Preliminaries

T: a complete consistent theory, in language L (countable) with infinite models.

that is, $T = Th(M)$, M: L-structure
infinite

L denotes also the set of formulas of language L

$M = (|M|; \dots)$, but M also denotes $|M|$ (for brevity)
$\uparrow$
$\emptyset \neq$ universe of M

usually we omit $|\cdot|$ in $|M|$.

$M \supseteq A$: a set of parameters.

$L_n(A) = \{\varphi(x_1, \dots, x_n, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in L, \bar{a} \subseteq A\}$

$L(A) = \bigcup_n L_n(A)$, also $L(A)$: language L extended by names for elements of A.

$L_n(A)$: Lindenbaum algebra.

[formally: on $L_n(A)$: $\varphi \sim \psi \Leftrightarrow T(A) \vdash \varphi \leftrightarrow \psi$
$\Leftrightarrow M \models \varphi \leftrightarrow \psi$

here: $\underbrace{T(A)}$ $= Th(M, a)_{a \in A}$
a complete theory in language $L(A)$.

$L_n(A)/_\sim$: a Boolean algebra

(Lindenbaum algebra)

$[\varphi]_\sim \wedge [\psi]_\sim = [\varphi \wedge \psi]_\sim$ etc.

shortly : $L_n(A)$ denotes also $L_n(A)/_\sim$.

$S_n(A) = \{$complete n-types over A, in $M\}$.
in variables $x_1, \dots, x_n$

consistent n-type over A $\rightsquigarrow$ proper filter in $L_n(A)$

An n-type $p(\bar{x})$ over A is <u>complete</u> if

$\cap|$

$L_n(A)$

- $p(\bar{x})$: consistent type
- $\forall \varphi(\bar{x}) \in L_n(A)$ $(\varphi(\bar{x}) \in p$ or $(\neg\varphi(\bar{x})) \in p)$

$S(A) := S_1(A)$
(default)

$S_n(A)$: topological space :

for $\varphi(\bar{x}) \in L_n(A)$

$$[\varphi] = \{p \in S_n(A) : \varphi \in p\}$$

basic open set [clopen]

$\|$

closed and open

$S_n(A)$: compact Hausdorff space, 0-dimensional
(i.e. basis of clopen sets)

complete n-types/A $\leftrightsquigarrow$ ultrafilters in $L_n(A)$

So $S_n(A) = S(L_n(A))$, the Stone space of ultrafilters in $L_n(A)$

- the ~~topo~~ topology on $S_n(A)$ = the Stone space topology.

For $p(\bar{x}) \in S_n(A)$

$$p(M) = \{\bar{a} \in M^n : \underbrace{\bar{a} \overset{\text{realizes } p}{\text{satisfies}} p}_{\bar{a} \models p \text{, i.e. } M \models \varphi(\bar{a}) \text{ for every } \varphi(\bar{x}) \in p(\bar{x})}\}$$

- The same notation for arbitrary type (also incomplete)
- A formula $\varphi(\bar{x}) \in L(M)$: a special case of a type $\{\varphi(\bar{x})\}$.

  $\varphi(M) = \ldots$

- When $p \in S_n(A)$, $\bar{a} \subseteq M$ and $\bar{a} \models p$, then

  $$p = tp^M(\bar{a}/A) = tp(\bar{a}/A) = \{\varphi(\bar{x}) \in L_n(A) : M \models \varphi(\bar{a})\}.$$

Example Assume $p(\bar{x})$: a consistent type over M.

Then $\exists N \succeq M$ $\underbrace{p \text{ is realized in } N}_{\text{i.e. } p(N) \neq \emptyset.}$

From now on "a type" means "a consistent type".

Def A type $p(\bar{x})$ over A is isolated, if:

$\exists\, \varphi(\bar{x}) \in L_n(A)$ ① $\varphi(\bar{x})$ is consistent (with T), i.e. $\varphi(M) \neq \emptyset$ $(\Leftrightarrow T(A) \vdash \exists \bar{x}\, \varphi(\bar{x}))$

(2) $\cancel{\varphi(M)}$

symbolically: $\varphi(\bar{x}) \vdash p(\bar{x})$ $\longrightarrow$ $\forall \psi(\bar{x}) \in p(\bar{x})$ $\quad \varphi(M) \subseteq \psi(M)$

$$\Updownarrow$$

$$T(A) \vdash \varphi(\bar{x}) \rightarrow \psi(\bar{x})$$

- When $p(\bar{x})$: a complete type over A, then:

  $p(\bar{x})$ is isolated $\Leftrightarrow$ p is isolated in $S_n(A)$ in the topological sense (i.e. $\{p\}$ is open)

Tarski–Vaught test

Assume $A \subseteq M$. Then $A = |N|$ for some $N \prec M$ iff

$$\forall \varphi(x) \in L_1(A)\, [\varphi(M) \neq \emptyset \Rightarrow \varphi(M) \cap A \neq \emptyset]$$

Construction of an elementary submodel of M containing A:

- $A_n \subseteq M$, $n < \omega$, increasing chain of sets

recursive construction:

$A_0 = A$

$A_n \subseteq A_{n+1} \subseteq M$ such that $\forall \psi(x) \in L_1(A_n)$

$$[\psi(M) \neq \emptyset \Rightarrow \psi(M) \cap A_{n+1} \neq \emptyset]$$

$A_\infty = \bigcup_{n<\omega} A_n$ satisfies TV-test.

Omitting types theorem

Assume $p_n(\bar{x}_n)$, $n<\omega$: a family of non-isolated types in theory T, over $\emptyset$. Then:

$(\exists M \models T)$ M omits every p_n [i.e. $p_n(M)=\emptyset$]

Assume $\underset{\overset{\cup|}{A}}{M}, N \models T$

Def. $f: A \to N$ is *elementary* ($f: A \xrightarrow{\equiv} N$) if:

$$\forall \bar{a} \subseteq A \ \forall \varphi(\bar{x}) \in L \ \left(M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a})) \right)$$

$(\Leftrightarrow tp^M(\bar{a}) = tp^N(f(\bar{a})))$

Elementary diagram of $A \subseteq M$:

$D_e(A) = T(A) = Th(M,a)_{a\in A}$

Remark $f: A \to N$ is elementary $\Leftrightarrow (N, f(a))_{a\in A} \models T(A)$

Atomic diagram of $A \subseteq M$:

$D_{at}(A) = \{\varphi \in D_{el}(A) : \varphi$ is a quantifier free sentence$\}$

$= \{\varphi(\bar{a}) \in L(A) : M \models \varphi(\bar{a})$ and $\varphi(\bar{a})$: q.f.-sentence$\}$

Remark $f: M \to N$ is a monomorphism (i.e: $f: M \xrightarrow{\cong} f(M) \underset{\uparrow \text{substructure}}{\subseteq} N$)

$\Leftrightarrow (N, f(a))_{a\in M} \models D_{at}(M)$.

Here always $f: M \longrightarrow N$ denotes a <u>monomorphism</u>.

$M \subseteq N$: M is a submodel (substructure) of N

$M \prec N$: M is an elementary submodel of N, i.e:

$$M \subseteq N \text{ and } id_M : M \xrightarrow{\equiv} N$$

<u>Remark</u> Assume $M \prec N$, $A \subseteq M$.

(1) Assume $p(\bar{x}) \subseteq L_n(A)$. Then

$p(\bar{x})$ is a consistent type in $M \Leftrightarrow p(\bar{x})$ is a consistent type in N

(2) Assume $A \subseteq B \subseteq M$

- If $p(\bar{x})$: a type over B, then $\underbrace{p\restriction_A \overset{def}{=} p(\bar{x}) \cap L(A)}_{\text{a type over } A}$
- Let $r: S_n(B) \longrightarrow S_n(A)$, $r(p) \overset{def}{=} p\restriction_A$.

  Then r: continuous and "onto".

(3) If $p(\bar{x})$: a type over A, then $\exists q(\bar{x}) \in S_n(A)$ $p(\bar{x}) \subseteq q(\bar{x})$

Saturation, universality, (strong) homogeneity.

Let $\varkappa \in CN$, $\varkappa \geq \aleph_0$.

<u>Def.</u> (1) M $\varkappa$-<u>saturated</u> (nasycony) if $\forall A \subseteq M_{|A| < \varkappa}$ $\forall p \in S_1(A)$ $p(M) \neq \emptyset$

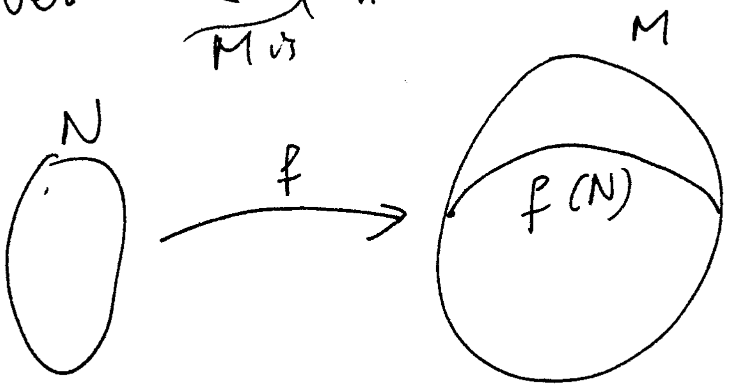
M is <u>saturated</u> if M is $\|M\|$-saturated

(2) M is $\varkappa$-universal if $\forall N \equiv M$ $(\|N\| \leq \varkappa \Rightarrow \exists f: N \xrightarrow{\equiv} M)$

$\uparrow$ elementarily equivalent

i.e. $Th(N) = Th(M)$

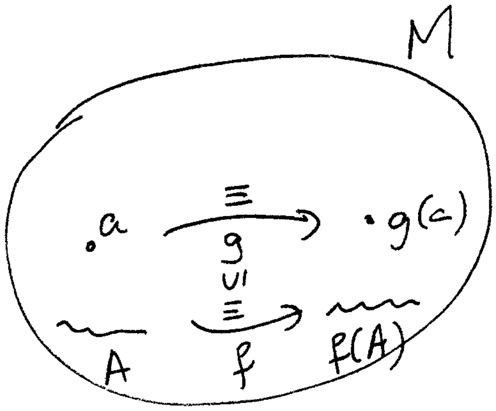
M: universal $\underset{\text{M is}}{\Longleftrightarrow}$ $\|M\|$-universal



$f : N \xrightarrow{\overset{\sim}{\equiv}} f(N) \prec M$

(3) M : κ-homogeneous if $\underset{|A|<\kappa}{\forall A \subseteq M}\ \forall a \in M\ \forall f : A \xrightarrow{\equiv} M$
$\exists \overset{\cap\!|}{g} : A \cup \{a\} \xrightarrow{\equiv} M$

homogeneous = $\|M\|$-homogeneous.



4. M strongly κ-homogeneous if $\underset{|A|<\kappa}{\forall A \subseteq M}\ \forall f : A \xrightarrow{\equiv} M$
$\exists \overset{\cap\!|}{g} : M \xrightarrow{\overset{\sim}{\equiv}} M$

strongly homogeneous =: strongly $\|M\|$-homogeneous.

5. M is κ-compact if ($\forall$ 1-type $p(x)$ over M)
$(|p| < \kappa \Rightarrow p(M) \neq \emptyset)$

Elementary chains of structures

Def $\langle M_\alpha : \alpha < \mu \rangle$, $\mu \in \mathrm{Ord}$, : an ~~e~~ elementary chain of structures if $(\forall \alpha < \beta < \mu)\, M_\alpha \prec M_\beta$.

Union of chain (when $\mu \in \mathrm{Lim}$)

$$M_\mu = \bigcup_{\alpha<\mu} M_\alpha \quad :$$

- $|M_\mu| := \overset{\nearrow}{\bigcup_{\alpha<\mu}} |M_\alpha|$
- $c \in L$ constant symbol $\quad c^{M_\mu} = c^{M_\alpha}$ for $\alpha < \mu$
- P : relation symbol

$$P^{M_\mu}(\underset{\underset{|M_\mu|}{\cap}}{a_1, \dots, a_n}) \iff M_\alpha \models P(a_1, \dots, a_n) \text{ for } \alpha < \mu$$

sufficiently large
[so that $\bar{a} \subseteq M_\alpha$]

- $f^{M_\mu}(\bar{a}) = b \iff M_\alpha \models f(\bar{a}) = b$ for $\alpha < \mu$ sufficiently large

Fact (Tarski) $M_\alpha \prec M_\mu$ for all $\alpha < \mu$.

Proof (1) $M_\alpha \subseteq M_\mu$ (substructure) : exercise

(2) $\forall \varphi(\bar{x}) \in L \ \forall \alpha < \mu \ \forall \bar{a} \subseteq M_\alpha \ (M_\alpha \models \varphi(\bar{a}) \iff M_\mu \models \varphi(\bar{a}))$

(a) φ atomic : $M_\alpha \subseteq M_\mu$ ✓

(b) $\varphi = \psi_1 \wedge \psi_2$, $\varphi = \neg \psi$: easy

(c) $\varphi(\bar{x}) = \exists y \, \psi(\bar{x}, y)$

$M_\alpha \models \varphi(\bar{a}) \Rightarrow M_\alpha \models \psi(\bar{a}, b)$ for some $b \in M_\alpha$

$\Downarrow$ ind. assumption for ψ

$M_\mu \models \psi(\bar{a}, b)$

$\Downarrow$

$M_\mu \models \varphi(\bar{a})$

$$\begin{array}{ccc} M_\mu \models \varphi(\bar a) \Longrightarrow & M_\mu \models \psi(\bar a,b) \text{ for some } & b\in M_\mu \\ \| & \Big\Downarrow \text{ ind. assumption} & \Downarrow \\ \exists y\,\psi(\bar a,y) & & b\in M_\beta \text{ for some } \\ & M_\beta\models\psi(\bar a,b) & \alpha\le\beta<\mu \\ & \Downarrow & \\ & M_\beta \models \varphi(\bar a) & \\ & \Downarrow\ M_\alpha \prec M_\beta & \\ & M_\alpha\models\varphi(\bar a) & \end{array}$$

Elementary directed systems of structures:

Let $(I, \leq)$: a directed set, i.e:

(1) $\leq$: partial order on I

(2) $(\forall a,b\in I)(\exists c\in I)(a\leq c \wedge b\leq c)$

Example J : a set $\rightsquigarrow$ $([J]^{<\omega}, \subseteq)$: directed set.

Directed system of structures:

- $\mathcal{M} = (M_i, f_{ij})_{i\leq j\in I}$

$$\overset{\uparrow}{\text{connecting functions}}\quad f_{ij} : M_i \longrightarrow M_j\ ,\ f_{ii} = id_{M_i}\ \text{such that}$$

$$(\forall\, i\leq j\leq t \in I)\ f_{it} = f_{jt}\circ f_{ij}$$

(compatibility)

$$\begin{tikzcd} & M_t \\ M_i \arrow[ur, "f_{it}"] \arrow[r, "f_{ij}"'] & M_j \arrow[u, "f_{jt}"'] \arrow[ul, phantom, "\#"] \end{tikzcd}$$

- System $\mathcal{M}$ is elementary if all f_{ij} are elementary.

Example Elementary chain $(M_\alpha)_{\alpha<\mu}$

$\mathcal{M} = (M_\alpha, f_{\alpha\beta})_{\alpha\le\beta<\mu}$ $f_{\alpha\beta} = id_{M_\alpha} : M_\alpha \xrightarrow{\equiv} M_\beta$
elementary directed system of structures

Direct limit of a directed system $\mathcal{M}$: $M_\infty = \varinjlim \mathcal{M}$.

$(M_\infty, f_{i\infty})_{i\in I}$, where $f_{i\infty} : M_i \longrightarrow M_\infty$ such that

(1) $\forall i \le j \in I$ $f_{i\infty} = f_{j\infty} \circ f_{ij}$ [compatible with connecting functions]

$$\begin{tikzcd}
M_j \arrow[rd, "f_{j\infty}"] & \\
{} \arrow[r, phantom, "\#"] & M_\infty \\
M_i \arrow[uu, "f_{ij}"] \arrow[ru, "f_{i\infty}"'] &
\end{tikzcd}$$

(2) $(\forall (M'_\infty, f'_{i\infty})_{i\in I}$ satisfying (1)$)$ $\exists! f : M_\infty \to M'_\infty$
(universality) $(\forall i \in I)$ $f'_{i\infty} = f \circ f_{i\infty}$

$$\begin{tikzcd}
& M_\infty \arrow[dd, "f"] \\
M_i \arrow[ru, "f_{i\infty}"] \arrow[rd, "f'_{i\infty}"'] \arrow[r, phantom, "\#"] & {} \\
& M'_\infty
\end{tikzcd}$$

Fact M_∞ exists (and is unique up to $\cong$).
If $\mathcal{M}$ is elementary, then $f_{i,\infty} : M_i \xrightarrow{\equiv} M_\infty$.

Proof 1. Construction of M_∞:

$S := \dot{\bigcup}_{i\in I} |M_i|$: formally disjoint union.

$\sim$ on S : an equivalence relation

$\overset{\overset{M_i}{\Downarrow}}{x} \sim \overset{\overset{M_j}{\Downarrow}}{y} \overset{\text{def}}{\iff} f_{it}(x)=f_{jt}(y)$ for some (≡ every) $t \geqslant i,j$

exercise: $\sim$ is transitive.

$|M_\infty| := S/\sim$

- $\sim\restriction_{|M_i|}$: the equality (because f_{ij} : 1-1 (monomorphism)
- $f_{i\infty}(\underset{\underset{M_i}{\Uparrow}}{x}) = x/\sim$, $f_{i\infty} : |M_i| \xrightarrow{1-1} |M_\infty|$.

L-structure on $|M_\infty|$:

- $c^{M_\infty} = c^{M_i}/\sim$
- $p^{M_\infty}(a_{i_1}/\sim, \dots, a_{i_n}/\sim) \iff M_t \models P(f_{i_1t}(a_{i_1}), \dots, f_{i_nt}(a_{i_n}))$

  $a_{i_j} \in M_{i_j}$ for $t \geqslant i_1, \dots, i_n$
- f^{M_∞} : similarly

the rest is an exercise.

How to extend elementary mappings?

~~Bat.~~ BAlg : Category of Boolean algebras

Comp$_0$: —"— of compact Hausdorff 0-dimensional spaces

$$F : \mathrm{BAlg} \longrightarrow \mathrm{Comp}_0 \qquad\qquad G : \mathrm{Comp}_0 \longrightarrow \mathrm{BAlg}$$

$$F(A) = \underset{}{\overset{\rotatebox{90}{$\in$}}{S(A)}} \qquad\qquad G(X) = \overset{\rotatebox{90}{$\in$}}{\mathrm{Clopen}(X)}.$$

F, G : contravariant functors "inverse" to each other

~~(F, G) is a duality of categories... (look it up)~~

Categories BAlg and Comp$_0$ are dually equivalent.

A, B : Boolean algebras

$$f : A \longrightarrow B \text{ homomorphism} \Rightarrow \overset{f^*}{\overset{\shortparallel}{F(f)}} : S(B) \longrightarrow S(A)$$

$$F(f)(p) = f^{-1}[p]$$

continuous.

$$\text{Assume } f : \underset{\rotatebox{-90}{$\subseteq$}}{A} \xrightarrow{\equiv} \underset{\rotatebox{-90}{$\subseteq$}}{B}. \text{ Then } \hat{f} : L_n(A) \longrightarrow L_n(B)$$

$$T \Dashv M, \quad N \models T$$

$$\hat{f}(\varphi(\bar{x}, \bar{a})) = \varphi(\bar{x}, f(\bar{a}))$$

homomorphism of Boolean algebras.

even: monomorphism.

We skip $\hat{}$ in $\hat{f}$, so:

$$f : L_n(A) \longrightarrow L_n(B) \quad \text{monomorphism}$$

$$\Downarrow$$

$$f^* : S_n(B) \longrightarrow S_n(A) \quad \text{epimorphism in } \mathrm{Comp}_0$$

i.e: <u>continuous onto</u>

Lemma (on extensions of elementary mappings)

Assume $M, N \models T$, $A \subseteq M$, $B \subseteq N$, $f : A \xrightarrow{\equiv} B$ "onto".

Assume $\underset{a}{\rotatebox{90}{$\in$}}$, $\underset{b}{\rotatebox{90}{$\in$}}$, $p = tp(a/A)$, $q = tp(b/B)$.

Then $f \cup \{\langle a, b\rangle\}$ is elementary $\iff f^*(q) = p$.

[here $f^* : S(B) \xrightarrow{\approx} S(A)$].
homeomorphism

Proof exercise.

Def. M is $(<\aleph_0)$-universal $\iff \forall n\, \forall p \in S_n(\emptyset)\ p(M) \neq \emptyset$.

Remark $M : \kappa$-universal $\Rightarrow M : (<\aleph_0)$-universal.

Proof Let $p \in S_n(\emptyset)$.

Choose a countable $N \models T$ with $p(N) \neq \emptyset$.

$M : \kappa$-universal $\Rightarrow \quad \exists f : N \xrightarrow{\equiv} M$

$\overset{\rotatebox{90}{$\in$}}{\bar{a}} \models p \overset{f}{\rightsquigarrow} f(a) \models p$.

Thm. (1) $M : \kappa$-saturated $\Rightarrow M : \kappa$-homogeneous and κ-universal.

(2) $M : \kappa$-~~universal~~ homogeneous and $(<\aleph_0)$-universal $\Rightarrow$ $M : \kappa$-saturated.

Proof. (1) κ-homogeneity of M:

Assume $f : A \xrightarrow{\equiv} M$, $A \subseteq M$, $|A| < \kappa$, $a \in M$.

We seek $b \in M$ s.t. $g = f \cup \{\langle a, b\rangle\}$ elementary

$$\Updownarrow \text{ Lemma}$$

$$f^*(tp(b/B)) = tp(a/A).$$

Let $\underset{\overset{\uparrow}{S_1(A)}}{p} = tp(a/A)$, $q = (f^*)^{-1}(p) \in S_1(B)$

$\overset{q}{\|}$

Let $\underset{\overset{\uparrow}{\text{good.}}}{b} \in M$ (exists by κ-saturation) of M

- $\underline{\kappa\text{-universality}}$ of M:

Assume $N \equiv M$, $\|N\| \leq \kappa$.

We seek $f : N \xrightarrow{\equiv} M$.

Let $\{a_\alpha : \alpha < \mu\}$: an enumeration of N, $\overset{\overset{\kappa}{\vee|}}{\mu} = \|N\|$.

We define $f(a_\alpha)$ by induction on $\alpha < \mu$:

- Suppose $f(a_\beta)$ defined for all $\beta < \alpha$ so that

  ~~and~~ $f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} M$

  Want to find $f(a_\alpha)$ so that

  $$f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} M.$$

~~By the Lemma it is enough that t~~

Let $p = tp(a_\alpha / \{a_\beta : \beta < \alpha\})$.

By the lemma it is enough to find $f(a_\alpha) \in M$ so that $f^*(tp(f(a_\alpha)/\{f(a_\beta) : \beta < \alpha\})) = p$.

So let $q = (f^*)^{-1}(p) \in S_1(\underbrace{\{f(a_\beta) : \beta < \alpha\}}_{\text{power} < \kappa})$

M κ-saturated $\Rightarrow$ q realized in M.

Let $f(a_\alpha) \in M$ s.t. $f(a_\alpha) \models q$.

(2) Assume M is κ-homogeneous & $(< \aleph_0)$-~~saturated~~ universal.

Want: M: κ-saturated.

So: Let $A \subseteq M$, $|A| < \kappa$, $p \in S_1(A)$. Show: $p(M) \neq \emptyset$.

<u>Induction on $|A|$</u>.

Case (a): <u>$|A| < \aleph_0$</u>.

$\exists N \succ M$ $p(N) \neq \emptyset$. So let $\overset{N}{b} \models p$.

Let $A^* = \underset{\{a_1,\dots,a_k\}}{\underset{\|}{A}} \cup \{b\}$

Let $q = tp^N(a_1,\dots,a_k,b) \in S_{k+1}(\emptyset)$

q is realized in M (by $(<\aleph_0)$-universality),

by $\langle \underbrace{a'_1,\dots,a'_k}_{A'}, b' \rangle$

Let $g : A \cup \{b\} \longrightarrow A' \cup \{b'\}$, $g(a_i) = a_i'$, $g(b) = b'$.

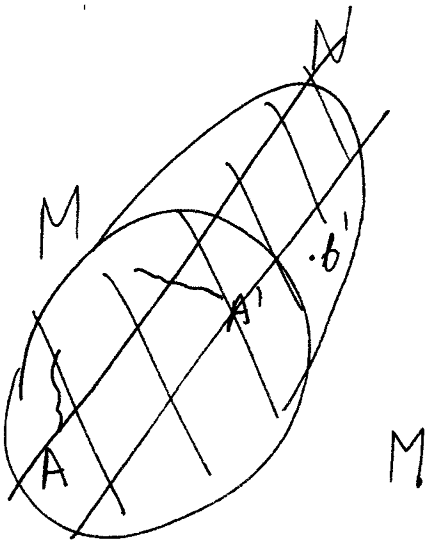
g: elementary.

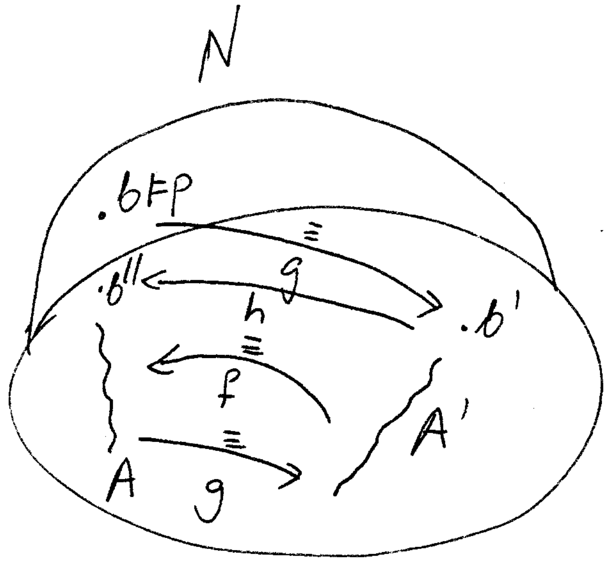
$\Rightarrow \; g\restriction_A : A \xrightarrow{\equiv} A'$

$\Downarrow \; f := (g\restriction_A)^{-1} : \underset{\cap|}{A'} \xrightarrow{\equiv} A$

M:
κ-homogeneous $\Rightarrow \; \exists \; h : A' \cup \{b'\} \xrightarrow{\equiv} A \cup \{b''\}$

for some $b'' \in M$.





$$\begin{tikzcd}
Ab \arrow[rr, "\equiv", "h\circ g"'] \arrow[dr, "g"', "\equiv"] & & Ab'' \\
& A'b' \arrow[ur, "\equiv", "h"'] &
\end{tikzcd}$$

Let $s = h \circ g$

$s\restriction A = \underbrace{(h\restriction_{A'})}_{\underset{(g\restriction_A)^{-1}}{\shortparallel}} \circ (g\restriction_A) = id_A.$

$\Downarrow$

~~$s^*(tp(b''/A)) = tp(b$~~

$s\restriction_A = id_A \Rightarrow \underset{\underset{id_{S(A)}}{\shortparallel}}{s^*} : S(A) \xrightarrow{\simeq} S(A)$

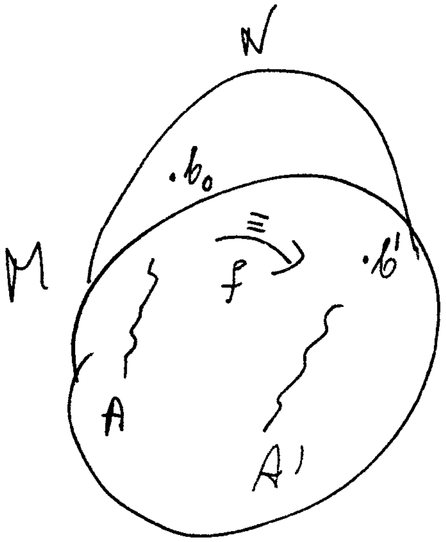
hence: $p = tp(b/A) \underset{\underset{\text{Lemma and } b'' \models p}{\uparrow}}{=} s^*(tp(b''/A)) \underset{\underset{s^* = id_{S(A)}}{\uparrow}}{=} tp(b''/A)$

Case (b) $|A| = \mu$, $\aleph_0 \leq \mu < \varkappa$.

$A = \{a_\alpha : \alpha < \mu\}$, $p \in S_1(A)$.

$p \restriction \emptyset \in S_1(\emptyset) \xRightarrow[M:(<\aleph_0)\text{-universal}]{} \exists b' \in M \ b' \models p \restriction \emptyset$.

$\exists N \succ M \ \exists \underset{\substack{\models \\ p}}{b_0} \in N$



Will find $A' = \{a'_\alpha : \alpha < \mu\} \subseteq M$

s.t. $f : Ab_0 \longrightarrow A'b'$

given by $f(a_\alpha) = a'_\alpha$

$f(b_0) = b'$

is <u>elementary</u>!

We find a'_α, $\alpha < \mu$ by induction on $\alpha < \mu$.

So suppose $\alpha < \mu$ and a'_β already defined for all $\beta < \alpha$

so that $\underbrace{f\restriction_{\{a_\beta : \beta < \alpha\} b_0}}_{=: f_0} : \{a_\beta : \beta < \alpha\} b_0 \xrightarrow{\equiv} \{a'_\beta : \beta < \alpha\} b'$

We look for a'_α.

Let $q = tp(a_\alpha / \{a_\beta : \beta < \alpha\} \cup \{b_0\})$

Then $(f_0^*)^{-1}(q) \in S(\underbrace{\{a'_\beta : \beta < \alpha\} \cup \{b'\}}_{\text{power } < \mu \leq |A|})$

By the lemma it is enough that $\underset{\wedge M.}{a'_\alpha} \models (f_0^*)^{-1}(q)$.

~~But M: ϰ-~~

~~Let A~~ By the inductive assumption on A:

$(f_0^*)^{-1}(q)$ is realized in M, so we are done with constructing A'.

Now: $f^{-1}: A' \xrightarrow{\equiv} A$ in $M \leftarrow \varkappa$-homogeneous, so

$\exists\ \hat{g}' : A'b' \xrightarrow{\equiv} Ab$ for some $b \in M$.

~~Let~~ Let $s = g \circ f$

$s : Ab_0^{\#} \xrightarrow{\equiv} Ab$

$$s\restriction_A = \underset{\underset{(f\restriction_A)^{-1}}{\parallel}}{(g\restriction_{A'})} \circ (f\restriction_A) = id_A,$$

so $\underset{p =}{} tp(b_0^{\#}/A) = tp(b/A)$

and p is realized in M.

Corollary

M is $\varkappa$-saturated $\Longleftarrow\!\!\Rightarrow$ M is $\varkappa$-homogeneous and ~~$\varkappa$-universal~~. and $\varkappa$-universal.

Proof $\Rightarrow$ by Thm (1).

$\Leftarrow$ $\varkappa$-homogeneous + $\varkappa$-universal $\Rightarrow$

$\varkappa$-homogeneous + $(<\aleph_0)$-universal $\underset{\text{Thm (2)}}{\Rightarrow}$ $\varkappa$-saturated

Properties of saturated models.

Thm. Assume $M, N \models T$ saturated models of the same power. Then $M \cong N$.

Proof $M = \{m_\alpha : \alpha < \varkappa\}$, $N = \{n_\alpha : \alpha < \varkappa\}$,
$\varkappa = \|M\| = \|N\|$. We find $f : M \xrightarrow{\cong} N$

back-and-forth method:

$$f = \bigcup^{\nearrow}_{\alpha<\varkappa} f_\alpha \qquad f_\alpha : M \overset{\equiv}{-\!\!\circ\!\!\!\rightarrow} N \qquad \text{s.t.:}$$

partial, elementary

(1) $m_\alpha \in \mathrm{Dom}\, f_{\alpha+1}$
$n_\alpha \in \mathrm{Rng}\, f_{\alpha+1}$, $|f_\alpha| \leq 2 \cdot |\alpha|$

(2) $f_0 = \emptyset$

(3) For $\delta \in \mathrm{Lim}$, $f_\delta = \bigcup^{\nearrow}_{\alpha<\delta} f_\alpha$.

(4) $f_{\alpha+1} = f_\alpha \cup \{\langle m_\alpha, \underset{N}{\underset{\uparrow}{n}} \rangle, \langle \underset{M}{\underset{\uparrow}{m}}, n_\alpha \rangle\}$

Inductive step:

Suppose we have f_α. Want: $f_{\alpha+1}$.

Let $A_\alpha = \mathrm{Dom}\, f_\alpha \subseteq M$, $B_\alpha = \mathrm{Rng}\, f_\alpha \subseteq N$.

$$\begin{array}{c} f_\alpha : A_\alpha \xrightarrow{\equiv} B_\alpha \\ \Downarrow \\ f_\alpha^* : S(B_\alpha) \xrightarrow{\cong} S(A_\alpha). \end{array}$$

"forth": Find $n \in N$ st. $f_\alpha \cup \{\langle m_\alpha, n\rangle\}$ elementary

$$\Updownarrow$$

$$(f_\alpha^*)^{-1}(tp(m_\alpha/A_\alpha)) = tp(n/B_\alpha).$$

Let $p = tp(m_\alpha/A_\alpha)$.

So $(f_\alpha^*)^{-1}(p) \in S(B_\alpha)$ is realized in N by some n.

"~~back~~": similarly.
back

<u>Thm</u> Assume $M, N \models T$ are homogeneous, of the same power and $\forall n<\omega\ \forall p \in S_n(\emptyset)\ (p(M)\neq\emptyset \Leftrightarrow p(N)\neq\emptyset)$.
Then $M \cong N$.

<u>**Lemma**</u> Under the assumptions of the Thm,

$$\forall A \subseteq M\ \exists f : A \xrightarrow{\equiv} N.$$

<u>Proof</u>. Induction on $|A|$.

Case (a) $|A| < \aleph_0$. $A = \{a_1, \dots, a_n\}$.

Let $p = tp(\langle a_1, \dots, a_n\rangle) \in S_n(\emptyset)$. realized in M
$\Downarrow$
realized in N
by some $\langle b_1, \dots, b_n\rangle \subseteq N$.

$f(a_i) = b_i$ is <u>good</u>.

Case (b) $|A| = \mu \geqslant \aleph_0$, $A = \{a_\alpha : \alpha < \mu\}$

We find $f(a_\alpha)$ by induction on $\alpha < \mu$.

Inductive step.

Suppose $\alpha < \mu$ and for every $\beta < \alpha$ we have $f(a_\beta)$

s.t. $f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} N$.

We shall find $f(a_\alpha) \in N$ s.t. $f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} N$.

Let $a_{<\alpha} := \{a_\beta : \beta < \alpha\}$. Likewise $a_{\leq\alpha}$.

$|a_{\leq\alpha}| < \mu = |A|$

By inductive assumption: $\exists g : a_{\leq\alpha} \xrightarrow{\equiv} N$.

Then $$f \circ g^{-1} : \underset{\substack{\cap\! | \\ N}}{g(a_{<\alpha})} \xrightarrow{\equiv} \underset{\substack{\cap\! | \\ N}}{f(a_{<\alpha})}$$

By homogeneity of N: $\exists f(a_\alpha) \in N$ s.t.

$$f \circ g^{-1} : \underbrace{g(a_{<\alpha})\, g(a_\alpha)}_{\substack{\| \\ g(a_{\leq\alpha})}} \xrightarrow{\equiv} f(a_{<\alpha})\, f(a_\alpha) = f(a_{\leq\alpha})$$

Then $f = (f \circ g^{-1}) \circ g : a_{\leq\alpha} \xrightarrow{\equiv} N$.

Proof of the theorem

$\kappa := \|M\| = \|N\|$

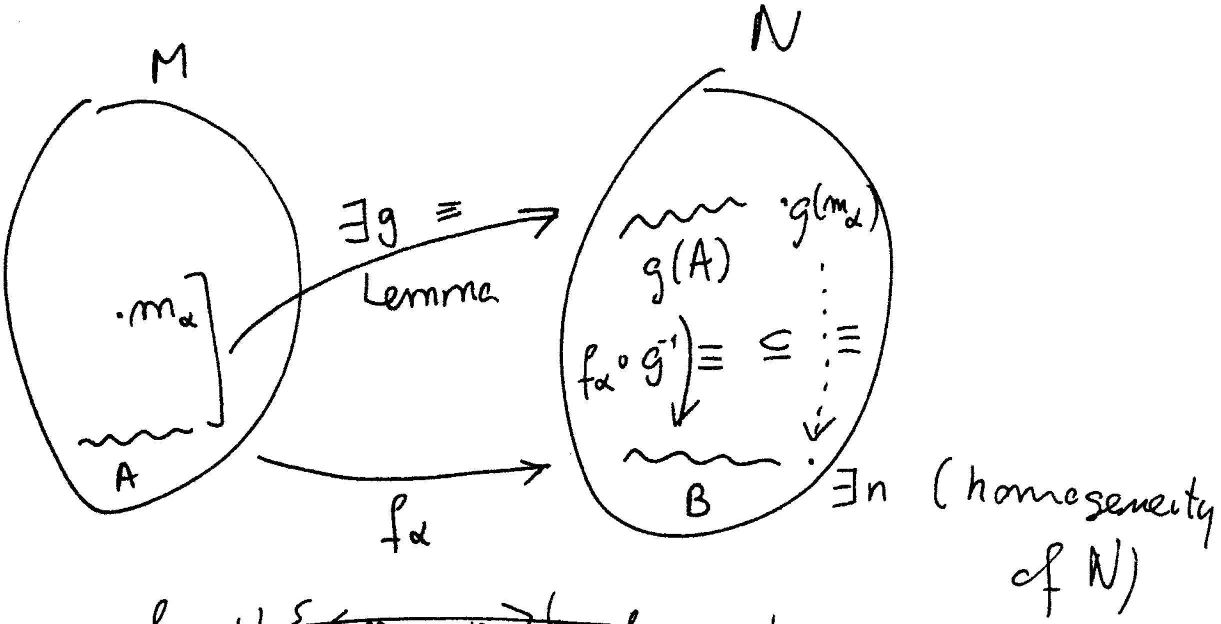
$f : M \xrightarrow{\cong} N$ constructed by back-and-forth method

$f = \bigcup\limits^{\uparrow}_{\alpha<\kappa} f_\alpha$, $f_\alpha : M \xrightarrow{\overset{\equiv}{\circ}} N$ (partial elementary), $\alpha < \kappa$

$|f_\alpha| \leq 2 \cdot |\alpha|$ + the same conditions as in the previous thm.

Inductive step $f_\alpha \longmapsto f_{\alpha+1}$

$A = \mathrm{Dom}\, f_\alpha$ $\quad B = \mathrm{Rng}\, f_\alpha$ $\quad \boxed{\text{"forth":}}$



~~$f_\alpha \cup \{\langle m_\alpha, n\rangle\}$ elementary~~

$$h = (f_\alpha \circ g^{-1})\restriction_{g(A)} \cup \{\langle g(m_\alpha), n\rangle\} \text{ elementary}$$

$$\underset{\substack{\cup\!| \\ f_\alpha}}{h \circ g} : A m_\alpha \xrightarrow{\equiv} B n \subseteq N$$

"back": similarly.

Constructions of models:

- saturated, $\Longrightarrow$ • (strongly) homogeneous $\not\Longleftarrow$

Thm. $\underbrace{\varkappa = 2^{<\varkappa},\ \varkappa \in Reg}_{\substack{\equiv \\ \varkappa^{<\varkappa} = \varkappa}},\ \varkappa > \aleph_0 \Rightarrow \exists M \models T$ saturated, of power $\varkappa$.

Proof

(*) $|S_1(A)| \le 2^{|A|+\aleph_0}$, because: $|L_1(A)| = |A| + \aleph_0$

Here: $|A| < \varkappa \Rightarrow |S_1(A)| \le \varkappa$.

Lemma $N \models T$, $\|N\| \le \varkappa \Rightarrow \underset{\text{the set}}{X_N} := \bigcup\{S_1(A) : A \subseteq N \,\&\, |A| < \varkappa\}$ has power $\le \varkappa$.

Pf • $|\{A \subset N : |A| \underset{\nwarrow}{<} \varkappa\}| \le \varkappa^{<\varkappa} = \varkappa$.

• $|S_1(A)| \le \varkappa$ for such A.

Proof of the thm.

M_α, $\alpha < \varkappa$: elementary chain of models of T of power $\varkappa$.

- M_0 : whatever
- $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$, when $\delta < \varkappa$ limit.
- $M_{\alpha+1} \succ M_\alpha$ such that $\forall p \in X_{M_\alpha}\ p(M_{\alpha+1}) \neq \emptyset$:

$T' = Th(M_\alpha, m)_{m \in M_\alpha} \cup$ ~~&~~ $\bigcup_{\beta<\varkappa} \{\varphi(\underset{\uparrow}{c_\beta}) : \varphi(x) \in p_\beta\}$,

where $X_{M_\alpha} = \{p_\beta : \beta < \varkappa\}$ new constant symbols,

and T' in language $L(M_\alpha) \cup \{\overset{\downarrow}{c_\beta} : \beta < \varkappa\}$.

T': consistent, has model of power κ : $M_{\alpha+1}$
such that $M_\alpha \prec M_{\alpha+1}$.

$M = \bigcup^{\uparrow}_{\alpha<\kappa} M_\alpha$: of power κ, saturated:

Let $A \subseteq M$, $|A|<\kappa$ and $p \in S_1^M(A)$

$\kappa \in Reg \Rightarrow A \subseteq M_\alpha$ for some $\alpha<\kappa$.

proof: $A = \{a_\beta : \beta<\mu\}$ for some $\overset{\overset{\text{CN}}{\downarrow}}{\mu}<\kappa$.

$\cdot\ \forall \beta<\mu\ \ \exists\, \alpha_\beta<\kappa\ \ a_\beta \in M_{\alpha_\beta}$

$\cdot\ \{\alpha_\beta : \beta<\mu\} \subseteq \kappa\ ,\ \mu < cf(\kappa)=\kappa$

$\Rightarrow \exists\, \underset{\underset{A \subseteq M_\alpha.}{\uparrow}}{\alpha}<\kappa\ \ \forall\beta<\mu\ \ \alpha_\beta<\alpha$

~~Let~~ $M_\alpha \prec M \Rightarrow p \in S_1^{M_\alpha}(A) = S_1^M(A)$

p realized in $M_{\alpha+1}$ by some $a \in M_{\alpha+1}$

$a \models p$ in $M_{\alpha+1} \underset{M_{\alpha+1} \prec M}{\Rightarrow} a \models p$ in M.

Monster model:

Let $\bar\kappa$: a large cardinal number.

"Ideal model" $\mathfrak{M} \models T$: saturated of power $\bar\kappa$

because: $\forall M \models T\ (\|M\|<\bar\kappa \Rightarrow \exists M' \prec \mathfrak{M}\ \ M \cong M')$.

Advantages of saturated model M:

(i) universality

(ii) strong homogeneity

~~More~~ More Weakly (a bit):

(1) $\overline{\varkappa}$-universality

(2) strong $\overline{\varkappa}$-homogeneity

$\mathrm{Aut}(M)$: the group of automorphisms of M

$\mathrm{Aut}(M/A) = \{ f \in \mathrm{Aut}(M) : f\restriction_A = \mathrm{id}_A \}$: automorphisms of M over A.

$A \subseteq M$

Lemma Assume M is strongly $\varkappa$-homogeneous, $\varkappa$-saturated, $A \subseteq M$, $|A| < \varkappa$. Then:

(1) For $a, b \in M$ ($tp(a/A) = tp(b/A) \iff a, b$ are in the same orbit of $\mathrm{Aut}(M/A)$ on M.

(2) $[\text{orbits } \mathrm{Aut}(M/A) \text{ on } M^n] \overset{1\text{-}1}{\underset{\text{onto}}{\longleftrightarrow}} S_n(A)$

Proof (1) $\Leftarrow$: $f \in \mathrm{Aut}(M/A)$, $f(a) = b$

$$\Downarrow$$

$$tp(a/A) = tp(b/A).$$

$\Rightarrow$: $tp(a/A) = tp(b/A) \Rightarrow f : Aa \xrightarrow{\equiv} Ab$

$f\restriction_A = \mathrm{id}_A$, $f(a) = b$

strong $\varkappa$-homogeneity, $|A| < \varkappa$ $\Rightarrow f \subseteq g \in \mathrm{Aut}(M)$, $g \in \mathrm{Aut}(M/A)$

$g(a) = b$: a, b in the same orbit of $\mathrm{Aut}(M/A)$

MT3/4

(2) $M^n \supseteq \mathcal{O} \xrightarrow[\varphi]{(1)} p_{\mathcal{O}} \in S_n(A)$

$\uparrow$ orbit of $\mathrm{Aut}(M/A)$

$p_{\mathcal{O}}$ $\shortparallel$ common type $tp(a/A)$ for $a \in \mathcal{O}$.

$$\begin{tikzcd} \text{\{orbits of } \mathrm{Aut}(M/A) \text{ on } M^n\} \arrow[d, "\varphi"] \\ S_n(A) \end{tikzcd}$$

$\mathcal{O}_1 \neq \mathcal{O}_2 \overset{(1)}{\Rightarrow} p_{\mathcal{O}_1} \neq p_{\mathcal{O}_2}$ $\boxed{\text{so } \varphi : 1\text{-}1}$

[if $p_{\mathcal{O}_1} = p_{\mathcal{O}_2}$ then let $a \in \mathcal{O}_1$, $b \in \mathcal{O}_2 \Rightarrow \exists g \in \mathrm{Aut}(M/A)$ $g(a) = b$ ↯.]

M: κ-saturated $\Rightarrow$ φ : "onto".

<u>Def</u> Let $\bar{\kappa}$: a (large) cardinal number, $\mathcal{M} \models T$ <u>monster model</u> (w.r. to $\bar{\kappa}$), if $\mathcal{M}$: $\bar{\kappa}$-saturated, strongly $\bar{\kappa}$-homogeneous

Thm. Assume $\aleph_0 \leq \kappa \in CN$. Then $\exists M$: κ-saturated ~~&~~ strongly $\bar{\kappa}$-saturated.

<u>Proof</u> $M = \bigcup^{\uparrow}_{\alpha < \kappa^+} M_\alpha$: union of elementary chain s.t.:

(1) $M_0 \models T$ any

(2) $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$ if $\delta \in Lim$,

(3) $M_{\alpha+1} \succ M_\alpha$ s.t:

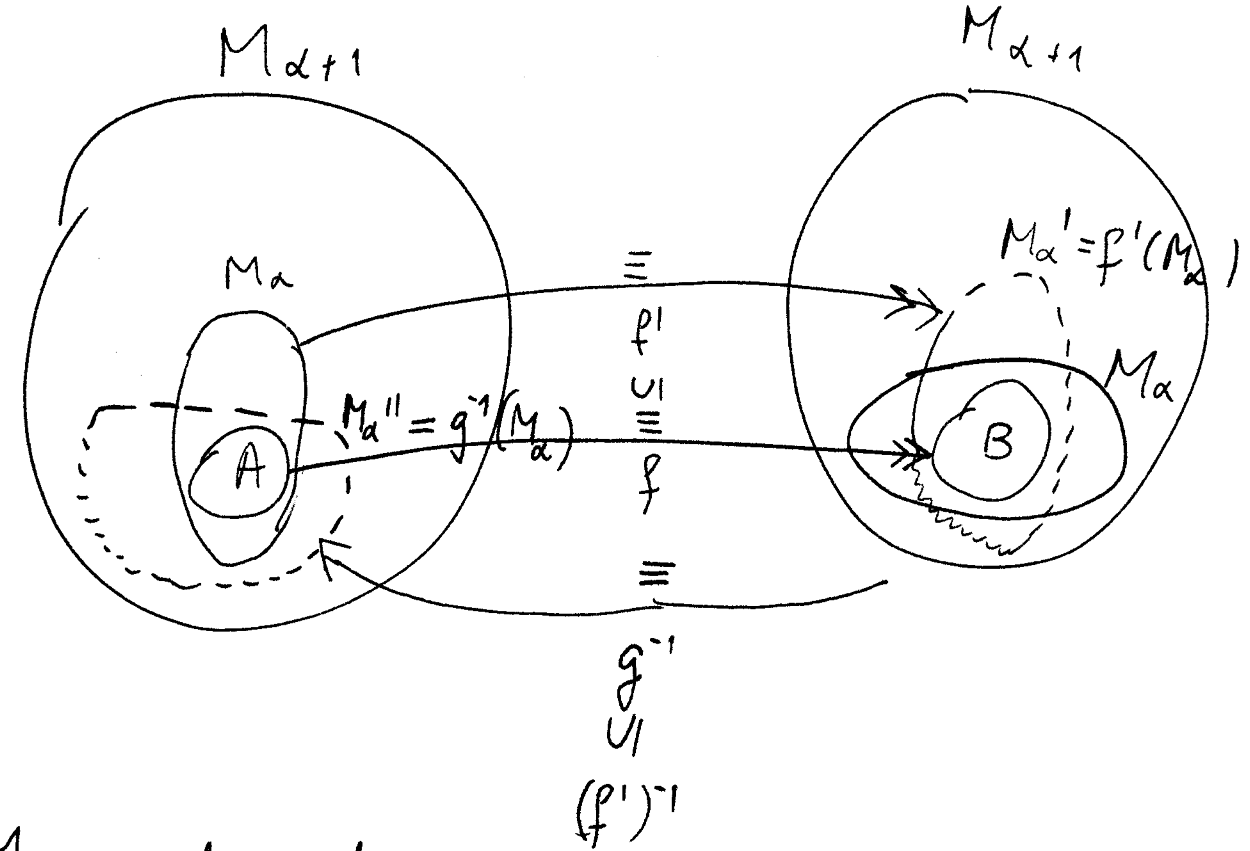
(a) $\forall p \in S_1(M_\alpha)$ p realized in $M_{\alpha+1}$

(b) $\left(\forall f: \underset{\underset{M_\alpha}{\cap\!\shortmid}}{A} \xrightarrow{\equiv} \underset{\underset{M_\alpha}{\cap\!\shortmid}}{B}\right)(\exists\, g \supseteq f)\left(g: \underset{\underset{M_\alpha}{\cup\!\shortmid}}{A'} \xrightarrow{\equiv} \underset{\underset{M_\alpha}{\cup\!\shortmid}}{B'} \text{ in } M_{\alpha+1}\right)$

It is enough that $M_{\alpha+1}$ is $\|M_\alpha\|^+$-saturated,

$\uparrow$ To satisfy (a), (b). $M_{\alpha+1} \succ M_\alpha$

Proof of (b) for such $M_{\alpha+1}$:



I. M: κ-saturated: <u>clear</u>

II M strongly κ-homogeneous:

Assume $A \subset M$, $|A| < \kappa$. Then $A \subseteq M_\alpha$, $B \subseteq M_\alpha$

$f: A \xrightarrow{\equiv} M$ for some $\alpha < \kappa$.

$B = f[A]$

- we construct

a sequence f_β, $\alpha \leq \beta < \varkappa^+$:.

- increasing $f_\beta : M \xrightarrow{\overset{\equiv}{\theta}} M$ partial elementary
- $f \subseteq f_\alpha$

(*) $M_\beta \subseteq \operatorname{dom} f_\beta \cap \operatorname{rng} f_\beta$.

- f_α constructed according to 3)(b)

$$f_\alpha : M_{\alpha+1} \xrightarrow{\overset{\equiv}{\theta}} M_{\alpha+1}$$

- $f_\beta : M_{\beta+1} \xrightarrow{\overset{\equiv}{\theta}} M_{\beta+1}$, as in 3.(b) when β : successor.
- $f_\delta = \bigcup_{\beta<\delta} f_\beta$ when δ limit. , still $f_\delta : M_\delta \xrightarrow{\overset{\equiv}{\theta}} M_\delta$.

$f_\infty = \bigcup^{\nearrow}_{\alpha \leq \beta < \varkappa^+} f_\beta$, $f_\beta \in \mathrm{Aut}(M)$, $f \subseteq f_\beta$.

<u>Assumptions</u> Let $\bar{\varkappa}$: a cardinal number <u>large enough</u> so that:

(1) We consider only small models of T

$\shortparallel$ of power $< \bar{\varkappa}$, or even $\ll \bar{\varkappa}$

(2) We work within a monster model $\mathfrak{M} \models T$ (w.r.to $\bar{\varkappa}$)

(3) We consider only small models $M \prec \mathfrak{M}$

$\shortparallel$ of power $< \bar{\varkappa}$, or even $\ll \bar{\varkappa}$, Consequences:

(1) For $M, N \prec \mathcal{M}$, $M \subseteq N \iff M \prec N$

(2) Convention: For $\bar{a} \subseteq \mathcal{M}$

$\models \varphi(\bar{a})$ means $\mathcal{M} \models \varphi(\bar{a})$

(3) For $A \subseteq M \prec \mathcal{M}$:

$S_n^M(A) = S_n^{\mathcal{M}}(A) =: S_n(A)$

<u>Notation</u> Assume $p(\bar{x})$, $q(\bar{x})$ types (small, over $\mathcal{M}$)

- $p(\bar{x}) \vdash q(\bar{x}) \iff p(\mathcal{M}) \subseteq q(\mathcal{M})$

  "p implies q"

- $p(\bar{x}) \underset{\substack{\uparrow \\ \text{equivalent}}}{\equiv} q(\bar{x}) \iff p \vdash q \;\&\; q \vdash p$

Special case: $p(\bar{x}) = \{\varphi(\bar{x})\}$.

$\varphi(\bar{x}) \vdash q(\bar{x})$: "φ isolates q".

<u>Remark</u>: Syntactically:

$$\underset{\substack{\nwarrow \\ \text{types over } A}}{p(\bar{x})} \vdash \underset{\nearrow}{q(\bar{x})} \iff \forall \varphi(\bar{x}) \in q \;\; \exists \underset{\text{finite}}{p_0(\bar{x}) \subseteq p(\bar{x})}$$

$$T(A) \vdash \bigwedge p_0(x) \rightarrow \varphi(x)$$

Remark (exercise)

$$p(\bar{x}) \vdash q(\bar{x}) \iff \underset{|A|^+\text{-saturated}}{\forall M \models T} \;\; p(M) \subseteq q(M).$$

Def. (reminder)

Let $p(\bar{x})$: a type over A.

p is isolated over $A \iff \exists \underset{\text{consistent (with T)}}{\varphi(\bar{x}) \in L(A)}\quad \varphi \vdash p$.

Thm (omitting types, Ehrenfeucht)

Assume $p_n(\bar{x}_n)$, $n<\omega$: a ~~fam~~ ~~non~~ a family of non-isolated types over $\emptyset$. Then $\exists M \models T\ \forall n\ \underbrace{p_n(M)=\emptyset,}_{M \text{ omits } p_n.}$

Lemma

Assume A is ctble, $p_n(\bar{x}_n)$, $n<\omega$: a family of non-isolated types over A, $\varphi(x) \in L_1(A)$, $\underbrace{\varphi(\mathfrak{M}) \neq \emptyset}_{\text{i.e. } \varphi \text{: consistent}}$.

Then $\exists c \in \varphi(\mathfrak{M})\ \forall n\ p_n$ non-isolated over $A \cup \{c\}$.

Proof [Lemma $\Rightarrow$ Thm]

By the Lemma: $\exists \underbrace{\{a_n : n<\omega\}}_{\overset{\shortparallel}{A}} \subseteq \mathfrak{M}$ s.t.

(1) A satisfies the TV-test

(2) p_n, $n<\omega$: non-isolated over A.

Construction of a_n, $n<\omega$: recursion on n.

Let $\{\varphi_n(x, \bar{y}) : n<\omega\}$: all formulas of L of this form.

Suppose $n<\omega$ and $\{a_i : i<n\} = a_{<n}$ already constructed
so that all $p_k, k<\omega$
still non-isolated over $a_{<n}$.

We consider a consistent formula $\varphi_n(x) \in L_1(a_{<n})$
By the Lemma we find $\underset{\overset{\shortparallel}{a_n}}{c} \in \varphi_n(\mathcal{M})$ so that
all p_k, $k<\omega$, still non-isolated over $a_{\leq n}$.

- the formulas $\varphi_n(x), n<\omega$ may be chosen so that after ω steps:
  $\forall \underset{\text{consistent}}{\varphi(x)} \in L(A)\ \exists n\ \varphi = \varphi_n$.

Then $A = \{a_n : n<\omega\}$ satisfies TV-test

$A = M \prec \mathcal{M}$, every p_k still non-isolated over A.

$p_k(M) = \emptyset$ [if not,
some $\underset{\overset{\cap}{M}}{\bar{m}} \models p_k$. then ~~[illegible]~~
$(\bar{x}_k = \bar{m}) \vdash p_k(\bar{x}_k)$ ↯)

Proof of the Lemma.

Let $p(\bar{x})$: one of the types $p_n(\bar{x}_n)$.

Let $h(\bar{x}, y, \bar{a}) \in L(A$

$h(\bar{x}, c, \bar{a}) \vdash p(\bar{x}) \iff h(\mathcal{M}, c, \bar{a}) \subseteq p(\mathcal{M})$.

$\iff \forall \psi(\bar{x}) \in p(\bar{x})\ h(\mathcal{M}, c, \bar{a}) \subseteq \psi(\mathcal{M})$

$\iff \forall \psi \in p\ \ \mathcal{M} \models \forall \bar{x}\,(h(\bar{x}, c, \bar{a}) \rightarrow \psi(\bar{x}))$

$\Longleftrightarrow \forall \psi \in p \quad \psi_h(y) \in tp(c/A)(y)$

where $\psi_h(y) = \forall \bar{x}\,(h(\bar{x}, y, \bar{a}) \longrightarrow \psi(\bar{x}))$

<u>hence</u>:

$$tp(c/A) = tp(c'/A) \Longrightarrow [\, h(\bar{x}, c, \bar{a}) \vdash p \Longleftrightarrow h(\bar{x}, c', \bar{a}) \vdash p \,]$$

$h(\bar{x}, c, \bar{a})$ consistent $\Longleftrightarrow (\exists \bar{x}\, h(\bar{x}, y, \bar{a})) \in tp(c/A)(y)$.

Let $X_{h,p} = \{ q \in S_1(A) : \text{For } c \models q,\ h(\bar{x}, c, \bar{a}) \vdash p(\bar{x}) \}$
and $h(\bar{x}, c, \bar{a})$ consistent.

$\uparrow$

"bad types"

Let $q \in S_1(A)$ then

$$q \in X_{h,p} \Longleftrightarrow q(y) \in S_1(A) \cap \overset{\substack{\text{For } c \models q\ \ h(\bar{x}, c, \bar{a}) \text{ consistent} \\ \shortparallel}}{[\exists \bar{x}\, h(\bar{x}, y, \bar{a})]} \cap \underset{\substack{\shortparallel \\ \text{For } c \models q\ \ h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})}}{\underbrace{\bigcap_{\psi \in p} [\psi_h(y)]}}$$

(*) $X_{h,p}$: nowhere dense in $S_1(A)$.

Proof of (*) : (a.a.)

Suppose $\theta(y) \in L_1(A)$ and $\emptyset \neq S_1(A) \cap [\theta] \subseteq X_{h,p}$.

Let $\underset{\substack{\cap \\ L(A)}}{\alpha(\bar{x})} = \exists y (h(\bar{x}, y, \bar{a}) \wedge \theta(y))$

- $\alpha(\bar{x})$: consistent :

Let $c \in \theta(\mathcal{M})$

$\Downarrow$ $[\theta] \subseteq X_{h,p}$)

$\mathcal{M} \models \exists \bar{x}\, h(\bar{x}, c, \bar{a})$.

Let $\bar{d} \subseteq \mathcal{M}$ s.t. $\mathcal{M} \models h(\bar{d}, c, \bar{a})$.

$\bar{d}$ satisfes in $\mathcal{M}$: $\underbrace{\exists y\, h(\bar{x}, y, \bar{a})}_{\overset{\shortparallel}{\alpha(\bar{x})}}$.

- $\alpha(\bar{x}) \vdash p(\bar{x})$, i.e. $\alpha(\mathcal{M}) \subseteq p(\mathcal{M})$.

Let $\bar{d} \in \alpha(\mathcal{M})$. So there is $c \in \mathcal{M}$ st

$\models h(\bar{d}, c, \bar{a}) \wedge \theta(c)$

$\Downarrow$ $[\theta] \subseteq X_{h,d}$

$\forall \psi \in p \ \models \psi_h(c)$

$\Downarrow$

$\forall \psi \in p \ h(\bar{x}, c, \bar{a}) \vdash \psi(\bar{x}) \Rightarrow h(\underset{\overset{\#}{\bar{d}}}{\bar{x}}, c, \bar{a}) \vdash p(\underset{\overset{\#}{d}}{\bar{x}})$ ↯.

as: $p(\bar{x})$ non-isolated

Let $\underset{\overset{\uparrow}{\text{meager}}}{X} = \bigcup_{h, p_n} X_{h,p_n} \subseteq S_1(A)$.

Let $q \in S_1(A) \cap [\varphi] \setminus X$

$c \models q$ good.

Pf (a.a) Suppose $p = p_n$ isolated over $A \cup \{c\}$.

$\exists\, h(\bar{x}, c, \bar{a})$ $\quad \underset{\text{Consistent}}{h(\bar{x}, c, \bar{a})} \vdash p(x) \Rightarrow \underset{\overset{\shortparallel}{tp(\bar{c}/A)}}{q} \in X_{h,p_n}$ ↯.

14.03.2022

Def T is quantifier eliminable if $\forall \varphi \in L\ \exists \underset{\substack{\text{open}\\ \equiv \text{q.f.}}}{\psi} \in L$

$$T \vdash \varphi \leftrightarrow \psi$$

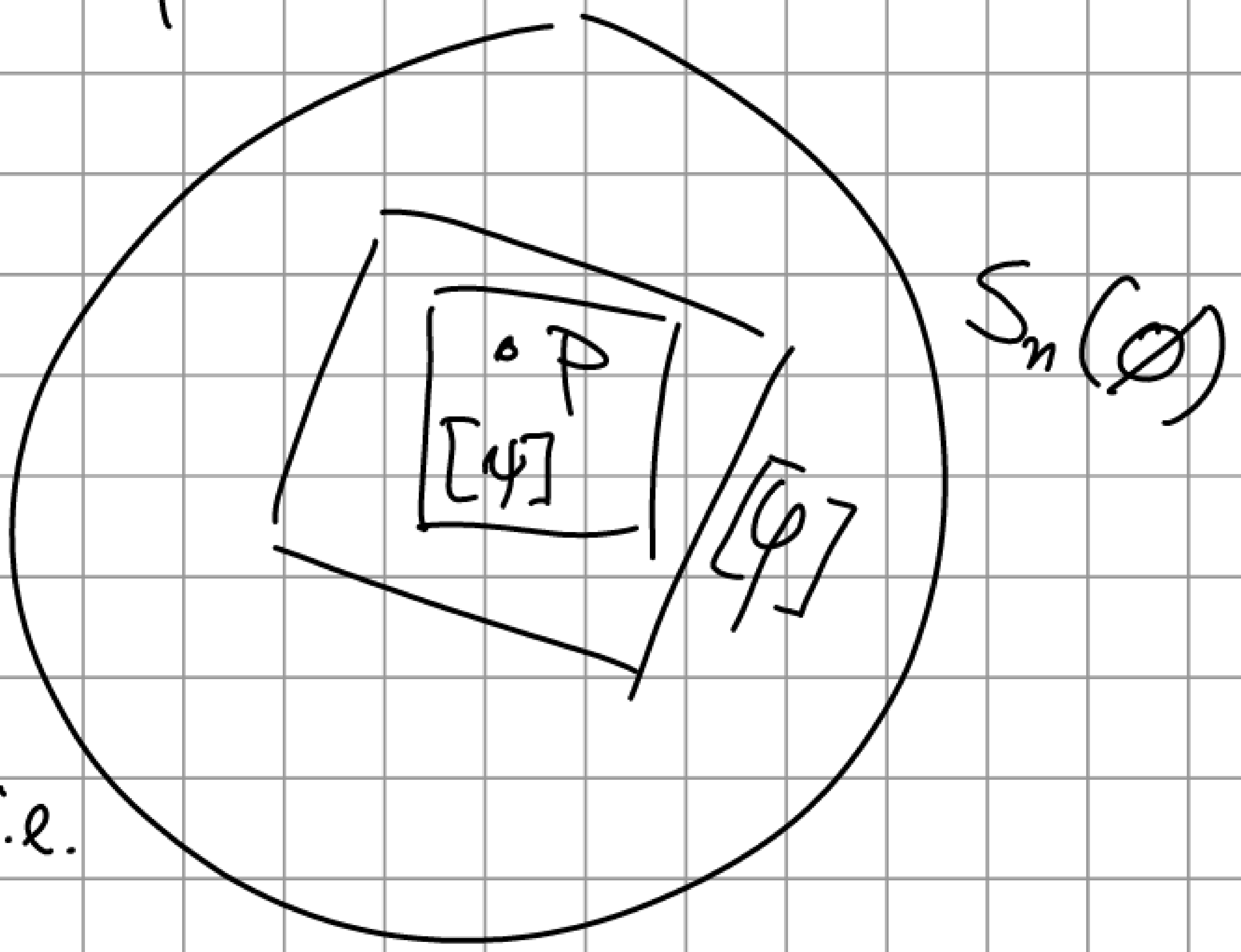
Def. For $p(\bar{x}) \in S_n(\emptyset)$ let $\underline{p_0(\bar{x})} = \{\varphi(\bar{x}) \in p(\bar{x}) : \varphi \text{ open}\}$.

Remark T is q.e. $\iff \forall n\, \forall p \in S_n(\emptyset)\quad p_0 \vdash p$

Proof „$\Rightarrow$" Obvious. „$\Leftarrow$" Let $\varphi(\bar{x}) \in L$.

- $\forall p \in [\varphi] \cap S_n(\emptyset)\ \exists \underset{\text{open}}{\psi} \in p\quad p \in [\psi] \subseteq [\varphi]$

Why? $\begin{array}{l} p_0 \vdash p \\ \qquad\ \ \cup \\ p_0 \vdash \varphi \end{array}$



by compactness

$\exists$ finite $p_0' \subseteq p_0$ s.t.

$p_0' \vdash \varphi$, i.e.

$$p_0'(\mathcal{M}) \subseteq \varphi(\mathcal{M})$$

$$\psi(\mathcal{M}) = \Big(\bigwedge_{\psi' \in p_0'} \psi'\Big)(\mathcal{M}) \subseteq \varphi(\mathcal{M}) \rightsquigarrow \mathcal{M} \models \psi(\bar{x}) \to \varphi(\bar{x})$$

$$\Updownarrow$$

$$\psi \vdash \varphi \Rightarrow [\psi] \cap S_n(\emptyset) \subseteq [\varphi] \cap S_n(\emptyset)$$

∎

Application $L = \{+, \cdot, 0, 1\}$: the language of rings.

ACF_p: the theory of algebraically closed fields of char p, in L.

Axioms:

1) field axioms

2) char $= p \neq 0$: $\underbrace{1 + \dots + 1}_{p} = 0$

2') $p = 0$: $\underbrace{1 + \dots + 1}_{n} = 0$ for $n \geq 1$

3) Every polynomial of deg n has a root:
$0 < n$

$$\forall y_{n-1}, y_{n-2}, \dots, y_0 \, \exists x \;\; x^n + y_{n-1} x^{n-1} + \dots + y_0 = 0.$$

Fact ACF_p is complete.

Proof Let $M, N \models ACF_p$. Enough to show that $M \equiv N$. Let $\varkappa > \|M\|, \|N\|$ and let $M' \succ M$, $N' \succ N$.

power $\varkappa$.

M', N': uncountable acl fields of the same power and char

$$\Downarrow \text{ algebra}$$

$$\begin{tikzcd} M' \arrow[r, phantom, "\cong"] & N' \arrow[r, Rightarrow] & M' \arrow[r, phantom, "\equiv"] \arrow[d, phantom, "\equiv"] \arrow[dr, phantom, "\Downarrow"] & N' \arrow[d, phantom, "\equiv"] \\ & & M \arrow[r, phantom, "\equiv"] & N \end{tikzcd}$$

<u>Fact</u> ACF_p is q.e. (Chevalley, Tarski)

<u>Proof</u> (in $\mathcal{M}$) We will show that $\forall p \in S_n(\emptyset)$ $p_0 \vdash p$ $(\Leftrightarrow p_0(\mathcal{M}) \subseteq p(\mathcal{M}))$. Let $\bar{a} \models p_0$, $\bar{b} \models p$, $\bar{a}, \bar{b} \subseteq \mathcal{M}$. It's enough to prove that $\exists f \in Aut(\mathcal{M})$ $f(\bar{a}) = \bar{b}$.

$\bar{a} = (a_1, \ldots, a_n)$
$\bar{b} = (b_1, \ldots, b_n)$

Let $\langle \bar{a} \rangle, \langle \bar{b} \rangle$: the subrings with 1 of $\mathcal{M}$ generated by $\bar{a}, \bar{b}$. $\bar{a}, \bar{b} \models p_0 \Rightarrow \langle \bar{a} \rangle \cong \langle \bar{b} \rangle$

$$a_i \xmapsto{f} b_i$$

Some algebraic magic.

$$\begin{tikzcd} & \langle \bar{a} \rangle \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cap"] \arrow[dr, Rightarrow, "\text{unique}"] & \langle \bar{b} \rangle \arrow[d, phantom, "\cap"] & \\ \mathcal{M} \supseteq & \langle \bar{a} \rangle_0 \arrow[r, phantom, "\cong"] \arrow[dr, bend right, "\text{(not unique)}"'] & \langle \bar{b} \rangle_0 \subseteq \mathcal{M} & \\ & \mathcal{M} \supseteq \underbrace{\langle \bar{a} \rangle_0^{alg}}_{F_a} \arrow[r, phantom, "\cong"] & \underbrace{\langle \bar{b} \rangle_0^{alg}}_{F_b} \subseteq \mathcal{M} & \end{tikzcd}$$

(fraction field)

$$\operatorname{trdeg}(\mathcal{M}/F_a) = \|\mathcal{M}\| = \operatorname{trdeg}(\mathcal{M}/F_b)$$

$$\Downarrow$$

$$\begin{tikzcd}
\mathcal{M} \arrow[d, "f"', "\cong"] \arrow[d, bend left=60, leftarrow] & \cong F_a(X_\alpha, \alpha<\lambda)^{alg} \arrow[d, phantom, "\wr\wr"] \\
\mathcal{M} & \cong F_b(x_\beta, \beta<\lambda)^{alg}
\end{tikzcd}$$

$$f \in \operatorname{Aut}(\mathcal{M}),\ f(\bar{a}) = \bar{b}.$$

■

Types in $T = ACF_p$

Let $\mathcal{M} \models T$: a monster model.

subfield (small) $\overset{\cup|}{K}$. We will describe $S_n(K)$.

Let $\bar{a} \subseteq \mathcal{M}$, $|\bar{a}| = n$.

$K[\bar{x}] \rhd I(\bar{a}/K) = \{ f \in K[\bar{x}] : f(\bar{a}) = 0 \}$

Remark 1) $tp(\bar{a}/K) = tp(\bar{a}'/K) \iff I(\bar{a}/K) = I(\bar{a}'/K)$

2) $\forall I \underset{\text{prime}}{\lhd} K[\bar{x}]\ \exists \bar{a} \subseteq \mathcal{M}\ I(\bar{a}/K) = I$.

Proof 1) "$\Rightarrow$" $tp(\bar{a}/K) = tp(\bar{a}'/K) \Rightarrow \exists f \in \operatorname{Aut}(\mathcal{M}/K)$

$I(\bar{a}/K) = I(\bar{a}'/K)$ $\quad f(\bar{a}) = \bar{a}'$

[alternatively: $f \in I(\bar{a}/K) \iff$ "$f(\bar{x}) = 0$" $\in tp(\bar{a}/K)$]

„$\Leftarrow$" Assume $I(\bar{a}/K) = I(\bar{a}'/K) = I$.

$$\Downarrow$$

$$K[\bar{a}] \underset{K}{\cong} K[\bar{x}]/I \cong K[\bar{a}']$$

$$\Downarrow$$

$$\exists f \in Aut(\mathcal{M}/K)\ f(\bar{a}) = \bar{a}'$$

$$\Downarrow$$

$$tp(\bar{a}/K) = tp(\bar{a}'/K).$$

2) $K \subseteq K[\bar{x}]/I = K[\bar{a}] \supseteq \bar{a} = \bar{x}/I$ and $I(\bar{a}/K) = I$.

$$\vdots$$

$\square$

$S_1(K) = \{ tp(a/K) : a \in \mathcal{M} \}$: a top. space.

$\rotatebox{-90}{$\in$}$

$p(x) = tp(a/K)$, $I_p = I(a/K) \triangleleft K[x]$

a) $I_p \neq \{0\}$, i.e. a is algebraic / K, so

$\rotatebox{-90}{$\in$}$

$0 \neq f$ "$f(x) = 0$" $\in p(x)$. In fact,

irreducible over K $\nearrow$ "$f(x) = 0$" $\vdash p(x)$ (isolates)

Let $a' \in \mathcal{M}$ s.t. $f(a')=0 \Rightarrow I(a'/K) \ni f \Rightarrow$ f generates $I(a'/K)$

$$p = tp(a/K) = tp(a'/K) \Leftarrow I(a/K) = I(a'/K) \Leftarrow$$

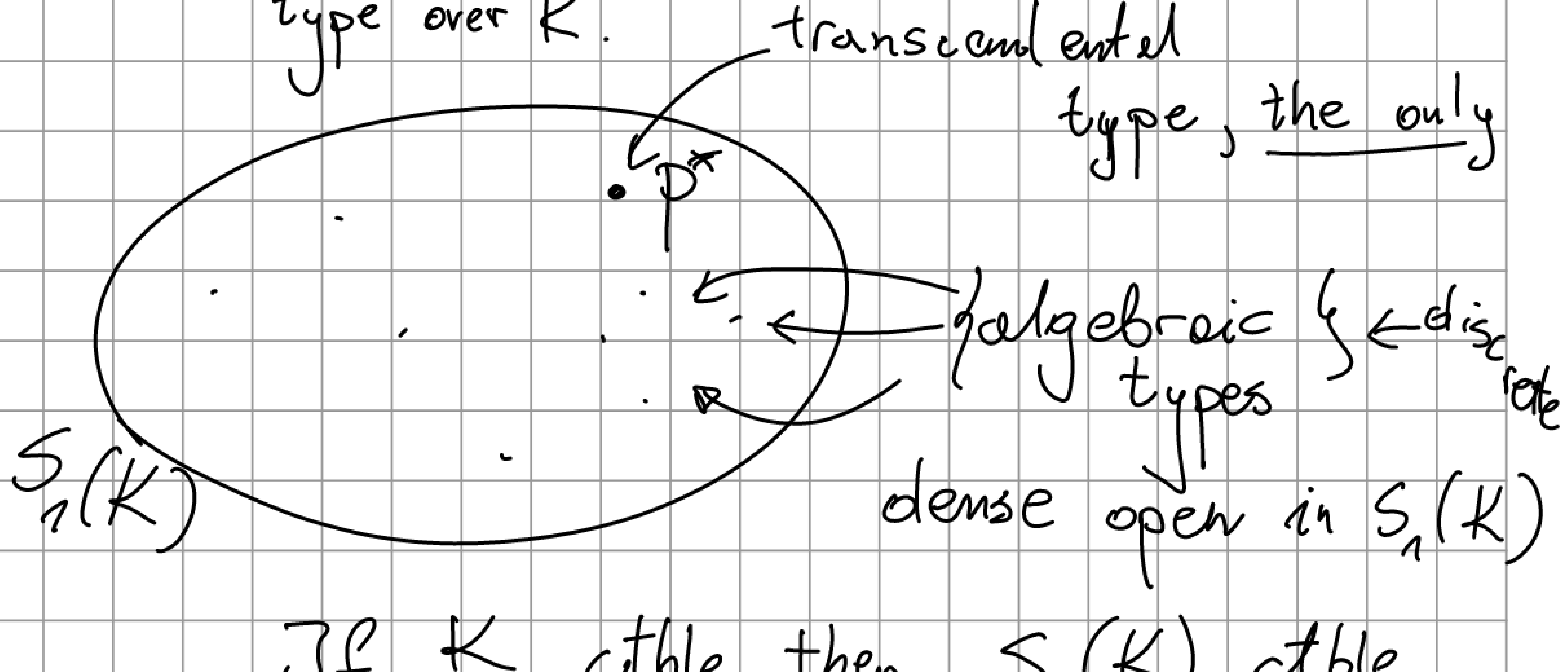
$p(x)$ here is called <u>algebraic</u>.

[More generally (T: arbitrary)

<u>Def</u> $\varphi(\bar{x}) \in L(A)$ is called <u>algebraic</u>, if $0 < |\varphi(\mathcal{M})| < \aleph_0$, similarly for q: a type.]

b) $I_p = \{0\}$: $p = tp(a/K)$ s.t. $a \in \mathcal{M}$ transcendental over K

↑ the transcendental type over K.



If K ctble then $S_1(K)$ ctble

Corollary ACF_p is $\aleph_0$- stable [recall: T is κ-stable $\iff \forall A \subseteq \mathcal{M}, |A| \leq \kappa$ $|S_1(A)| \leq \kappa$]

Proof Let $\underset{\text{ctble}}{A} \subseteq \mathcal{M}$.

$$\underset{\substack{\uparrow \\ \text{ctble} \\ \text{subfield}}}{A \subseteq K} \subseteq \mathcal{M} \quad |S_1(A)| \leq |S_1(K)| = \aleph_0$$

Remark T is totally transcendental $\iff T$: $\aleph_0$-stable

Proof "$\Rightarrow$" from def, "$\Leftarrow$": (A.a.) Let $\kappa > \aleph_0$.

Suppose $|A| \leq \kappa < |S_1(A)|$ for some $A \subseteq \mathcal{M}$.

Shall find $\underset{\text{ctble}}{A_0} \subseteq A$ with $|S_1(A_0)| \geq 2^{\aleph_0}$.

(def) $\varphi(x) \in L(A)$ is <u>large</u> iff $|S_1(A) \cap [\varphi]| > \kappa$

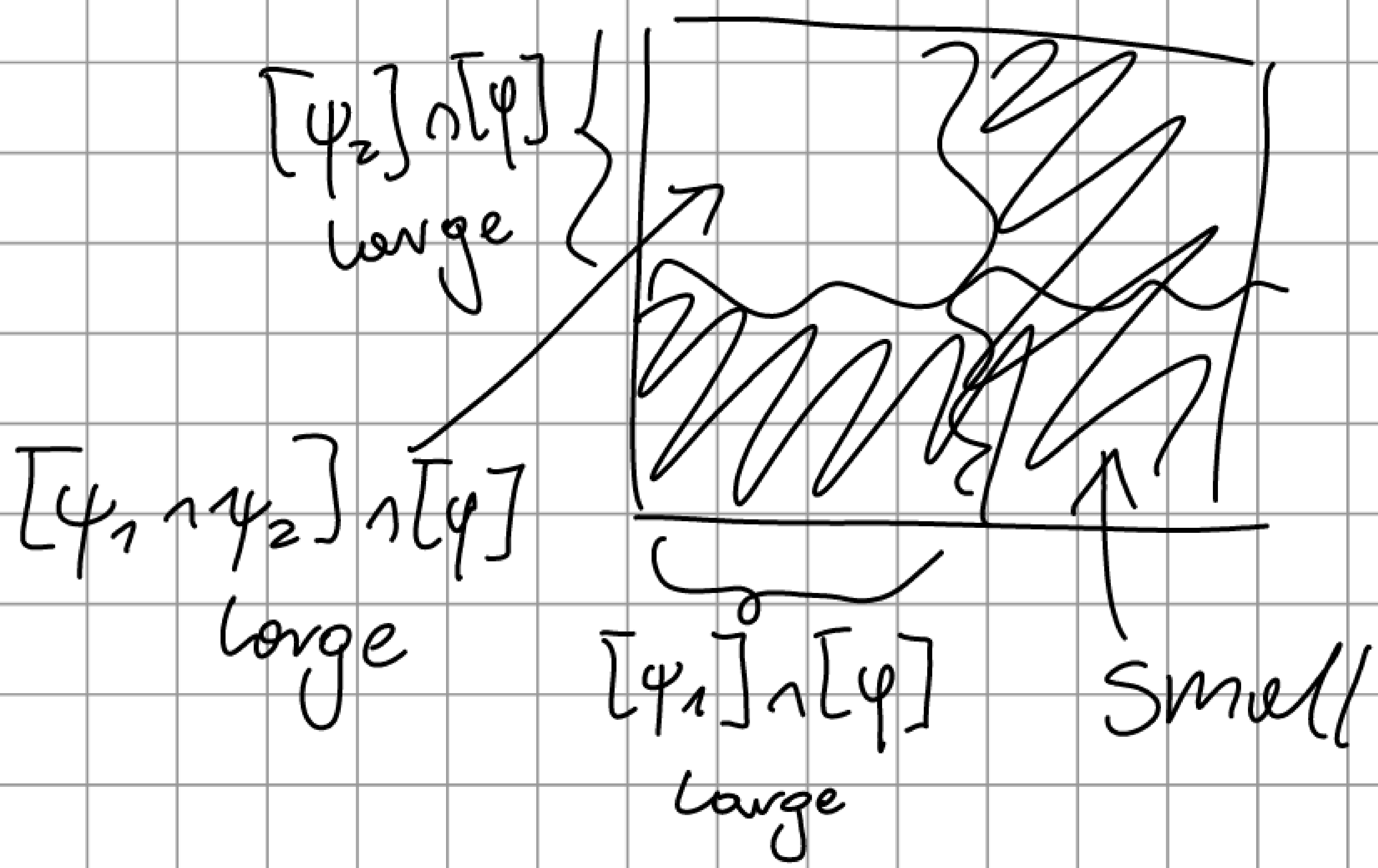
otherwise: $\varphi(x)$ small.

(a) "$x = x$" is large

(b) if $\varphi(x)$: large, then $\exists \psi_1, \psi_2 \in L(A)$ large s.t. $\varphi(\mathcal{M}) = \psi_1(\mathcal{M}) \dot{\cup} \psi_1(\mathcal{M})$

Pf. (b) <u>if not</u> then $p^* = \{\psi \in L_1(A) : \psi \wedge \varphi$ is large$\}$ is a complete type is $S_1(A) \cap [\varphi]$

1° p^*: consistent: $\psi_1, \psi_2 \in p^* \Longrightarrow \psi_1 \wedge \psi_2 \in p^*$



2° p^*: complete OK

$p^* \in S_1(A)\cap[\varphi]$: the only large type.

$$\underbrace{S_1(A)\cap[\varphi]}_{>\varkappa} = \underbrace{\{p^*\}\cup \bigcup_{\substack{\psi\in L_1(A)\\ \text{small}\\ \underbrace{\psi\vdash\varphi}_{\leq\varkappa}}} \underbrace{\left(S_1(A)\cap[\varphi]\right)}_{\leq\varkappa}}_{\leq\varkappa}$$

↯

c) a tree of large formulas in $L_1(A)$ $\varphi_\eta(x)$, $\eta \in 2^{<\omega}$ st. $\varphi_\eta(\mathcal{M}) = \varphi_{\eta 0}(\mathcal{M}) \underset{\substack{\uparrow \\ \text{by } (b)}}{\odot} \varphi_{\eta 1}(\mathcal{M})$

Let $A_0 \subseteq A$: the set of all params of $\varphi_\eta, \eta \in 2^{<\omega}$

Then $|A_0| \leqslant \aleph_0$.

For $\eta = 2^{\omega}$: $p^0_\eta = \{\varphi_{\eta|n}(x) : x < \omega\}$: a consistent 1-type over A_0

$$\begin{array}{c} p^0_\eta \\ \cap \\ p_\eta \in S_1(A_0) \end{array}$$

When $\nu \neq \eta$ then $p_\eta \neq p_\nu$

$\Downarrow$

$$|S_1(A_0)| \geqslant 2^{\aleph_0} > \aleph_0 \quad ↯$$