c) \in \text{Av}(I/C)$~~ (4)

Pf. Let $\varphi(x, m) \in q$. Let $a_{\omega+k} \in \mathcal{B} \setminus \text{Ba}_{\omega+k}^m$ for all $k=0, 1, \dots < \omega$



$J = I \cup I'$: also indiscernible / \mathcal{B}

But $I' \in \varphi(J, m)$, thus infinite in J , thus cofinite in J , thus cofinite in I , thus $\varphi(x, m) \in \text{Av}(I/M)$ \square

Claim I is based on A .

Pf Let $C \in \mathcal{M}$ and $\varphi(x, c) \in \text{Av}(I/C)$. Want: $\varphi(x, c)$ dnf / A .

Let $c' \in M$ with $\text{tp}(c'/AI) = \text{tp}(c/AI)$. So $\varphi(x, c)$ dnf / $A \Leftrightarrow \varphi(x, c')$ dnf / A ,
 \uparrow
 κ^+ -saturated

$\varphi(x, c') \in \text{Av}(I/c') \subseteq \text{Av}(I/M) = q$ n.f. ext. of p

So $\varphi(x, c')$ dnf / A \square

Choose $f \in \text{Aut}(\mathcal{M}/A)$ with $f(a_0) = a$. So $a \in f[I] \leftarrow$ an A -indisc. set based on A .

Lemma \square

Lemma Assume $|B| > \kappa := (|A|+2)^{\kappa_0}$. Then $\exists I \subseteq B$
 \uparrow
 infinite, indiscernible / A

Proof For a type $p(x)$ let $\vec{R}(p) = \langle (R_\Delta(p), M_{\text{tp}_\Delta(p)}) : \Delta \subseteq L \text{ finite} \rangle$

$\forall p \in S(A) \exists A_0 \subseteq A$ c.f.d.e. $\vec{R}(p) = \vec{R}(p|_{A_0})$.

Let $B_0 \subseteq B$ c.f.d.e. Let $p \in S(A \cup B_0)$ where $B_0 \subseteq B$ s.t. c.f.d.e.

$p(B) = \{b \in B : b \models p\}$ has power $> \kappa$ and

$\vec{R}(p)$ is minimal possible (lexicographically) (over all B_0, p)

$\forall B_1 \subseteq B$ c.f.d.e. $\exists p' \in S(AB_1)$ ($\vec{R}(p') = \vec{R}(p)$ & $|p'(B)| > \kappa$) \square

If μ is a measure let $\mathcal{P} = \{p' \in S(AB_n) : p \subseteq p'\}$

$$p' \supseteq p \Rightarrow \vec{R}(p') \subseteq \vec{R}(p)$$

$$\exists p' \in \mathcal{P} \text{ with } |\mathcal{P}'(B)| > \aleph \text{ (as } p(B) = \bigcup_{\substack{p' \in \mathcal{P} \\ |P| \leq \aleph}} p'(B) \text{)}$$

$$\text{Then } \vec{R}(p') = \vec{R}(p)$$

Uniqueness of p' : $Mlt_{\Delta}(p') = Mlt_{\Delta}(p)$ for every $\Delta \in L$ finite

If there's other p'' like that then

$$p' \neq p'' \Rightarrow \exists \Delta \text{ formula } \Rightarrow Mlt_{\Delta}(p') < Mlt_{\Delta}(p)$$

Let $I = \{b_n : n < \omega\} \in B$ st. $\forall n \ p_n = tp(b_n / AB, b_{<n})$ is as in (*)

clearly I is good.

Lemma Assume $I = \{a_n : n < \omega\}$ indiscernible / A , based on A .

$$\text{Assume } \varphi(x, y) \in L \text{ and } n < \omega \text{ for } \varphi'(x, \bar{y}_{2n}) = \bigvee_{W \subseteq 2n} \bigwedge_{i \in W} \varphi(x, y_i)$$

Then for $n \geq n_{\varphi}$ $\varphi'(x, \bar{a}_{<n})$ is almost over A .

[that is, it is over $acl^{eq}(A)$]

\uparrow recall
 \downarrow
 $\{f(\varphi'(x, \bar{a}_{<n})) : f \in Aut(\mathcal{U}/A)\}$
is finite

Theorem (Shelah, finite equivalence relation thm)

Assume $p \neq q \in S(\mathcal{M})$ def / A . Then $\exists E \in FE(A) \ p(x) \cup q(y) \vdash \neg E(x, y)$

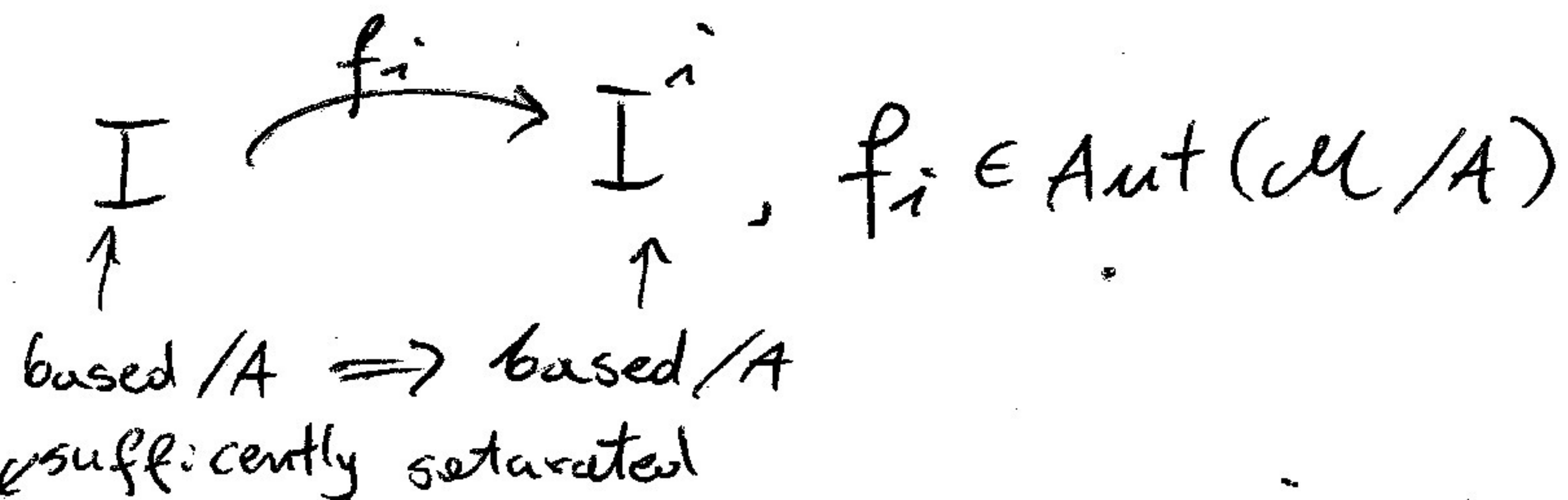
13.06.2022 Lemma $I = \{a_n : n < \omega\}$ indiscernible / A , based on A , $\varphi(x, y) \in L^1$

$\varphi'(x, \bar{y}_{2n}) = \bigvee_{W \subseteq 2n} \bigwedge_{i \in W} \varphi(x, y_i)$. Then for $n \geq n_\varphi$, $\varphi'(x, \bar{a}_{2n})$ is almost over A
 "
 $\langle y_1, \dots, y_{2n-1} \rangle$ $|W| \geq n$

Proof (A.a.) Let $\kappa = (|A|+2)^{\aleph_0}$. Let $\{\bar{a}_{2n}^i : i < (2^\kappa)^+\}$ s.t.

$tp(\bar{a}_{2n}^i / A) = tp(\bar{a}_{2n} / A)$ and $\{\varphi'(x, \bar{a}_{2n}^i) : i < (2^\kappa)^+\}$ are pairwise non-equivalent.

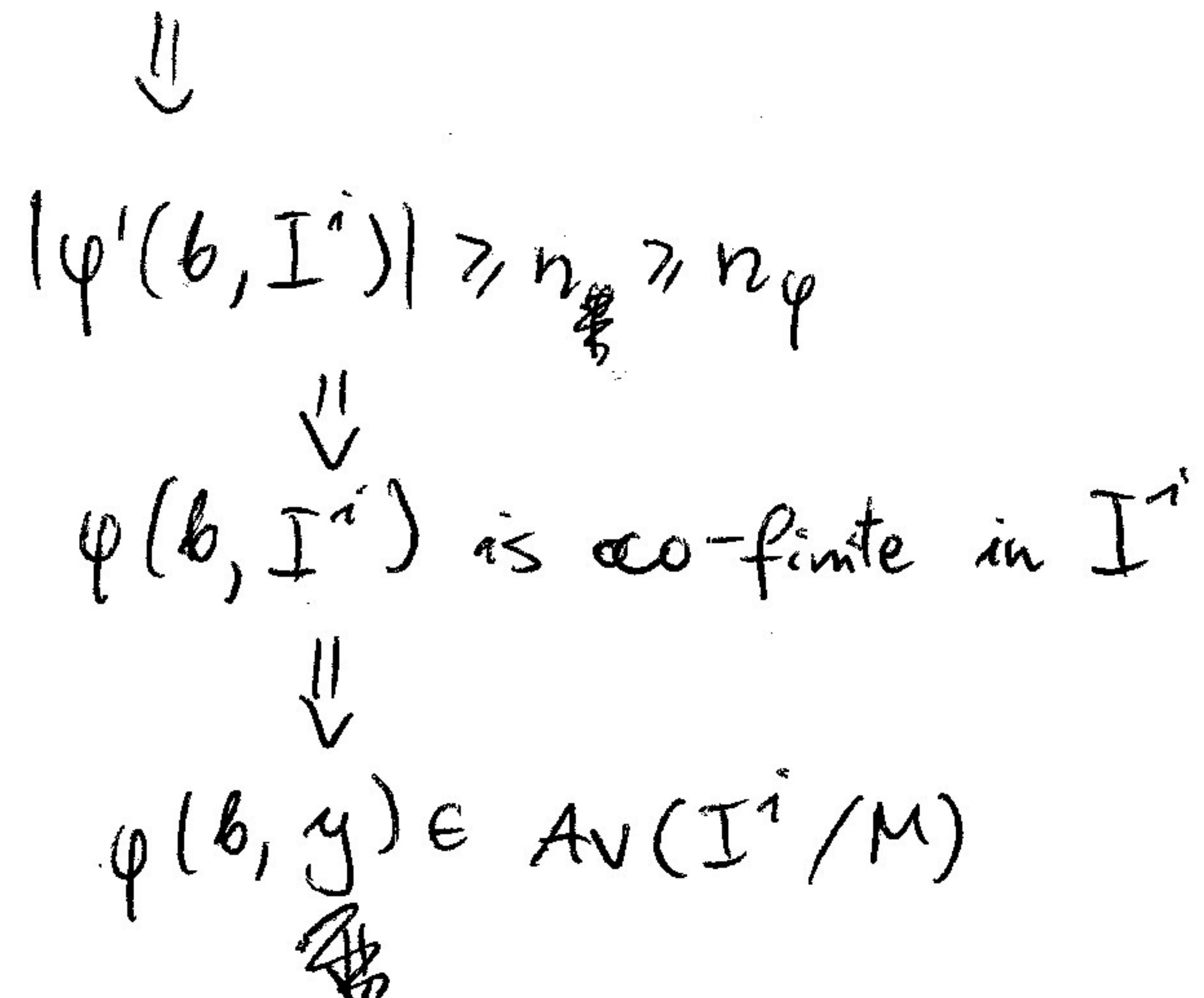
$$\bar{a}_{2n}^i \in I^i = \{a_n^i : i < \omega\}$$



Let $M \supseteq A \cup \bigcup_i I_i$. Let $p_i = \text{Av}(I^i / M)$, $i < (2^\kappa)^+$

Claim If $i \neq j < (2^\kappa)^+$, then $p_i \neq p_j$

Pf Let $b \in M$ s.t. $\underbrace{\varphi'(b, \bar{a}_{2n}^i)} \wedge \neg \varphi(b, \bar{a}_{2n}^j)$



Likewise $\neg \varphi(b, y) \in \text{Av}(I^j / M)$ □

Define \sim on M :

$$b \sim_0 b' \Leftrightarrow \exists y \text{ infinite, indiscernible } / A, \text{ both } \langle b y^n \rangle, \langle b' y^n \rangle \text{ are indiscernible } / A$$

Let \sim be the transitive closure of \sim_0 .

$$\text{So: } b \sim b' \Leftrightarrow \exists b = b_0, \dots, b_n = b' \wedge b_i \sim_0 b_{i+1}$$

Subclaim $b \sim b', p \in S(M)$ $\text{dnf}/A \Rightarrow \psi(x, b) \in p \Leftrightarrow \psi(x, b') \in p$ 2

Pf May assume $b \sim_0 b'$ (A.c.) ^{finiteness $\psi(x, y) \in L$} Suppose $\psi(x, b) \in p \wedge \neg \psi(x, b') \in p$.
 Let $c \models p$. $\underbrace{\psi(c, y) \in \text{Av}(J/Ac)}_{\text{consider this}}$ or $\neg \psi(c, y) \in \text{Av}(J/Ac)$

Then there is $b^* \in J$ s.t. $\models \psi(c, b^*)$.

Look at $J' = \{b^*\} \cup J$. $b^* \neq b' \in J'$: infinite indiscernible / A
 and $\underbrace{\psi(x, b^*) \wedge \neg \psi(x, b')} \in p$, so p forks / A ⊥

but this forks / A (see problem from list 3)

Cases:

1° \sim has $\leq \kappa$ classes on M .

Let $B \subseteq M$, $|B| \leq \kappa$, B contains representatives of all \sim -classes,
 A^M

by previous claim, for $i \neq j$ $p_i|B \neq p_j|B$. So $|S(B)| \geq (2^\kappa)^+$,
 but $|S(B)| \leq 2^\kappa$ ⊥

2° \sim has $> \kappa$ classes on M .

Let $\{b_\alpha : \alpha < \kappa^+\} \subseteq M$, pairwise ~~non-~~ ^{non-} ~~non-~~ ^{non-} equivalent

By the previous lemma: $\exists J \subseteq \{b_\alpha : \alpha < \kappa^+\}$
 infinite, indiscernible / A

$b_\alpha, b_\beta \in J$ for some $\alpha < \beta < \kappa^+$

\Downarrow
 $b_\alpha \sim_0 b_\beta = b_\alpha \sim b_\beta$ ⊥

Thm (Shelah) $p \neq q \in S(\mathcal{U})$, both dnf/A

\Downarrow
 $\exists E \in FE(A) \ p(x) \cup q(y) \vdash \neg \exists (x, y)$

Proof $\psi(x, b) \in p, \neg \psi(x, b) \in q$. Let $I \subseteq \mathcal{U}$ indiscernible / A, based / A
 with $b \in I = \{a_n : n < \omega\}$

This implies (by the claim):

$$\forall c \in I (\varphi(x, c) \in p \ \& \ \neg \varphi(x, c) \in q)$$

$$[\text{or else } \varphi(x, b) \wedge \neg \varphi(x, c), \neg \varphi(x, b) \wedge \varphi(x, c)]$$

fork for every $c \in I$

Let $m \succ m_\varphi$. $\varphi'(x, \bar{a}_{\leq n})$ is almost / A, $\neg \varphi'(x, \bar{a}_{\leq n}) \in q$

So $\exists E \in FE(A)$ s.t. $\varphi'(\mathcal{M}, \bar{a}_{\leq n}) =$ a union of E -classes.

$$p(x) \cup q(y) \vdash \varphi'(x, \bar{a}_{\leq n}) \wedge \neg \varphi'(y, \bar{a}_{\leq n}) \vdash \neg E(x, y) \quad \square$$

Def $p \in S(A)$ is stationary if $\forall B \supseteq A \exists! q \in S(B)$

Corollary (1) $p \in S(\text{acl}^{eq}(A)) \Rightarrow p$: stationary $p \upharpoonright^{\text{nf}}$

(2) $p \in S(M) \Rightarrow p$: stationary

Proof (1) Let $B \supseteq \text{acl}^{eq}(A)$. Suppose $p \upharpoonright^{\text{nf}} S(B)$.

Then $q_i \subseteq q_i' \in S(\mathcal{M})$
 \nwarrow still dnf / $\text{acl}^{eq}(A)$

$$q_1' \neq q_2' \Rightarrow \exists E \in FE(\text{acl}^{eq}(A)) \ q_1'(x) \cup q_2'(y) \vdash \neg E(x, y) \quad (t)$$

Let a_1, \dots, a_k : names of E -classes. $a_1, \dots, a_k = \text{acl}^{eq}(\text{acl}^{eq}(A)) = \text{acl}^{eq}(A)$
 \uparrow
 \mathcal{M}

$$"x/E = a_i" \in L(\text{acl}^{eq}(A)), \ i = 1, \dots, k$$

for some $1 \leq i < j \leq k$ $"x/E = a_i" \in q_1'$, $"x/E = a_j" \in q_2'$

By (t): $i \neq j$. But $q_1' \upharpoonright \text{acl}^{eq}(A) = p = q_2' \upharpoonright \text{acl}^{eq}(A)$, so $i = j$. \downarrow

(2) $M = \text{acl}^{eq}(M)$

\square

Corollary (1) $A \subseteq B$, $p \in S(B)$. Then $p \text{ dnf } A \Leftrightarrow \forall \Delta \subseteq L \text{ finite } R_\Delta(p) = R_\Delta(p|A)$

(2) $p \in S(A)$ is stationary $\Leftrightarrow \forall \Delta \in L \text{ finite } \text{Mlt}_\Delta(p) = 1$

(3) $p \in S(A)$ is stationary $\Rightarrow (\forall \Delta \in L \text{ finite } \forall B \supseteq A \ R_{\Delta,2}(p|B) = R_{\Delta,2}(p))$
 and $p|B$ is definable over A (unique extension of p is $S(B)$)

Proof \Leftarrow clear. \Rightarrow : $p \subseteq p' \in S(\text{acl}^{\text{eq}}(B) \cup \text{acl}^{\text{eq}}(A))$ ~~s.t. $p' \text{ dnf } A$~~

Then (*) $\forall \Delta \subseteq L \text{ finite } R_\Delta(p') = R_\Delta(p)$ (exercise)

$$R_\Delta(p'|\text{acl}^{\text{eq}}(A)) = R_\Delta(p|A)$$

~~By~~ We can assume $p' \text{ dnf } A$, hence $p' \text{ dnf } \text{acl}^{\text{eq}}(A)$

Therefore we may assume $A = \text{acl}^{\text{eq}}(A)$, $B = \text{acl}^{\text{eq}}(B)$

Shall prove $\forall \Delta \subseteq L \text{ finite } R_\Delta(p') = R_\Delta(p'|\text{acl}^{\text{eq}}(A))$

Let $q = p|_A$. $A = \text{acl}^{\text{eq}}(A)$, so q is stationary and

$$q \in p \in S(B) = S(\text{acl}^{\text{eq}}(B)).$$

Let Δ : finite. Want: $R_\Delta(p) = R_\Delta(q)$
 \leftarrow
obvious

\geq : Let $k = R_\Delta(q)$, $k \in \mathbb{N}$, wlog $\Delta = \{1, \dots, k\}$, closed under

:
□

Comment: similarly one can show that in this situation (i.e. when $A = \text{acl}^{\text{eq}}(A)$, $B = \text{acl}^{\text{eq}}(B)$) $\text{Mlt}_\Delta(p) = \text{Mlt}_\Delta(q)$

Pf (3) As in (1) ("lifting") we show that $R_{\Delta,2}(p|B) = R_{\Delta,2}(p)$
 for every $\Delta \subseteq L$ (but easier, binary tree)
 \leftarrow
finite

By sketch proof of definability lemma $p|B$ is definable A □

Pf (2) \Leftarrow : Let $B \supseteq A$. Suppose

$$S(B) \ni q \supseteq \text{nf } p \in S(A)$$

$$S(B) \ni q' \supseteq$$

By (1) $R_\Delta(q) = R_\Delta(q') = R_\Delta(p)$ for every $\Delta \subseteq L$ finite

Let $\delta(x, \bar{c}) \in q, \neg \delta(x, \bar{c}) \in q'$. Let $\Delta = \{\delta(x, \bar{c})\} \subseteq L$

$$R_\Delta(q) = R_\Delta(q') = R_\Delta(p) \Rightarrow \text{Mlt}_\Delta(p) > 1 \quad \Downarrow$$

\Rightarrow : Let $q \notin p \in S(A)$: stationary.

$$\text{nf } q \in S(\mathcal{M}), \text{ so } \text{Mlt}_\Delta(q) = 1$$

by the comment $\text{Mlt}_\Delta(q) = \text{Mlt}_\Delta(p)$

Corollary Assume $p \in S(A), \text{RM}(p) < \infty, p \subseteq q \in S(B)$. Then

$$q \text{ dnf } /A \Leftrightarrow \text{RM}(p) = \text{RM}(q)$$

Corollary Assume $p \in S(\mathcal{M})$.

$$(1) \text{RM}(p) < \infty \Rightarrow \text{Mlt}_{\text{RM}}(p) = 1$$

$$(2) \forall \Delta \subseteq L \text{ finite } \text{Mlt}_\Delta(p) = 1$$

Corollary $p \in S(B) \text{ or } A$. Then $p \text{ dnf } /A \Leftrightarrow p$ is definable / $\text{ed}^{\text{eq}}(A)$

FORKING INDEPENDENCE

$A, B, C \subseteq \mathcal{M}, a \in \mathcal{M}$

Def $a \perp_C B$ [a is independent from B over C] if $\text{tp}(a/B \cup C)$

Def $A \perp_C B$ if $\forall \bar{a} \subseteq A \bar{a} \perp_C B$ dnf / C

\perp : has properties