

# Preliminaries

MT1/1

$T$ : a complete consistent theory, in language  $L$   
with infinite models (countable)

that is,  $T = \text{Th}(M)$ ,  $M$ :  $L$ -structure.  
infinite

$L$  denotes also the set of formulas of language  $L$

$M = (|M|; \dots)$ , but  $M$  also denotes  $|M|$ .  
 $\emptyset \neq \uparrow$  universe of  $M$  (for brevity)

usually we omit  $| \cdot |$  in  $|M|$ .

$M \supseteq A$ : a set of parameters.

$L_n(A) = \{ \varphi(x_1, \dots, x_n, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in L, \bar{a} \subseteq A \}$

$L(A) = \bigcup_n L_n(A)$ , also  $L(A)$ : language  $L$   
extended by names for elements of  $A$ .

$L_n(A)$ : Lindenbaum algebra.

[formally: on  $L_n(A)$ :  $\varphi \sim \psi \Leftrightarrow T(A) \vdash \varphi \leftrightarrow \psi$   
 $\Leftrightarrow M \models \varphi \leftrightarrow \psi$

here:  $T(A) = \text{Th}(M, a)_{a \in A}$

a complete theory  
in language  $L(A)$ .



$L_m(A)/\sim$  : a Boolean algebra  
(Lindenbaum algebra)

$$[\varphi]_{\sim} \wedge [\psi]_{\sim} = [\varphi \wedge \psi]_{\sim} \text{ etc.}$$

shorthand:  $L_m(A)$  denotes also  $L_m(A)/\sim$ .

$S_m(A) = \{ \text{complete } n\text{-types over } A, \text{ in } \mathcal{M}_n \}$   
in variables  $x_1, \dots, x_n$

consistent  $n$ -type over  $A \iff$  proper filter in  $L_m(A)$

An  $n$ -type  $p(\bar{x})$  over  $A$  is complete if

$$L_m(A) \begin{cases} \cdot p(\bar{x}) : \text{consistent type} \\ \cdot \forall \varphi(\bar{x}) \in L_m(A) (\varphi(\bar{x}) \in p \text{ or } (\neg\varphi(\bar{x})) \in p) \end{cases}$$

$$S(A) := S_1(A)$$

(default)

$S_m(A)$  : topological space :

for  $\varphi(\bar{x}) \in L_m(A)$

$$[\varphi] = \{ p \in S_m(A) : \varphi \in p \}$$

basic open set [clopen]

closed and open

$S_m(A)$  : compact Hausdorff space, 0-dimensional  
(i.e. basis of clopen sets)

complete  $n$ -types /  $A \leftrightarrow$  ultrafilters in  $L_n(A)$  MT1/3

So  $S_n(A) = S(L_n(A))$ , the Stone space  
of ultrafilters in  $L_n(A)$

- the ~~type~~ topology on  $S_n(A)$  = the Stone space topology.

For  $p(\bar{x}) \in S_n(A)$

$$p(M) = \{ \bar{a} \in M^n : \bar{a} \text{ satisfies } p \}$$

$\bar{a} \models p$ , i.e.  $M \models \varphi(\bar{a})$  for every  $\varphi(\bar{x}) \in p(\bar{x})$

- The same notation for arbitrary type (also incomplete)

- A formula  $\varphi(\bar{x}) \in L(M)$ : a special case of a type  $\{ \varphi(\bar{x}) \}$ .

$$\varphi(M) = \dots$$

- When  $p \in S_n(A)$ ,  $\bar{a} \subseteq M$  and  $\bar{a} \models p$ , then

$$p = \text{tp}^M(\bar{a}/A) = \{ \varphi(\bar{x}) \in L_n(A) : M \models \varphi(\bar{a}) \}$$

Example Assume  $p(\bar{x})$ : a consistent type over  $M$ .

Then  $\exists N \supseteq M$   $p$  is realized in  $N$

i.e.  $p(N) \neq \emptyset$ .



From now on "a type" means "a consistent type". MT1/4

Def A type  $p(\bar{x})$  over  $A$  is isolated, if:

$$\exists \varphi(\bar{x}) \in L_n(A) \left\{ \begin{array}{l} \textcircled{1} \varphi(\bar{x}) \text{ is consistent (wrt } T), \text{ i.e.} \\ \varphi(M) \neq \emptyset \Leftrightarrow T(A) \vdash \exists \bar{x} \varphi(\bar{x}) \end{array} \right.$$

symbolically:  $\varphi(\bar{x}) \vdash p(\bar{x}) \rightarrow \left\{ \begin{array}{l} \textcircled{2} \varphi(\bar{x}) \\ \forall \psi(\bar{x}) \in p(\bar{x}) \quad \varphi(M) \subseteq \psi(M) \\ \updownarrow \\ T(A) \vdash \varphi(\bar{x}) \rightarrow \psi(\bar{x}) \end{array} \right.$

• When  $p(\bar{x})$ : a complete type over  $A$ , then:

$p(\bar{x})$  is isolated  $\Leftrightarrow p$  is isolated in  $S_n(A)$   
in the topological sense  
(i.e.  $\{p\}$  is open)

Tarski - Vaught test

Assume  $A \subseteq M$ . Then  $A = |N|$  for some  $N \prec M$  iff

$$\forall \varphi(x) \in L_1(A) \quad [\varphi(M) \neq \emptyset \Rightarrow \varphi(M) \cap A \neq \emptyset]$$

Construction of an elementary submodel of  $M$  containing  $A$ :

•  $A_n \subseteq M$ ,  $n < \omega$ , increasing chain of sets

recursive construction:

$$A_0 = A$$

$$A_n \subseteq A_{n+1} \subseteq M \text{ such that } \forall \psi(x) \in L_1(A_n)$$

$$[\psi(M) \neq \emptyset \Rightarrow \psi(M) \cap A_{n+1} \neq \emptyset]$$

$$A_\infty = \bigcup_{n < \omega} A_n \text{ satisfies TV-test.}$$

# Omitting types theorem

MT1/5

Assume  $p_n(\bar{x}_n)$ ,  $n < \omega$ : a family of non-isolated types in theory  $T$ , over  $\emptyset$ . Then:

$(\exists M \models T)$   $M$  omits every  $p_n$  [i.e.  $p_n(M) = \emptyset$ ]

Assume  $M, N \models T$   
 $\underset{A}{\cup}$

Def.  $f: A \rightarrow N$  is elementary ( $f: A \xrightarrow{\equiv} N$ ) if:

$$\forall \bar{a} \in A \forall \varphi(\bar{x}) \in L (M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a})))$$

$$(\Leftrightarrow) \text{tp}^M(\bar{a}) = \text{tp}^N(f(\bar{a}))$$

Elementary diagram of  $A \subseteq M$ :

$$D_e(A) = T(A) = \text{Th}(M, a)_{a \in A}$$

Remark  $f: A \rightarrow N$  is elementary  $\Leftrightarrow (N, f(a))_{a \in A} \models T(A)$

Atomic diagram of  $A \subseteq M$ :

$$D_{\text{at}}(A) = \{ \varphi \in D_{\text{el}}(A) : \varphi \text{ is a quantifier free sentence} \}$$
$$= \{ \varphi(\bar{a}) \in L(A) : M \models \varphi(\bar{a}) \text{ and } \varphi(\bar{a}) : \text{q.f.-sentence} \}$$

Remark  $f: M \rightarrow N$  is a monomorphism (i.e.:

$$f: M \xrightarrow{\cong} f(M) \subseteq N$$

↑ substructure

$$\Leftrightarrow (N, f(a))_{a \in M} \models D_{\text{at}}(M).$$



Here always  $f: M \rightarrow N$  denotes a monomorphism. MT 1/6

$M \subseteq N$  :  $M$  is a submodel (substructure) of  $N$

$M < N$  :  $M$  is an elementary submodel of  $N$ , i.e.:

$$M \subseteq N \text{ and } \text{id}_M: M \xrightarrow{\equiv} N$$

Remark Assume  $M < N$ ,  $A \subseteq M$ .

(1) Assume  $p(\bar{x}) \subseteq L_n(A)$ . Then

$p(\bar{x})$  is a consistent type in  $M \Leftrightarrow p(\bar{x})$  is a consistent type in  $N$

(2) Assume  $A \subseteq B \subseteq M$

• If  $p(\bar{x})$ : a type over  $B$ , then  $p \upharpoonright_A \stackrel{\text{def}}{=} p(\bar{x}) \cap L(A)$   
a type over  $A$

Let  $r: S_n(B) \rightarrow S_n(A)$ ,  $r(p) \stackrel{\text{def}}{=} p \upharpoonright_A$ .

Then  $r$ : continuous and "onto".

(3) If  $p(\bar{x})$ : a type over  $A$ , then  $\exists q(\bar{x}) \in S_n(A)$   $p(\bar{x}) \subseteq q(\bar{x})$

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Saturation, universality, (strong) homogeneity.

Let  $\kappa \in \mathbb{C}N$ ,  $\kappa \neq \aleph_0$ .

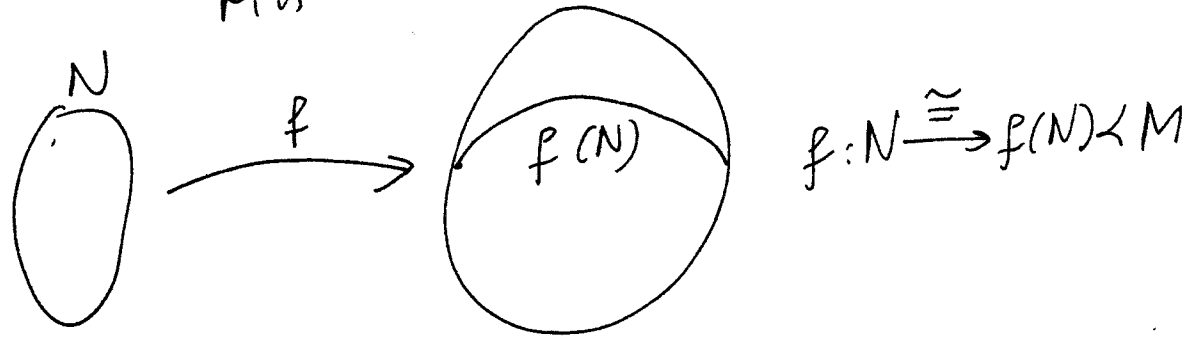
Def. (1)  $M$   $\kappa$ -saturated if  $\forall A \subseteq M \forall p \in S_n(A)$   $p(M) \neq \emptyset$   
(nasyrony)  $|A| < \kappa$

$M$  is saturated if  $M$  is  $\|M\|$ -saturated

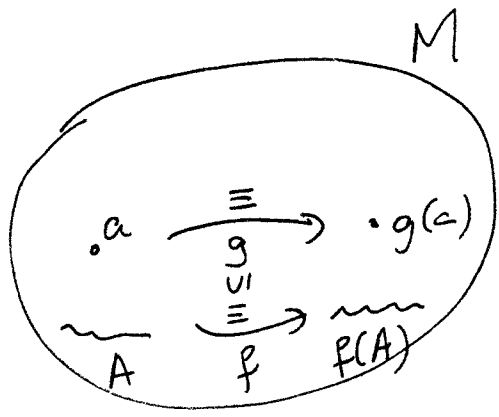
(2)  $M$  is  $\kappa$ -universal if  $\forall N \equiv M$  ( $\|N\| \leq \kappa \Rightarrow \exists f: N \xrightarrow{\equiv} M$ )  
elementarily equivalent  
i.e.  $\text{Th}(N) = \text{Th}(M)$

$M$ : universal  $\Leftrightarrow \text{||M||-universal}$

MT1/7



(3)  $M$ :  $\kappa$ -homogeneous if  $\forall A \subseteq M \forall a \in M \forall f: A \xrightarrow{\cong} M$   
 $|A| < \kappa \quad \exists g: A \cup \{a\} \xrightarrow{\cong} M$   
 homogeneous =  $\text{||M||-homogeneous}$ .



4.  $M$  strongly  $\kappa$ -homogeneous if  $\forall A \subseteq M \forall f: A \xrightarrow{\cong} M$   
 $|A| < \kappa \quad \exists g: M \xrightarrow{\cong} M$

strongly homogeneous = strongly  $\text{||M||-homogeneous}$ .

5.  $M$  is  $\kappa$ -compact if  $(\forall 1$ -type  $p(x)$  over  $M)$   
 $(|p| < \kappa \Rightarrow p(M) \neq \emptyset)$

### Elementary chains of structures

Def  $\langle M_\alpha : \alpha < \mu \rangle, \mu \in \text{Ord}$ , : an elementary chain of structures if  $(\forall \alpha < \beta < \mu) M_\alpha \prec M_\beta$ .

Union of chain (when  $\mu \in \text{Lim}$ )

$$M_\mu = \bigcup_{\alpha < \mu} M_\alpha ?$$

•  $|M_\mu| := \bigcup_{\alpha < \mu} |M_\alpha|$

•  $c \in L$  constant symbol

$c^{M_\mu} = c^{M_\alpha}$  for  $\alpha < \mu$

•  $P$  : relation symbol

$P^{M_\mu}(a_1, \dots, a_n) \Leftrightarrow M_\alpha \models P(a_1, \dots, a_n)$  for  $\alpha < \mu$   
 $\bigcap_{\alpha < \mu} |M_\alpha|$  sufficiently large  
 [so that  $\bar{a} \subseteq M_\alpha$ ]

•  $f^{M_\mu}(\bar{a}) = b \Leftrightarrow M_\alpha \models f(\bar{a}) = b$  for  $\alpha < \mu$   
 sufficiently large

Fact (Tarski)  $M_\alpha \prec M_\mu$  for all  $\alpha < \mu$ .

Proof (1)  $M_\alpha \subseteq M_\mu$  (substructure): exercise

(2)  $\forall \varphi(\bar{x}) \in L \forall \alpha < \mu \forall \bar{a} \subseteq M_\alpha (M_\alpha \models \varphi(\bar{a}) \Leftrightarrow M_\mu \models \varphi(\bar{a}))$

(a)  $\varphi$  atomic :  $M_\alpha \subseteq M_\mu \checkmark$

(b)  $\varphi = \psi_1 \wedge \psi_2, \varphi = \neg \psi$  : easy

(c)  $\varphi(\bar{x}) = \exists y \psi(\bar{x}, y)$

$M_\alpha \models \varphi(\bar{a}) \Rightarrow M_\alpha \models \psi(\bar{a}, b)$  for some  $b \in M_\alpha$

$\Downarrow$  ind. assumption for  $\psi$

$M_\mu \models \psi(\bar{a}, b)$

$\Downarrow$   
 $M_\mu \models \varphi(\bar{a})$



$$M_\mu \models \varphi(\bar{a}) \Rightarrow M_\mu \models \psi(\bar{a}, b) \text{ for some } b \in M_\mu$$

$$\exists y \psi(\bar{a}, y)$$

$\Downarrow$  ind. assumption

$$b \in M_\beta \text{ for some } \alpha \leq \beta < \mu$$

$$M_\beta \models \psi(\bar{a}, b)$$

$\Downarrow$

$$M_\beta \models \varphi(\bar{a})$$

$$\Downarrow M_\alpha < M_\beta$$

$$M_\alpha \models \varphi(\bar{a})$$

Elementary directed systems of structures:

Let  $(I, \leq)$ : a directed set, i.e.:

(1)  $\leq$ : partial order on  $I$

(2)  $(\forall a, b \in I)(\exists c \in I)(a \leq c \wedge b \leq c)$

Example  $J$ : a set  $\mapsto ([J]^{<\omega}, \leq)$ : directed set.

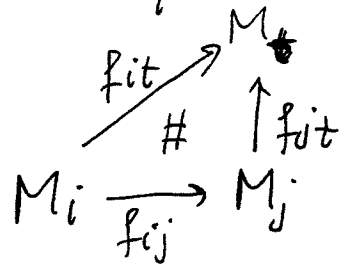
Directed system of structures:

$$\mathcal{M} = (M_i, f_{ij})_{i \leq j \in I}$$

connecting functions  $f_{ij}: M_i \rightarrow M_j$ ,  $f_{ii} = id_{M_i}$ . such that

$$(\forall i \leq j \leq t \in I) f_{it} = f_{jt} \circ f_{ij}$$

(compatibility)



System  $\mathcal{M}$  is elementary if all  $f_{ij}$  are elementary.

# Example Elementary chain $(M_\alpha)_{\alpha < \mu}$

MT1/10

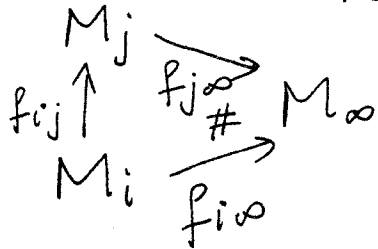
$\mathcal{M} = (M_\alpha, f_{\alpha\beta})_{\alpha \leq \beta < \mu}$   $f_{\alpha\beta} = id_{M_\alpha} : M_\alpha \xrightarrow{\cong} M_\beta$   
elementary directed system of structures

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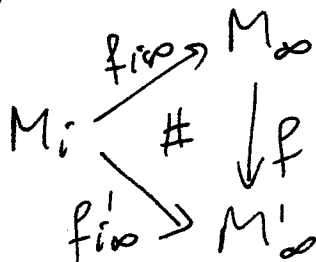
Direct limit of a directed system  $\mathcal{M} : M_\infty = \varinjlim \mathcal{M}$

$(M_\infty, f_{i\infty})_{i \in I}$ , where  $f_{i\infty} : M_i \rightarrow M_\infty$  such that

(1)  $\forall i \leq j \in I$   $f_{i\infty} = f_{j\infty} \circ f_{ij}$  [compatible with connecting functions]



(2)  $(\forall (M'_\infty, f'_{i\infty})_{i \in I})$  satisfying (1)  $\exists ! f : M_\infty \rightarrow M'_\infty$   
(universality)  $(\forall i \in I) f'_{i\infty} = f \circ f_{i\infty}$



Fact  $M_\infty$  exists (and is unique up to  $\cong$ ).

If  $\mathcal{M}$  is elementary, then  $f_{i,\infty} : M_i \xrightarrow{\cong} M_\infty$ .

Proof 1. Construction of  $M_\infty$ :

$S := \dot{\bigcup}_{i \in I} |M_i|$  : formally disjoint union.

$\sim$  on  $S$  : an equivalence relation

$$M_i \quad M_j \quad \text{MTI/II}$$

$$\downarrow \quad \downarrow \quad \text{def}$$

$$x \sim y \Leftrightarrow f_{it}(x) = f_{jt}(y) \text{ for some } (= \text{every})$$

$$t \geq i, j$$

exercise:  $\sim$  is transitive.

$$|M_\infty| := S/\sim$$

- $\sim \upharpoonright |M_i|$ : the equality (because  $f_{ij}$ : 1-1 (monomorphism))
- $f_{i\infty}(x) = x/\sim$ ,  $f_{i\infty}: |M_i| \xrightarrow{1-1} |M_\infty|$ .

L-structure on  $|M_\infty|$ :

- $c^{M_\infty} = c^{M_i}/\sim$
- $P^{M_\infty}(a_{i_1}/\sim, \dots, a_{i_m}/\sim) \Leftrightarrow M_t \models P(f_{i_1 t}(a_{i_1}), \dots, f_{i_m t}(a_{i_m}))$   
 $a_{ij} \in M_{ij}$  for  $t \geq i_1, \dots, i_m$
- $f^{M_\infty}$ : similarly

the rest is an exercise.

How to extend elementary mappings?

MT2/1

~~Def.~~  $\mathbf{BA}_{lg}$ : Category of Boolean algebras

$\mathbf{Comp}_0$ :  $\mathbb{T}$  — of compact Hausdorff 0-dimensional spaces

$$F: \mathbf{BA}_{lg} \rightarrow \mathbf{Comp}_0$$

$$G: \mathbf{Comp}_0 \rightarrow \mathbf{BA}_{lg}$$

$$F(A) = S(A)$$

$$G(X) = C(\text{open}(X))$$

$F, G$ : contravariant functors "inverse" to each other

~~$(F, G)$  is a duality of categories. (look it up)~~  
Categories  $\mathbf{BA}_{lg}$  and  $\mathbf{Comp}_0$  are dually equivalent.

$A, B$ : Boolean algebras

$$f: A \rightarrow B \text{ homomorphism} \Rightarrow F(f): S(B) \rightarrow S(A)$$

$$F(f)(p) = f^{-1}[p]$$

continuous.

$$\text{Assume } f: A \xrightarrow{\cong} B$$

$$\begin{matrix} \cap & & \cap \\ T \neq M & , & N \neq T \end{matrix}$$

$$\text{Then } \hat{f}: L_n(A) \rightarrow L_n(B)$$

$$\hat{f}(\varphi(\bar{x}, \bar{a})) = \varphi(\bar{x}, f(\bar{a}))$$

homomorphism

of Boolean algebras.

even: monomorphism.

We skip  $\hat{\quad}$  in  $\hat{f}$ , so:

$$f: L_n(A) \rightarrow L_n(B) \text{ monomorphism}$$

$$f^*: S_n(B) \rightarrow S_n(A) \text{ epimorphism in } \mathbf{Comp}_0$$

i.e.: continuous onto

Lemma (on extensions of elementary mappings) MT2/2

Assume  $M, N \models T$ ,  $A \subseteq M$ ,  $B \subseteq N$ ,  $f: A \xrightarrow{\equiv} B$  "onto".

Assume  $\overset{\psi}{\underset{a}{\#}}, \overset{\psi}{\underset{b}{\#}}$ ,  $p = \text{tp}(a/A)$ ,  $q = \text{tp}(b/B)$ .

Then  $f \cup \{Ka, b\}$  is elementary  $\Leftrightarrow f^*(q) = p$ .

[here  $f^*: S(B) \xrightarrow{\cong} S(A)$   
homeomorphism]

Proof exercise.

Def.  $M$  is  $(< \kappa_0)$ -universal  $\Leftrightarrow \forall n \forall p \in S_n(\emptyset) p(M) \neq \emptyset$ .

Remark  $M: \kappa$ -universal  $\Rightarrow M: (< \kappa_0)$ -universal.

Proof Let  $p \in S_n(\emptyset)$ .

Choose a countable  $N \models T$  with  $p(N) \neq \emptyset$ .

$M: \kappa$ -universal  $\Rightarrow \exists f: N \xrightarrow{\equiv} M$   
 $\overset{\psi}{\underset{a}{\#}} \neq p \mapsto \overset{\psi}{\underset{f(a)}{\#}} \neq p$ .

Thm. (1)  $M: \kappa$ -saturated  $\Rightarrow M: \kappa$ -homogeneous  
and  $\kappa$ -universal.

(2)  $M: \kappa$ -~~universal~~ <sup>homogeneous</sup> and  $(< \kappa_0)$ -universal  $\Rightarrow$   
 $M: \kappa$ -saturated.

Proof. (1)  $\kappa$ -homogeneity of  $M$ :

Assume  $f: A \xrightarrow{\equiv} M$ ,  $A \subseteq M$ ,  $|A| < \kappa$ ,  $a \in M$ .

We seek  $b \in M$  s.t.  $g = f \cup \{ \langle a, b \rangle \}$  elementary

MT2/3

↕ Lemma

$$f^*(tp(b/B)) = tp(a/A)$$

⊗ Let  $p = tp(a/A)$ ,  $q = (f^*)^{-1}(p) \in S_1(B)$

↑  
 $S_1(A)$

Let  $b \in M$  (exists by  $\kappa$ -saturation)  
↑  
good. of  $M$

•  $\kappa$ -universality of  $M$ :

Assume  $N \equiv M$ ,  $\|N\| \leq \kappa$ .

We seek  $f: N \xrightarrow{\equiv} M$ .

Let  $\{a_\alpha : \alpha < \mu\}$ : an enumeration of  $N$ ,  $\mu = \|N\|$ .

We define  $f(a_\alpha)$  by induction on  $\alpha < \mu$ :

• Suppose  $f(a_\beta)$  defined for all  $\beta < \alpha$  so that

$$f: \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} M$$

Want to find  $f(a_\alpha)$  so that

$$f: \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} M.$$

~~By the Lemma it is enough that~~

Let  $p = tp(a_\alpha / \{a_\beta : \beta < \alpha\})$ .

By the lemma it is enough to find  $f(a_\alpha) \in M$

so that  $f^*(tp(f(a_\alpha) / \{f(a_\beta) : \beta < \alpha\})) = p$ .

So let  $q_f = (f^*)^{-1}(p) \in S_{\kappa}(\underbrace{\{f(a_\beta) : \beta < \alpha\}}_{\text{power} < \kappa})$

(MT2/4)

$M$   $\kappa$ -saturated  $\Rightarrow q_f$  realized in  $M$ .

Let  $f(a_\alpha) \in M$  s.t.  $f(a_\alpha) \neq q_f$ .

(2) Assume  $M$  is  $\kappa$ -homogeneous &  $(< \aleph_0)$ -~~saturated~~ <sup>universal</sup>.

Want:  $M$ :  $\kappa$ -saturated.

So: Let  $A \subseteq M$ ,  $|A| < \kappa$ ,  $p \in S_{\kappa}(A)$ . Show:  $p(M) \neq \emptyset$ .

Induction on  $|A|$ .

Case (a):  $|A| < \aleph_0$ .

$N$

$\exists N \supseteq M$   $p(N) \neq \emptyset$ . So let  $b \in p$ .

Let  $A^* = A \cup \{b\}$   
 $\quad \quad \quad \cup \{a_1, \dots, a_k\}$

Let  $q_f = t_p^N(a_1, \dots, a_k, b) \in S_{\kappa+1}(\emptyset)$

$q_f$  is realized in  $M$  ( $(< \aleph_0)$ -universality),

by  $\langle \underbrace{a'_1, \dots, a'_k}_{A'}, b' \rangle$

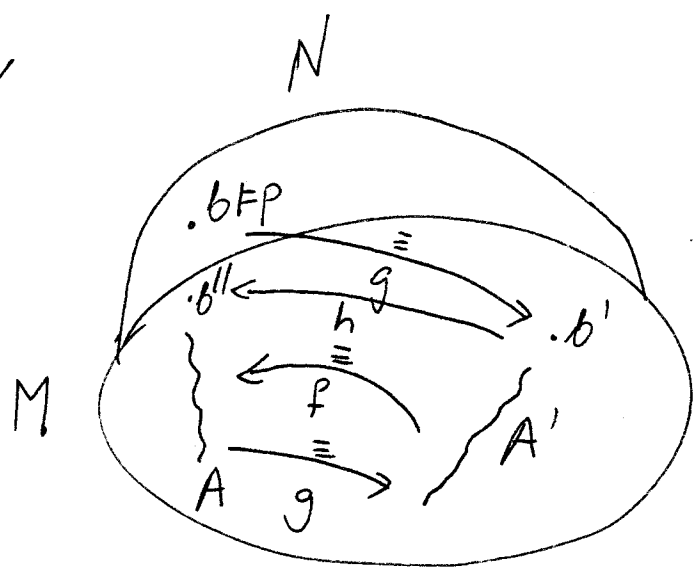
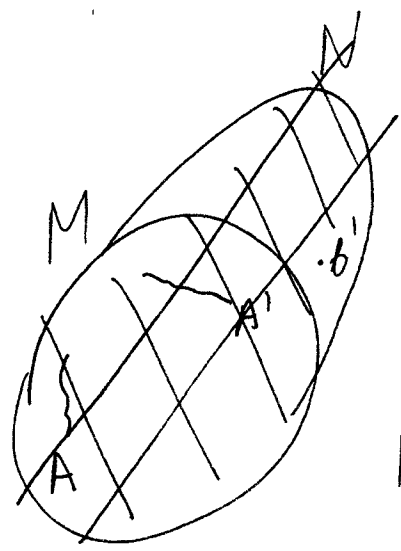
Let  $g: A \cup \{b\} \rightarrow A' \cup \{b'\}$ ,  $g(a_i) = a'_i$ ,  $g(b) = b'$ .

$g$ : elementary.

$$\Rightarrow g \uparrow_A : A \xrightarrow{\cong} A'$$

$$\Downarrow f := (g \uparrow_A)^{-1} : A' \xrightarrow{\cong} A$$

M:  $\kappa$ -homogeneous  $\Rightarrow \exists h : A' \cup \{b''\} \xrightarrow{\cong} A \cup \{b''\}$   
for some  $b'' \in M$ .



$$\begin{array}{ccc} A \cup b & \xrightarrow{\cong} & A \cup b'' \\ g \downarrow \cong & & \cong \uparrow h \\ A' \cup b' & & \end{array}$$

Let  $s = h \circ g$   
 $s \uparrow_A = \underbrace{(h \uparrow_{A'})}_{\cong} \circ (g \uparrow_A) = \text{id}_A$   
 $\Downarrow$

$$s^*(\cancel{tp(b''/A)}) = \cancel{tp(b''/A)}$$

$$s \uparrow_A = \text{id}_A \Rightarrow s^* : S(A) \xrightarrow{\cong} S(A)$$

$\cong$   
 $\text{id}_{S(A)}$

$$\text{hence: } p = tp(b/A) \underset{\uparrow \text{Lemma}}{=} s^*(tp(b''/A)) \underset{\uparrow}{=} tp(b''/A)$$

and  $b'' \neq p$   $s^* = \text{id}_{S(A)}$



Case (b)  $|A| = \mu$ ,  $x_0 \leq \mu < \kappa$ .

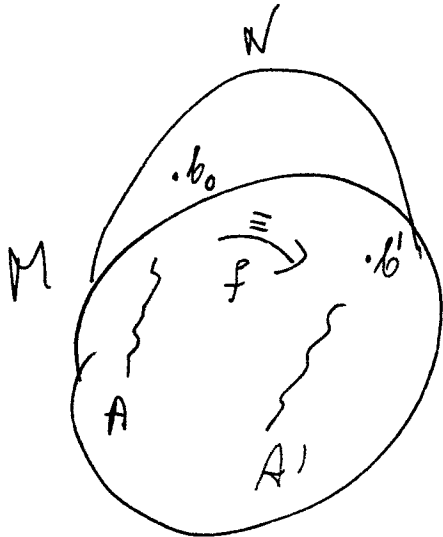
$$A = \{a_\alpha : \alpha < \mu\}, p \in S_1(A).$$

$$p \upharpoonright \emptyset \in S_1(\emptyset) \implies \exists b' \in M \quad b' \neq p \upharpoonright \emptyset,$$

$M: \langle x_0 \rangle$ -universal

$$\exists N \supset M \quad \exists b_0 \in N$$

$\begin{matrix} \pi \\ \downarrow \\ p \end{matrix}$



Will find  $A' = \{a'_\alpha : \alpha < \mu\} \subseteq M$

s.t.

$$f: A b_0 \longrightarrow A' b'$$

given by  $f(a_\alpha) = a'_\alpha$   
 $f(b_0) = b'$

is elementary!

We find  $a'_\alpha, \alpha < \mu$  by induction on  $\alpha < \mu$ .

So suppose  $\alpha < \mu$  and  $a'_\beta$  already defined for all  $\beta < \alpha$

so that  $\boxed{p \upharpoonright \{a_\beta : \beta < \alpha \cup b_0\}} = \{a_\beta : \beta < \alpha \cup b_0\} \equiv \{a'_\beta : \beta < \alpha \cup b'\}$

$\equiv: f_0$

We look for  $a'_\alpha$ .

Let  $q = \text{tp}(a_\alpha / \{a_\beta : \beta < \alpha \cup \{b_0\}\})$

Then  $(f_0^*)^{-1}(q) \in S(\underbrace{\{a'_\beta : \beta < \alpha \cup \{b'\}\}}_{\text{power} < \mu \leq |A|})$

power  $< \mu \leq |A|$

By the lemma it is enough that  $a'_\alpha \neq (f_0^*)^{-1}(q)$ .

But  $M: \kappa$ -

$M$ .

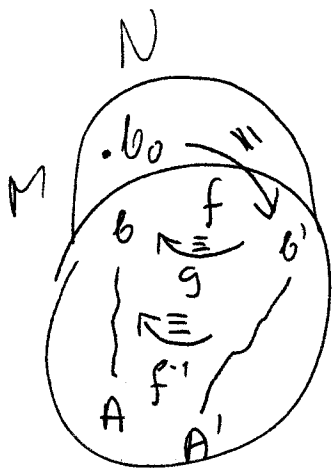
~~Let~~ By the inductive assumption on  $A$ :

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$(f_0^*)^{-1}(q)$  is realized in  $M$ , so we are done with constructing  $A'$ .

Now:  $f^{-1}: A' \xrightarrow{\cong} A$  in  $M \leftarrow \kappa$ -homogeneous, so

$\exists \tilde{g}: A'b' \xrightarrow{\cong} Ab$  for some  $b \in M$ .



~~Let~~ Let  $s = g \circ f$

$$s: Ab_0 \xrightarrow{\cong} Ab$$

$$s \uparrow_A = (g \uparrow_{A'}) \circ (f \uparrow_A) = id_A,$$

$$\parallel \quad (f \uparrow_A)^{-1} \quad \text{so } tp(b_0^{\#}/A) = tp(b/A)$$

and  $p$  is realized in  $M$ .

### Corollary

$M$  is  $\kappa$ -saturated  $\Leftrightarrow M$  is  $\kappa$ -homogeneous and  $\kappa$ -universal and  $\kappa$ -universal.

Proof  $\Rightarrow$  by Thm (1).

$\Leftarrow$   $\kappa$ -homogeneous +  $\kappa$ -universal  $\Rightarrow$

$\kappa$ -homogeneous +  $(\langle \cdot \rangle_0)$ -universal  $\Rightarrow$   $\kappa$ -saturated  
Thm (2)

Properties of saturated models.

Thm. Assume  $M, N \models T$  saturated models of the same power. Then  $M \cong N$ .

Proof  $M = \{m_\alpha : \alpha < \kappa\}$ ,  $N = \{n_\alpha : \alpha < \kappa\}$ ,  
 $\kappa = \|M\| = \|N\|$ . We find  $f: M \xrightarrow{\cong} N$

back-and-forth method:

$$f = \bigcup_{\alpha < \kappa} f_\alpha \quad f_\alpha: M \xrightarrow{\cong} N \quad \text{s.t.}$$

partial, elementary

$$(1) \quad \begin{array}{l} m_\alpha \in \text{Dom } f_{\alpha+1} \\ n_\alpha \in \text{Rng } f_{\alpha+1} \end{array} \quad , \quad |f_\alpha| \leq 2 \cdot |\alpha|$$

$$(2) \quad f_0 = \emptyset$$

$$(3) \quad \text{For } \delta \in \text{Lim}, \quad f_\delta = \bigcup_{\alpha < \delta} f_\alpha.$$

$$(4) \quad f_{\alpha+1} = f_\alpha \cup \left\{ \left\langle \underset{\substack{\uparrow \\ N}}{m_\alpha}, m \right\rangle, \left\langle m, \underset{\substack{\uparrow \\ M}}{n_\alpha} \right\rangle \right\}$$

Inductive step:

Suppose we have  $f_\alpha$ . Want:  $f_{\alpha+1}$ .

Let  $A_\alpha = \text{Dom } f_\alpha \subseteq M$ ,  $B_\alpha = \text{Rng } f_\alpha \subseteq N$ .

$$f_\alpha: A_\alpha \xrightarrow{\cong} B_\alpha$$

$$\downarrow$$

$$f_\alpha^*: S(B_\alpha) \xrightarrow{\cong} S(A_\alpha).$$

"forth": Find  $n \in N$  st.  $f_\alpha \cup \{ \langle m_\alpha, n \rangle \}$  elementary MT2/9

$$\begin{array}{c} \Downarrow \\ (f_\alpha^*)^{-1}(tp(m_\alpha/A_\alpha)) = tp(n/B_\alpha). \end{array}$$

Let  $p = tp(m_\alpha/A_\alpha)$ .

So  $(f_\alpha^*)^{-1}(p) \in S(B_\alpha)$  is realized in  $N$  by some  $n$ .

"~~back~~": similarly.  
back

Thm Assume  $M, N \models T$  are homogeneous, of the same power and  $\forall n < \omega \forall p \in S_n(\emptyset) (p(M) \neq \emptyset \Leftrightarrow p(N) \neq \emptyset)$ .  
Then  $M \cong N$ .

Lemma Under the assumptions of the Thm,

$$\forall A \subseteq M \exists f: A \xrightarrow{\cong} N.$$

Proof. Induction on  $|A|$ .

Case (a)  $|A| < \aleph_0$ .  $A = \{a_1, \dots, a_n\}$ .

Let  $p = tp(\langle a_1, \dots, a_n \rangle) \in S_n(\emptyset)$ . realized in  $M$   
 $\Downarrow$   
realized in  $N$

by some  $\langle b_1, \dots, b_n \rangle \in N$ .  
 $f(a_i) = b_i$  is good.

Case (b)  $|A| = \mu \geq \aleph_0$ ,  $A = \{a_\alpha : \alpha < \mu\}$

We find  $f(a_\alpha)$  by induction on  $\alpha < \mu$ .

Inductive step.

Suppose  $\alpha < \mu$  and for every  $\beta < \alpha$  we have  $f(a_\beta)$

$$\text{s.t. } f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} N.$$

We shall find  $f(a_\alpha) \in N$  s.t.  $f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} N.$

Let  $a_{<\alpha} := \{a_\beta : \beta < \alpha\}$ . Likewise  $a_{\leq \alpha}$ .

$$|a_{\leq \alpha}| < \mu = |A|$$

By inductive assumption:  $\exists g : a_{\leq \alpha} \xrightarrow{\equiv} N.$

$$\text{Then } f \circ g^{-1} : \underbrace{g(a_{<\alpha})}_N \xrightarrow{\equiv} \underbrace{f(a_{<\alpha})}_N$$

By homogeneity of  $N$ :  $\exists f(a_\alpha) \in N$  s.t.

$$f \circ g^{-1} : \underbrace{g(a_{<\alpha}) \cup g(a_\alpha)}_{g(a_{\leq \alpha})} \xrightarrow{\equiv} f(a_{<\alpha}) \cup f(a_\alpha) = f(a_{\leq \alpha})$$

$$\text{Then } f = (f \circ g^{-1}) \circ g : a_{\leq \alpha} \xrightarrow{\equiv} N.$$

Proof of the theorem

$$\kappa := \|M\| = \|N\|$$

$f : M \xrightarrow{\cong} N$  constructed by back-and-forth method

$$f = \bigcup_{\alpha < \kappa} f_\alpha, \quad f_\alpha : M \xrightarrow{\cong} N \text{ (partial elementary), } \alpha < \kappa$$

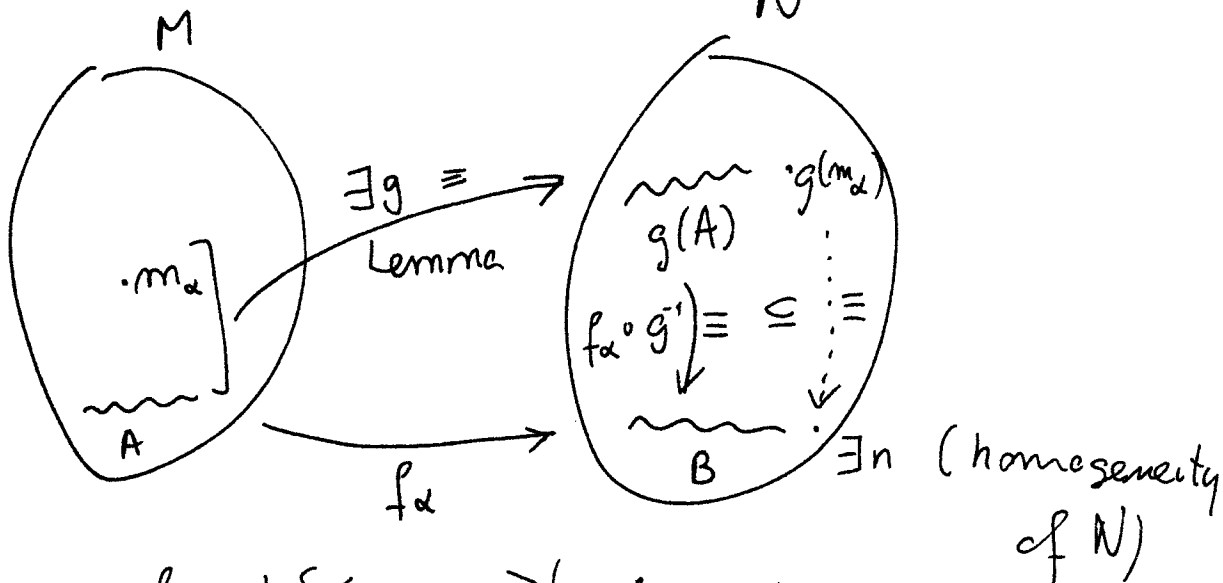
$|f_\alpha| \leq 2 \cdot |\alpha|$  + the same conditions as in the previous thm.

inductive step  $f_\alpha \mapsto f_{\alpha+1}$

$A = \text{Dom } f_\alpha$

$B = \text{Rng } f_\alpha$

"forth"



~~$f_\alpha \cup \{ \langle m_\alpha, n \rangle \}$  elementary~~

$h = (f_\alpha \circ g^{-1}) \upharpoonright_{g(A)} \cup \{ \langle g(m_\alpha), n \rangle \}$  elementary

$h \circ g : A \cup m_\alpha \xrightarrow{\equiv} B \cup n \subseteq N$   
 $\cup$   
 $f_\alpha$

"back": similarly.

Constructions of models:

MT3/1

- Saturated,  $\Rightarrow$  • (strongly) homogeneous  $\bar{\equiv}$

Thm.  $\kappa = 2^{<\kappa}$ ,  $\kappa \in \text{Reg}$ ,  $\kappa > \aleph_0 \Rightarrow \exists M \models T$   
 saturated, of power  $\kappa$ .

$\parallel$   
 $\kappa^{<\kappa} = \kappa$

Proof

(\*)  $|S_i(A)| \leq 2^{|A| + \aleph_0}$ , because:  $|L_i(A)| = |A| + \aleph_0$

Here:  $|A| < \kappa \Rightarrow |S_1(A)| \leq \kappa$ .

Lemma  $N \models T$ ,  $\|N\| \leq \kappa \Rightarrow X_N := \bigcup \{S_i(A) : A \subseteq N \text{ \& } |A| < \kappa\}$   
 the set has power  $\leq \kappa$ .

Pf  $|\{A \subset N : |A| < \kappa\}| \leq \kappa^{<\kappa} = \kappa$ .

- $|S_i(A)| \leq \kappa$  for such  $A$ .

Proof of the thm.

$M_\alpha$ ,  $\alpha < \kappa$ : elementary chain of models of  $T$  of power  $\kappa$ .

- $M_0$ : whatever

•  $M_\delta = \bigcup_{\alpha < \delta} M_\alpha$ , when  $\delta < \kappa$  limit.

•  $M_{\alpha+1} \supset M_\alpha$  such that  $\forall p \in X_{M_\alpha} p(M_{\alpha+1}) \neq \emptyset$ :

$$T' = \text{Th}(M_\alpha, m)_{m \in M_\alpha} \cup \bigcup_{\beta < \kappa} \{ \varphi(c_\beta) : \varphi(x) \in p_\beta \},$$

where  $X_{M_\alpha} = \{ p_\beta : \beta < \kappa \}$

$\uparrow$   
 new constant symbols,

and  $T'$  in language  $L(M_\alpha) \cup \{ c_\beta : \beta < \kappa \}$ .

$T^1$ : consistent, has model of power  $\kappa$ :  $M_{\alpha+1}$   
such that  $M_\alpha < M_{\alpha+1}$ .

$M = \bigcup_{\alpha < \kappa} M_\alpha$ : of power  $\kappa$ , saturated:

Let  $A \subseteq M$ ,  $|A| < \kappa$ , and  $p \in S_1^M(A)$   
 $\kappa \in \text{Reg} \Rightarrow A \subseteq M_\alpha$  for some  $\alpha < \kappa$ . CW  
↓

proof:  $A = \{a_\beta : \beta < \mu\}$  for some  $\mu < \kappa$ .

$\forall \beta < \mu \exists \alpha_\beta < \kappa \ a_\beta \in M_{\alpha_\beta}$

$\{\alpha_\beta : \beta < \mu\} \subseteq \kappa$ ,  $\mu < \text{cf}(\kappa) = \kappa$

$\Rightarrow \exists \alpha < \kappa \ \forall \beta < \mu \ \alpha_\beta < \alpha$   
↑  
 $A \subseteq M_\alpha$ .

~~Let~~  $M_\alpha < M \Rightarrow p \in S_1^{M_\alpha}(A) = S_1^M(A)$

$p$  realized in  $M_{\alpha+1}$  by some  $a \in M_{\alpha+1}$

$a \models p$  in  $M_{\alpha+1} \Rightarrow a \models p$  in  $M$ .

$M_{\alpha+1} < M$

Monster model:

Let  $\bar{\kappa}$ : a large cardinal number.

"Ideal model"  $M \models T$ : saturated of power  $\bar{\kappa}$

because:  $\forall M \models T$  ( $\|M\| < \bar{\kappa} \Rightarrow \exists M' < M \ M \cong M'$ ).



# Advantages of saturated model $M$ :

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(i) universality

(ii) strong homogeneity

~~More~~ <sup>More</sup> Weakly (a bit):

(1)  $\bar{\kappa}$ -universality

(2) strong  $\bar{\kappa}$ -homogeneity

$\text{Aut}(M)$ : the group of automorphisms of  $M$

$\text{Aut}(M/A) = \{ f \in \text{Aut}(M) : f|_A = \text{id}_A \}$ : automorphisms of  $M$  over  $A$   
 $A \subseteq M$

Lemma Assume  $M$  is strongly  $\kappa$ -homogeneous,  $\kappa$ -saturated,  $A \subseteq M$ ,  $|A| < \kappa$ . Then:

(1) For  $a, b \in M$  ( $\text{tp}(a/A) \stackrel{!}{=} \text{tp}(b/A) \Leftrightarrow a, b$  are in the same orbit of  $\text{Aut}(M/A)$  on  $M$ ).

(2) [orbits  $\text{Aut}(M/A)$  on  $M^n$ ]  $\overset{1:1}{\underset{\text{onto}}{\longleftrightarrow}} S_n(A)$

Proof (1)  $\Leftarrow$ :  $f \in \text{Aut}(M/A)$ ,  $f(a) = b$

$$\Downarrow \text{tp}(a/A) = \text{tp}(b/A)$$

$\Rightarrow$ :  $\text{tp}(a/A) = \text{tp}(b/A) \Rightarrow f: Aa \xrightarrow{\cong} Ab$

strong  $\kappa$ -homogeneity  $f|_A = \text{id}_A, f(a) = b$

$|A| < \kappa \Rightarrow f \in g \in \text{Aut}(M), g \in \text{Aut}(M/A)$   
 $g(a) = b$ :  $a, b$  in the same orbit of  $\text{Aut}(M/A)$

$$(2) M^n \supseteq \mathcal{O} \xrightarrow[\varphi]{(1)} p_{\mathcal{O}} \in S_n(A)$$

$\uparrow$   
 orbit of  
 $\text{Aut}(M/A)$

$\parallel$   
 common  
 type  $tp(a/A)$   
 for  $a \in \mathcal{O}$ .

$\mathcal{O} : \{ \text{orbits of } \text{Aut}(M/A) \text{ on } M^n \}$

$$\downarrow \varphi$$

$$S_n(A)$$

$$\mathcal{O}_1 \neq \mathcal{O}_2 \xrightarrow{(1)} p_{\mathcal{O}_1} \neq p_{\mathcal{O}_2} \quad \boxed{\text{so } \varphi: 1-1}$$

[if  $p_{\mathcal{O}_1} = p_{\mathcal{O}_2}$  then let  $a \in \mathcal{O}_1, b \in \mathcal{O}_2 \Rightarrow \exists g \in \text{Aut}(M/A)$

$M: \kappa$ -saturated  $\Rightarrow \varphi: \text{"onto"}$   $g(a) = b \quad \checkmark$

Def Let  $\bar{\kappa} : \text{a (large) cardinal number,}$

$M \models T$  monster model, if  $M: \bar{\kappa}$ -saturated,  
 (w.r. to  $\bar{\kappa}$ ) strongly  $\bar{\kappa}$ -homogeneous

Thm. Assume  $\aleph_0 \leq \kappa \in \mathcal{C.N.}$  Then

$\exists M: \kappa$ -saturated ~~is~~ strongly  $\bar{\kappa}$ -saturated.

Proof  $M = \bigcup_{\alpha < \kappa^+} M_\alpha$  : union of elementary chain  
 s.t.:

(1)  $M_0 \models T$  any

(2)  $M_\delta = \bigcup_{\alpha < \delta} M_\alpha$  if  $\delta \in \text{Lim}$ ,

(3)  $M_{\alpha+1} \succ M_\alpha$  s.t.:

(a)  $\forall p \in S_1(M_\alpha)$   $p$  realized in  $M_{\alpha+1}$

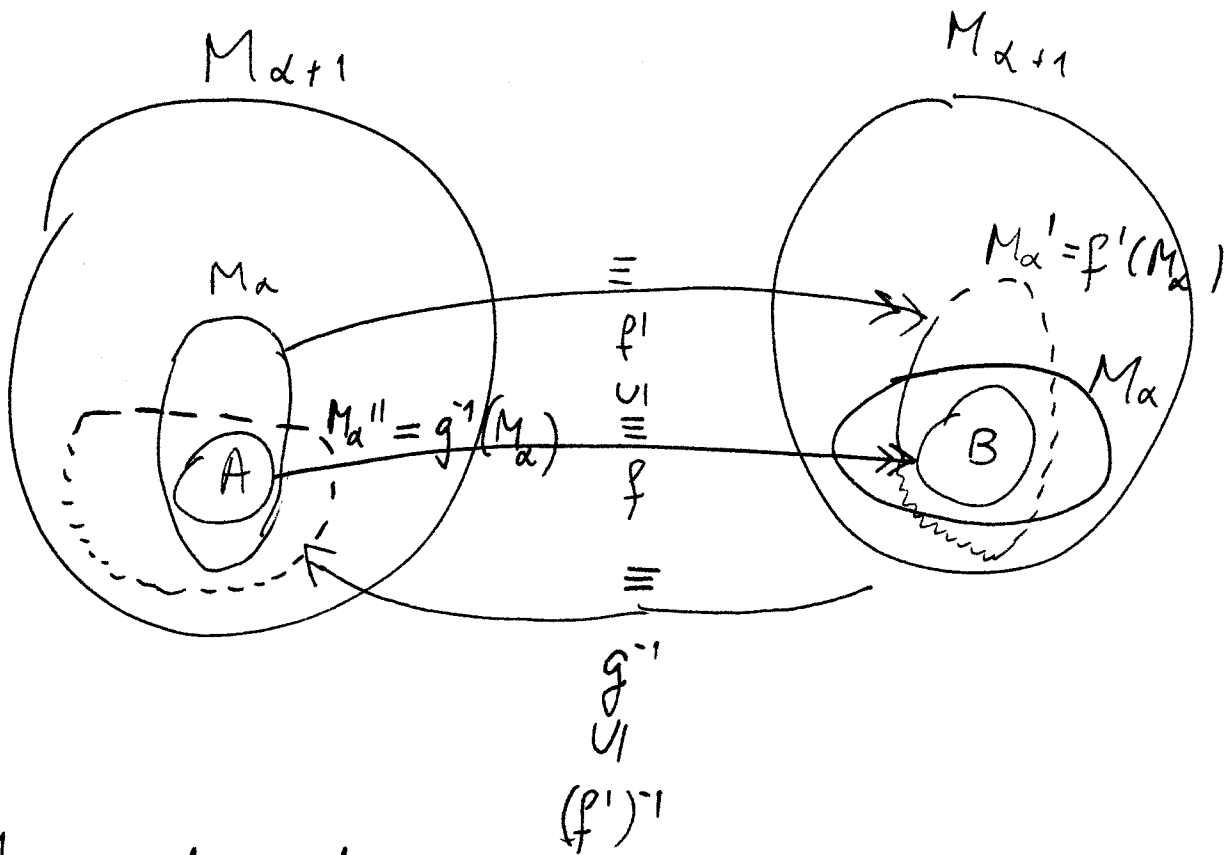
$$(b) \left( \forall f: A \xrightarrow{\equiv} B \right) \left( \exists g \geq f \right) \left( g: A' \xrightarrow{\equiv} B' \text{ in } M_{\alpha+1} \right)$$

$\begin{matrix} \cap & & \cap \\ M_\alpha & & M_\alpha \end{matrix} \qquad \begin{matrix} \cup & & \cup \\ M_\alpha & & M_\alpha \end{matrix}$

MT3/5.

It is enough ~~to~~ that  $M_{\alpha+1}$  is  $\|M_\alpha\|^+$ -saturated,  
 To satisfy (a), (b).  $M_{\alpha+1} \succ M_\alpha$

Proof of (b) for such  $M_{\alpha+1}$ :



I.  $M$ :  $\kappa$ -saturated: clear

II.  $M$  strongly  $\kappa$ -homogeneous:

Assume  $A \subset M$ ,  $|A| < \kappa$ . Then  $A \subseteq M_\alpha$ ,  $B \subseteq M_\alpha$   
 $f: A \xrightarrow{\equiv} B$  for some  $\alpha < \kappa$ .  
 $B = f[A]$

• we construct

a sequence  $f_\beta$ ,  $\alpha \leq \beta < \kappa^+$  :-

• increasing  $f_\beta : M \xrightarrow{\equiv} M$

•  ~~$f_\alpha$~~   $f_\alpha \subseteq f_{\alpha+1}$  partial elementary

(\*)  $M_\beta \subseteq \text{dom } f_\beta \cap \text{rng } f_\beta$ .

•  ~~$f_\alpha$~~   $f_\alpha$  constructed according to 3)(b)

$$f_\alpha : M_{\alpha+1} \xrightarrow{\equiv} M_{\alpha+1}$$

•  $f_\beta : M_{\beta+1} \xrightarrow{\equiv} M_{\beta+1}$ , as in 3.(b)

when  $\beta$  : successor.

•  $f_\delta = \bigcup_{\beta < \delta} f_\beta$  when  $\delta$  limit, still  $f_\delta : M_\delta \xrightarrow{\equiv} M_\delta$ .

$$f_\infty = \bigcup_{\alpha \leq \beta < \kappa^+} f_\beta, \quad f_\beta \in \text{Aut}(M), \quad f \subseteq f_\beta.$$

Assumptions let  $\bar{\kappa}$  : a cardinal number large enough  
so that:

(1) We consider only small models of  $T$

||  
of power  $< \bar{\kappa}$ , or even  $\ll \bar{\kappa}$

(2) We work within a monster model  $\mathcal{M} \models T$  (w.r. to  $\bar{\kappa}$ )

(3) We consider only small models  $M \prec \mathcal{M}$

||  
of power  $< \bar{\kappa}$ , or even  $\ll \bar{\kappa}$ ,

Consequences:

(1) For  $M, N < \mathcal{M}$ ,  $M \subseteq N \Leftrightarrow M < N$

(2) Convention: For  $\bar{a} \in \mathcal{M}$   
 $\models \varphi(\bar{a})$  means  $\mathcal{M} \models \varphi(\bar{a})$

(3) For  $A \subseteq M < \mathcal{M}$ :  
 $S_m^M(A) = S_m^{\mathcal{M}}(A) =: S_m(A)$

Notation Assume  $p(\bar{x}), q(\bar{x})$  types (small, over  $\mathcal{M}$ )

•  $p(\bar{x}) \vdash q(\bar{x}) \Leftrightarrow p(\mathcal{M}) \subseteq q(\mathcal{M})$   
 "p implies q"

•  $p(\bar{x}) \equiv q(\bar{x}) \Leftrightarrow p \vdash q \ \& \ q \vdash p$   
 ↑  
 equivalent

Special case:  $p(\bar{x}) = \{ \varphi(\bar{x}) \}$ .

$\varphi(\bar{x}) \vdash q(\bar{x})$ : "φ isolates q".

Remark: Syntactically:

$$p(\bar{x}) \vdash q(\bar{x}) \Leftrightarrow \forall \varphi(\bar{x}) \in q \ \exists p_0(\bar{x}) \subseteq p(\bar{x}) \text{ finite} \\ \uparrow \quad \uparrow \\ \text{types over } A \quad T(A) \vdash \bigwedge p_0(x) \rightarrow \varphi(x)$$

Remark (exercise)

$$p(\bar{x}) \vdash q(\bar{x}) \Leftrightarrow \forall M \models T \text{ IA-saturated } p(M) \subseteq q(M).$$

Def. (reminder)

Let  $p(\bar{x})$ : a type over  $A$ .

$p$  is isolated over  $A \iff \exists \varphi(\bar{x}) \in L(A)$   $\varphi \vdash p$ .  
consistent (with  $T$ )

Thm (omitting types, Ehrenfeucht)

Assume  $p_n(\bar{x}_n), n < \omega$ : ~~non~~ a family of non-isolated types over  $\emptyset$ . Then  $\exists M \models T \forall n \underbrace{p_n(M) = \emptyset}$ ,  
 $M$  omits  $p_n$ .

Lemma

Assume  $A$  is stable,  $p_n(\bar{x}_n), n < \omega$ : a family of non-isolated types over  $A$ ,  $\varphi(\bar{x}) \in L_1(A)$ ,  $\varphi(M) \neq \emptyset$ .

Then  $\exists c \in \varphi(M) \forall n$   $p_n$  non-isolated over  $A \cup \{c\}$ .  
i.e.  $\varphi$ : consistent

Proof [Lemma  $\Rightarrow$  Thm]

By the lemma:  $\exists \underbrace{\{a_n : n < \omega\}}_A \subseteq M$  s.t.

(1)  $A$  satisfies the TV-test

(2)  $p_n, n < \omega$ : non-isolated over  $A$ .

Construction of  $a_n, n < \omega$ : recursion on  $n$ .

Let  $\{\varphi_n(x, \bar{y}) : n < \omega\}$ : all formulas of  $L$  of this form.

Suppose  $n < \omega$  and  $\{a_i : i < n\} = a_{<n}$  already  
 so that all  $p_k, k < \omega$  still non-isolated over  $a_{<n}$ .

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We consider a consistent formula  $\varphi_n(x) \in L_1(a_{<n})$

By the Lemma we find  $c \in \varphi_n(\mathcal{M})$  so that  
 $\downarrow$   
 $a_n$

all  $p_k, k < \omega$ , still non-isolated over  $a_{\leq n}$ .

• the formulas  $\varphi_n(x), n < \omega$  may be chosen so that  
 after  $\omega$  steps:

$$\forall \varphi(x) \in L(A) \exists n \varphi = \varphi_n.$$

consistent

Then  $A = \{a_n : n < \omega\}$  satisfies TV-test

$A = M \prec \mathcal{M}$ , every  $p_k$  still non-isolated  
 over  $A$ .

$$p_k(M) = \emptyset \text{ [if not,}$$

some  $\bar{m} \models p_k$ . then  $\bar{m} \in \bar{a}_0$

$$(\bar{x}_k = \bar{m}) \vdash p_k(\bar{x}_k) \downarrow)$$

### Proof of the Lemma.

Let  $p(\bar{x})$ : one of the types  $p_n(\bar{x}_n)$ .

Let  $h(\bar{x}, y, \bar{a}) \in L(A)$

$$h(\bar{x}, c, \bar{a}) \vdash p(\bar{x}) \Leftrightarrow h(\mathcal{M}, c, \bar{a}) \subseteq p(\mathcal{M}).$$

$$\Leftrightarrow \forall \psi(\bar{x}) \in p(\bar{x}) \quad h(\mathcal{M}, c, \bar{a}) \subseteq \psi(\mathcal{M})$$

$$\Leftrightarrow \forall \psi \in p \quad \mathcal{M} \models \forall \bar{x} (h(\bar{x}, c, \bar{a}) \rightarrow \psi(\bar{x}))$$

$$\Leftrightarrow \forall \psi \in p \quad \psi_h(y) \in t_p(c/A)(y)$$

$$\text{where } \psi_h(y) = \forall \bar{x} (h(\bar{x}, y, \bar{a}) \rightarrow \psi(\bar{x}))$$

hence:

$$t_p(c/A) = t_p(c'/A) \Rightarrow [h(\bar{x}, c, \bar{a}) \vdash p \Leftrightarrow h(\bar{x}, c', \bar{a}) \vdash p]$$

$$h(\bar{x}, c, \bar{a}) \text{ consistent} \Leftrightarrow (\exists \bar{x} h(\bar{x}, y, \bar{a})) \in t_p(c/A)(y).$$

$$\text{Let } X_{h,p} = \{q \in S_1(A) : \text{For } c \models q, h(\bar{x}, c, \bar{a}) \vdash p(\bar{x}) \text{ and } h(\bar{x}, c, \bar{a}) \text{ consistent.}\}$$

"bad types"

$$\text{Let } q \in S_1(A) \text{ then} \quad \text{For } c \models q, h(\bar{x}, c, \bar{a}) \text{ consistent}$$

$$q \in X_{h,p} \Leftrightarrow q(y) \in S_1(A) \cap [\exists \bar{x} h(\bar{x}, y, \bar{a})] \cap \bigcap_{\psi \in p} [\psi_h(y)]$$

$$\text{For } c \models q, h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$$

(\*)  $X_{h,p}$ : nowhere dense in  $S_1(A)$ .

Proof of (\*): (a.a.)

Suppose  $\theta(y) \in L_1(A)$  and  $\emptyset \neq S_1(A) \cap [\theta] \subseteq X_{h,p}$ .

$$\text{Let } \alpha(\bar{x}) = \exists y (h(\bar{x}, y, \bar{a}) \wedge \theta(y))$$

$$\uparrow \\ L(A)$$



•  $\alpha(\bar{x})$  : consistent :

MT3 / M

Let  $c \in \Theta(\mathcal{M})$

$\Downarrow [\theta] \subseteq X_{n,p}$

$\mathcal{M} \models \exists \bar{x} h(\bar{x}, c, \bar{a})$ .

Let  $\bar{d} \in \mathcal{M}$  s.t.  $\mathcal{M} \models h(\bar{d}, c, \bar{a})$ .

$\bar{d}$  satisfies in  $\mathcal{M}$  :  $\underbrace{\exists y h(\bar{x}, y, \bar{a})}_{\alpha(\bar{x})}$

•  $\alpha(\bar{x}) \vdash p(\bar{x})$ , i.e.  $\alpha(\mathcal{M}) \subseteq p(\mathcal{M})$ .

Let  $\bar{d} \in \alpha(\mathcal{M})$ . So there is  $c \in \mathcal{M}$  s.t.

$\models h(\bar{d}, c, \bar{a}) \wedge \theta(c)$

$\Downarrow [\theta] \subseteq X_{n,d}$

$\forall \psi \in p \models \psi_n(c)$

$\forall \psi \in p \models h(\bar{x}, c, \bar{a}) \vdash \psi(\bar{x}) \Rightarrow h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$   
 $\neq \bar{d} \Rightarrow \frac{\perp}{\bar{d}}$  (y)

as:  $p(\bar{x})$  non-isolated

Let  $X = \bigcup_{h,p_n} X_{h,p_n} \subseteq S_1(A)$ .

meager. Let  $q_i \in S_1(A) \cap [\varphi] \setminus X$

$c \models q_i$  good.

(pf) (a.e) Suppose  $p = p_n$  isolated over  $A \cup \Sigma c \mathcal{L}$ .

$\exists h(\bar{x}, c, \bar{a}) \models p(\bar{x}) \Rightarrow q_i \in X_{h,p_n} \Downarrow$   
 consistent  $\vdash_{\bar{c}/A}$