Preliminaries

T : a complete consistent theory, in language L (countable) with infinite models.

that is, $T = Th(M)$, M : L-structure infinite.

L denotes also the set of formulas of language L

$M = (|M|; \dots)$, but M also denotes $|M|$ (for brevity)

$\emptyset \neq$ universe of M

usually we omit $|\cdot|$ in $|M|$.

$M \supseteq A$: a set of parameters.

$L_n(A) = \{\varphi(x_1, \dots, x_n, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in L, \bar{a} \subseteq A\}$

$L(A) = \bigcup_n L_n(A)$, also $L(A)$: language L extended by names for elements of A.

$L_n(A)$: Lindenbaum algebra.

[formally: on $L_n(A)$: $\varphi \sim \psi \Leftrightarrow T(A) \vdash \varphi \leftrightarrow \psi$
$\Leftrightarrow M \models \varphi \leftrightarrow \psi$

here: $T(A) = Th(M, a)_{a \in A}$

a complete theory in language $L(A)$.

$L_n(A)/_\sim$: a Boolean algebra
(Lindenbaum algebra)

$[\varphi]_\sim \wedge [\psi]_\sim = [\varphi \wedge \psi]_\sim$ etc.

shortly : $L_n(A)$ denotes also $L_n(A)/_\sim$.

$S_n(A) = \{$complete n-types over A, in $M\}$.
in variables $x_1, \dots, x_n$

consistent n-type over A $\rightsquigarrow$ proper filter in $L_n(A)$

An n-type $p(\bar{x})$ over A is <u>complete</u> if
$\cap$
$L_n(A)$ $\begin{cases} \cdot\ p(\bar{x}) : \text{consistent type} \\ \cdot\ \forall \varphi(\bar{x}) \in L_n(A) \ (\varphi(\bar{x}) \in p \text{ or } (\neg\varphi(\bar{x})) \in p) \end{cases}$

$S(A) := S_1(A)$
(default)

$S_n(A)$: topological space :
for $\varphi(\bar{x}) \in L_n(A)$

$$[\varphi] = \{ p \in S_n(A) : \varphi \in p \}$$

basic open set [clopen]
$\shortparallel$
closed and open

$S_n(A)$: compact Hausdorff space, 0-dimensional
(i.e. basis of clopen sets)

complete n-types $/A$ $\leftrightsquigarrow$ ultrafilters in $L_n(A)$

So $S_n(A) = S(L_n(A))$, the Stone space of ultrafilters in $L_n(A)$

- the ~~topo~~ topology on $S_n(A)$ $=$ the Stone space topology.

For $p(\bar{x}) \in S_n(A)$

realizes p

$p(M) = \{\bar{a} \in M^n : \underbrace{\bar{a} \text{ satisfies } p}\}$

$\bar{a} \models p$, i.e. $M \models \varphi(\bar{a})$ for every $\varphi(\bar{x}) \in p(\bar{x})$

- The same notation for arbitrary type (also incomplete)
- A formula $\varphi(\bar{x}) \in L(M)$: a special case of a type $\{\varphi(\bar{x})\}$.

$\varphi(M) = \ldots$

- When $p \in S_n(A)$, $\bar{a} \subseteq M$ and $\bar{a} \models p$, then

$$p = tp^M(\bar{a}/A) = tp(\bar{a}/A) = \{\varphi(\bar{x}) \in L_n(A) : M \models \varphi(\bar{a})\}.$$

<u>Example</u> Assume $p(\bar{x})$: a consistent type over M.

Then $\exists N \succcurlyeq M$, $\underbrace{p \text{ is realized in } N}$

i.e. $p(N) \neq \emptyset$.

From now on "a type" means "a consistent type".

<u>Def</u> A type $p(\bar{x})$ over A is <u>isolated</u>, if:

$\exists \varphi(\bar{x}) \in L_n(A)$

(1) $\varphi(\bar{x})$ is consistent (with T), i.e. $\varphi(M) \neq \emptyset$ ($\Leftrightarrow T(A) \vdash \exists \bar{x}\, \varphi(\bar{x})$)

(2) ~~$\varphi(M)$~~ $\forall \psi(\bar{x}) \in p(\bar{x})$ $\varphi(M) \subseteq \psi(M)$

$\Updownarrow$

$T(A) \vdash \varphi(\bar{x}) \to \psi(\bar{x})$

symbolically: $\varphi(\bar{x}) \vdash p(\bar{x})$

- When $p(\bar{x})$: a complete type over A, then:

  $p(\bar{x})$ is isolated $\Leftrightarrow$ p is isolated in $S_n(A)$ in the topological sense (i.e. $\{p\}$ is open)

<u>Tarski–Vaught test</u>

Assume $A \subseteq M$. Then $A = |N|$ for some $N \prec M$ <u>iff</u>

$\forall \varphi(x) \in L_1(A)\, [\varphi(M) \neq \emptyset \Rightarrow \varphi(M) \cap A \neq \emptyset]$

Construction of an elementary submodel of M containing A:

- $A_n \subseteq M$, $n < \omega$, increasing chain of sets

recursive construction:

$A_0 = A$

$A_n \subseteq A_{n+1} \subseteq M$ such that $\forall \psi(x) \in L_1(A_n)$

$[\psi(M) \neq \emptyset \Rightarrow \psi(M) \cap A_{n+1} \neq \emptyset]$

$A_\infty = \bigcup_{n<\omega} A_n$ satisfies TV-test.

Omitting types theorem

Assume $p_n(\bar{x}_n)$, $n<\omega$: a family of non-isolated types in theory T, over $\emptyset$. Then:

$(\exists M \models T)$ M omits every p_n [i.e. $p_n(M)=\emptyset$]

Assume $\underset{\underset{A}{\cup\!\!\shortmid}}{M}, N \models T$

Def. $f: A \longrightarrow N$ is *elementary* ($f: A \xrightarrow{\equiv} N$) if:

$\forall \bar{a} \subseteq A \; \forall \varphi(\bar{x}) \in L \; \big(M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a})) \big)$

$(\Leftrightarrow \; tp^M(\bar{a}) = tp^N(f(\bar{a})) \,)$

Elementary diagram of $A \subseteq M$:

$D_e(A) = T(A) = Th(M, a)_{a\in A}$

Remark $f: A \longrightarrow N$ is elementary $\Leftrightarrow (N, f(a))_{a\in A} \models T(A)$

Atomic diagram of $A \subseteq M$:

$D_{at}(A) = \{\varphi \in D_{el}(A) : \varphi$ is a quantifier free sentence$\}$

$= \{\varphi(\bar{a}) \in L(A) : M \models \varphi(\bar{a})$ and $\varphi(\bar{a})$: q.f.-sentence$\}$

Remark $f: M \longrightarrow N$ is a monomorphism (i.e: $f: M \xrightarrow{\cong} f(M) \underset{\uparrow \text{substructure}}{\subseteq} N$)

$\Leftrightarrow (N, f(a))_{a\in M} \models D_{at}(M)$.

Here always $f: M \to N$ denotes a <u>monomorphism</u>.

$M \subseteq N$: M is a submodel (substructure) of N

$M \prec N$: M is an elementary submodel of N, i.e:

$$M \subseteq N \text{ and } id_M : M \xrightarrow{\equiv} N$$

<u>Remark</u> Assume $M \prec N$, $A \subseteq M$.

(1) Assume $p(\bar{x}) \subseteq L_n(A)$. Then

$p(\bar{x})$ is a consistent type in $M \Leftrightarrow p(\bar{x})$ is a consistent type in N

(2) Assume $A \subseteq B \subseteq M$

- If $p(\bar{x})$: a type over B, then $p\restriction_A \overset{def}{=} p(\bar{x}) \cap L(A)$ a type over A
- Let $r: S_n(B) \to S_n(A)$, $r(p) \overset{def}{=} p\restriction_A$.

  Then r: continuous and "onto".

(3) If $p(\bar{x})$: a type over A, then $\exists q(\bar{x}) \in S_n(A)\ p(\bar{x}) \subseteq q(\bar{x})$

Saturation, universality, (strong) homogeneity.

Let $\varkappa \in CN$, $\varkappa \geq \aleph_0$.

<u>Def.</u> (1) M $\varkappa$-<u>saturated</u> (nasycony) if $\underset{|A|<\varkappa}{\forall A \subseteq M}\ \forall p \in S_1(A)\ p(M) \neq \emptyset$

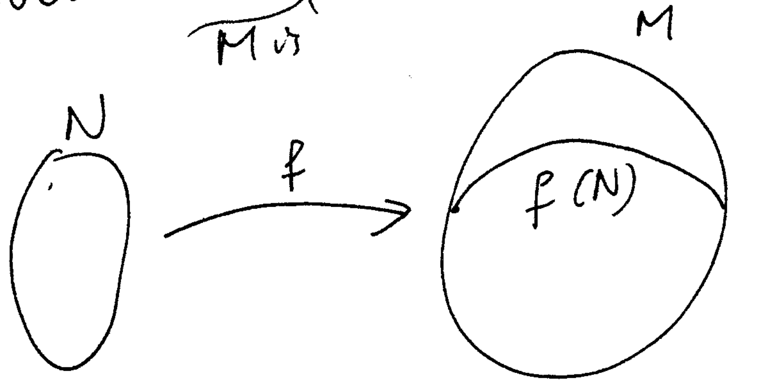
M is <u>saturated</u> if M is $\|M\|$-saturated

(2) M is $\varkappa$-universal if $\forall N \equiv M\ (\|N\| \leq \varkappa \Rightarrow \exists f : N \xrightarrow{\equiv} M)$

elementarily equivalent

i.e. $Th(N) = Th(M)$

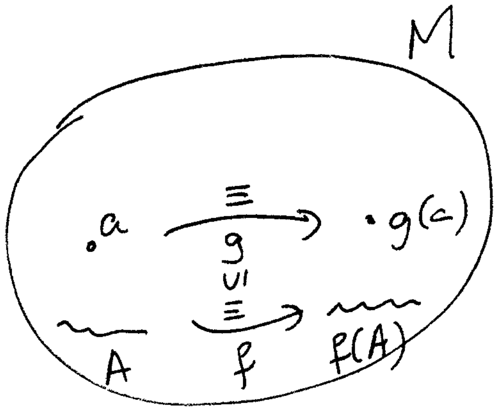
M: universal $\underset{M \text{ is}}{\iff}$ $\|M\|$-universal



$f: N \xrightarrow{\cong} f(N) \prec M$

(3) M : $\varkappa$-homogeneous if $\forall A \subseteq M$, $|A| < \varkappa$, $\forall a \in M$ $\forall f: A \xrightarrow{\equiv} M$ $\exists g \supseteq f: A \cup \{a\} \xrightarrow{\equiv} M$

homogeneous = $\|M\|$-homogeneous.



4. M strongly $\varkappa$-homogeneous if $\forall A \subseteq M$, $|A| < \varkappa$, $\forall f: A \xrightarrow{\equiv} M$ $\exists g \supseteq f: M \xrightarrow{\cong} M$

strongly homogeneous $=:$ strongly $\|M\|$-homogeneous.

5. M is $\varkappa$-compact if ($\forall$ 1-type $p(x)$ over M)

$(|p| < \varkappa \Rightarrow p(M) \neq \emptyset)$

<u>Elementary chains of structures</u>

<u>Def</u> $\langle M_\alpha : \alpha < \mu \rangle$, $\mu \in \mathrm{Ord}$, : an elementary chain of structures if $(\forall \alpha < \beta < \mu)\ M_\alpha \prec M_\beta$.

Union of chain (when $\mu \in \mathrm{Lim}$)

$$M_\mu = \bigcup_{\alpha < \mu} M_\alpha \ :$$

- $|M_\mu| := \bigcup\limits_{\alpha<\mu}^{\nearrow} |M_\alpha|$
- $c \in L$ constant symbol $\quad c^{M_\mu} = c^{M_\alpha}$ for $\alpha < \mu$
- P : relation symbol

$$\underset{\substack{\cap \\ |M_\mu|}}{P^{M_\mu}(a_1, \ldots, a_n)} \iff M_\alpha \models P(a_1, \ldots, a_n) \text{ for } \alpha < \mu$$

sufficiently large
[so that $\bar{a} \subseteq M_\alpha$]

- $f^{M_\mu}(\bar{a}) = b \iff M_\alpha \models f(\bar{a}) = b$ for $\alpha < \mu$ sufficiently large

Fact (Tarski) $M_\alpha \prec M_\mu$ for all $\alpha < \mu$.

<u>Proof</u> (1) $M_\alpha \subseteq M_\mu$ (substructure) : exercise

(2) $\forall \varphi(\bar{x}) \in L \ \forall \alpha < \mu \ \forall \bar{a} \subseteq M_\alpha \ (M_\alpha \models \varphi(\bar{a}) \Leftrightarrow M_\mu \models \varphi(\bar{a}))$

(a) φ atomic : $M_\alpha \subseteq M_\mu$ ✓

(b) $\varphi = \psi_1 \wedge \psi_2$, $\varphi = \neg\psi$: easy

(c) $\varphi(\bar{x}) = \exists y \, \psi(\bar{x}, y)$

$M_\alpha \models \varphi(\bar{a}) \Rightarrow M_\alpha \models \psi(\bar{a}, b)$ for some $b \in M_\alpha$

$\Downarrow$ ind. assumption for ψ

$M_\mu \models \psi(\bar{a}, b)$

$\Downarrow$

$M_\mu \models \varphi(\bar{a})$

$M_\mu \models \varphi(\bar{a}) \Rightarrow M_\mu \models \psi(\bar{a}, b)$ for some $b \in M_\mu$

$\varphi(\bar{a}) = \exists y\, \psi(\bar{a}, y)$

$\Downarrow$ ind. assumption

$b \in M_\beta$ for some $\alpha \leq \beta < \mu$

$M_\beta \models \psi(\bar{a}, b)$

$\Downarrow$

$M_\beta \models \varphi(\bar{a})$

$\Downarrow$ $M_\alpha \prec M_\beta$

$M_\alpha \models \varphi(\bar{a})$

<u>Elementary directed systems of structures:</u>

Let $(I, \leq)$: a <u>directed set</u>, i.e.:

(1) $\leq$: partial order on I

(2) $(\forall a, b \in I)(\exists c \in I)(a \leq c \wedge b \leq c)$

<u>Example</u> J : a set $\rightsquigarrow$ $([J]^{<\omega}, \subseteq)$: directed set.

Directed system of structures:

- $\mathcal{M} = (M_i, f_{ij})_{i \leq j \in I}$

f_{ij}: connecting functions $f_{ij} : M_i \longrightarrow M_j$, $f_{ii} = id_{M_i}$ such that

$(\forall i \leq j \leq t \in I)$ $f_{it} = f_{jt} \circ f_{ij}$
(compatibility)

$$\begin{tikzcd} & M_t \\ M_i \arrow[ur, "f_{it}"] \arrow[r, "f_{ij}"'] & M_j \arrow[u, "f_{jt}"'] \end{tikzcd}$$

- System $\mathcal{M}$ is <u>elementary</u> if all f_{ij} are elementary.

<u>Example</u> Elementary chain $(M_\alpha)_{\alpha<\mu}$

$\mathcal{M} = (M_\alpha, f_{\alpha\beta})_{\alpha\leq\beta<\mu}$ $\quad f_{\alpha\beta} = id_{M_\alpha} : M_\alpha \xrightarrow{\equiv} M_\beta$

elementary directed system of structures

Direct limit of a directed system $\mathcal{M}$: $M_\infty = \varinjlim \mathcal{M}$.

$(M_\infty, f_{i\infty})_{i\in I}$, where $f_{i\infty} : M_i \longrightarrow M_\infty$ such that

(1) $\forall i \leq j \in I \quad f_{i\infty} = f_{j\infty} \circ f_{ij}$ [compatible with connecting functions]

$$\begin{tikzcd}
M_j \arrow[dr, "f_{j\infty}"] & \\
 & M_\infty \\
M_i \arrow[uu, "f_{ij}"] \arrow[ur, "f_{i\infty}"'] \arrow[u, phantom, "\#"] &
\end{tikzcd}$$

(2) $(\forall (M'_\infty, f'_{i\infty})_{i\in I}$ satisfying (1)$)$ $\exists! f : M_\infty \to M'_\infty$

(universality) $\quad (\forall i \in I)\ f'_{i\infty} = f \circ f_{i\infty}$

$$\begin{tikzcd}
 & M_\infty \arrow[dd, "f"] \\
M_i \arrow[ur, "f_{i\infty}"] \arrow[dr, "f'_{i\infty}"'] \arrow[r, phantom, "\#"] & {} \\
 & M'_\infty
\end{tikzcd}$$

<u>Fact</u> M_∞ exists (and is unique up to $\cong$).

If $\mathcal{M}$ is elementary, then $f_{i,\infty} : M_i \xrightarrow{\equiv} M_\infty$.

<u>Proof</u> 1. Construction of M_∞:

$S := \dot{\bigcup\limits_{i\in I}} |M_i|$: formally disjoint union.

$\sim$ on S : an equivalence relation

$$\underset{\substack{M_i\\ \Downarrow}}{x} \sim \underset{\substack{M_j\\ \Downarrow}}{y} \overset{\text{def}}{\iff} f_{it}(x) = f_{jt}(y) \text{ for some } (\equiv \text{every}) \ t \geq i, j$$

exercise: $\sim$ is transitive.

$$|M_\infty| := S/\sim$$

- $\sim \restriction_{|M_i|}$: the equality (because f_{ij} : 1-1 (monomorphism)
- $f_{i\infty}(\underset{\substack{\Uparrow\\ M_i}}{x}) = x/\sim$, $\quad f_{i\infty} : |M_i| \xrightarrow{1-1} |M_\infty|$.

L-structure on $|M_\infty|$:

- $c^{M_\infty} = c^{M_i}/\sim$
- $p^{M_\infty}(a_{i_1}/\sim, \ldots, a_{i_n}/\sim) \iff M_t \models P(f_{i_1t}(a_{i_1}), \ldots, f_{i_nt}(a_{i_n}))$

  $a_{i_j} \in M_{i_j}$ $\qquad$ for $t \geq i_1, \ldots, i_n$

- f^{M_∞} : similarly

the rest is an exercise.

How to extend elementary mappings?

~~Bat.~~ BAlg : Category of Boolean algebras

Comp_0 : —— // —— of compact Hausdorff 0-dimensional spaces

$F : \mathrm{BAlg} \to \mathrm{Comp}_0$ $\qquad$ $G : \mathrm{Comp}_0 \to \mathrm{BAlg}$

$\quad\; F(\mathbb{A}) = S(\mathbb{A})$ $\qquad\qquad$ $G(X) = \mathrm{Clopen}(X)$.

F, G : contravariant functors "inverse" to each other

~~(F, G) is a duality of categories... (look it up)~~

Categories BAlg and Comp_0 are dually equivalent.

$\mathbb{A}, \mathbb{B}$: Boolean algebras

$f : \mathbb{A} \to \mathbb{B}$ homomorphism $\Rightarrow$ $\overset{f^*}{\overset{\shortparallel}{F(f)}} : S(\mathbb{B}) \to S(\mathbb{A})$

$F(f)(p) = f^{-1}[p]$

continuous.

Assume $f : A \xrightarrow{\equiv} B$. Then $\hat{f} : L_n(A) \to L_n(B)$

$\qquad\quad\; \cap\!| \qquad\;\; \cap\!|$

$T \models M \;,\; N \models T$

$\hat{f}(\varphi(\bar{x}, \bar{a})) = \varphi(\bar{x}, f(\bar{a}))$

homomorphism of Boolean algebras.

even: monomorphism.

We skip $\hat{}$ in $\hat{f}$, so:

$f : L_n(A) \to L_n(B)$ monomorphism

$\Downarrow$

$f^* : S_n(B) \to S_n(A)$ epimorphism in Comp_0

i.e: <u>continuous onto</u>

Lemma (on extensions of elementary mappings)

Assume $M, N \models T$, $A \subseteq M$, $B \subseteq N$, $f: A \xrightarrow{\equiv} B$ "onto".

Assume $\underset{M}{a} \in$, $\underset{N}{b} \in$, $p = tp(a/A)$, $q = tp(b/B)$.

Then $f \cup \{\langle a, b \rangle\}$ is elementary $\iff f^*(q) = p$.

[here $f^*: S(B) \xrightarrow{\approx} S(A)$].
homeomorphism

Proof exercise.

Def. M is $(<\aleph_0)$-universal $\iff \forall n\, \forall p \in S_n(\emptyset)\ p(M) \neq \emptyset$.

Remark $M: \varkappa$-universal $\Rightarrow M: (<\aleph_0)$-universal.

Proof Let $p \in S_n(\emptyset)$.

Choose a countable $N \models T$ with $p(N) \neq \emptyset$.

$M: \varkappa$-universal $\Rightarrow \exists f: N \xrightarrow{\equiv} M$

$$\underset{\bar{a} \models p}{N} \quad \overset{f}{\rightsquigarrow} \quad \underset{f(\bar{a}) \models p.}{M}$$

Thm. (1) $M: \varkappa$-saturated $\Rightarrow M: \varkappa$-homogeneous and $\varkappa$-universal.

(2) $M: \varkappa$-~~universal~~ homogeneous and $(<\aleph_0)$-universal $\Rightarrow$ $M: \varkappa$-saturated.

Proof. (1) $\varkappa$-homogeneity of M:

Assume $f: A \xrightarrow{\equiv} M$, $A \subseteq M$, $|A| < \varkappa$, $a \in M$.

We seek $b \in M$ s.t. $g = f \cup \{\langle a, b \rangle\}$ elementary

$$\Updownarrow \text{ Lemma}$$

$$f^*(tp(b/B)) = tp(a/A).$$

Let $\underset{\substack{\uparrow \\ S_1(A)}}{p} = tp(a/A)$, $q = (f^*)^{-1}(p) \in S_1(B)$

Let $\underset{\substack{\uparrow \\ \text{good.}}}{b} \in M$ (exists by κ-saturation) of M

- $\underline{\kappa\text{-universality}}$ of M:

Assume $N \equiv M$, $\|N\| \leq \kappa$.

We seek $f : N \xrightarrow{\equiv} M$.

Let $\{a_\alpha : \alpha < \mu\}$: an enumeration of N, $\underset{\vee\!\mid}{\overset{\kappa}{\mu}} = \|N\|$.

We define $f(a_\alpha)$ by induction on $\alpha < \mu$:

- Suppose $f(a_\beta)$ defined for all $\beta < \alpha$ so that

  ~~and~~ $f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} M$

  Want to find $f(a_\alpha)$ so that

  $$f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} M.$$

~~By the Lemma it is enough that t~~

Let $p = tp(a_\alpha / \{a_\beta : \beta < \alpha\})$.

By the lemma it is enough to find $f(a_\alpha) \in M$ so that $f^*(\, tp(f(a_\alpha) / \{f(a_\beta) : \beta < \alpha\})) = p$.

So let $q = (f^*)^{-1}(p) \in S_1(\underbrace{\{f(a_\beta) : \beta < \alpha\}}_{\text{power} < \kappa})$

M κ-saturated $\Rightarrow$ q realized in M.

Let $f(a_\alpha) \in M$ s.t. $f(a_\alpha) \models q$.

(2) Assume M is κ-homogeneous & $(<\aleph_0)$-~~saturated~~ universal.

Want: M: κ-saturated.

So: Let $A \subseteq M$, $|A| < \kappa$, $p \in S_1(A)$. Show: $p(M) \neq \emptyset$.

Induction on $|A|$.

Case (a): $|A| < \aleph_0$.

$\exists N \succ M \quad p(N) \neq \emptyset$. So let $b \models p$.

Let $A^* = \underset{\{a_1,\dots,a_k\}}{\overset{\|}{A}} \cup \{b\}$

Let $q = tp^N(a_1,\dots,a_k,b) \in S_{k+1}(\emptyset)$

q is realized in M (by $(<\aleph_0)$-universality),

by $\langle \underbrace{a_1',\dots,a_k'}_{A'}, b' \rangle$

Let $g : A \cup \{b\} \longrightarrow A' \cup \{b'\}$, $g(a_i) = a_i'$, $g(b) = b'$.

g: elementary.

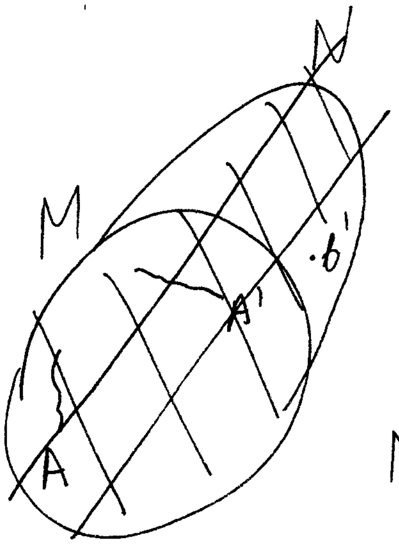
$\Rightarrow g\restriction_A : A \xrightarrow{\equiv} A'$

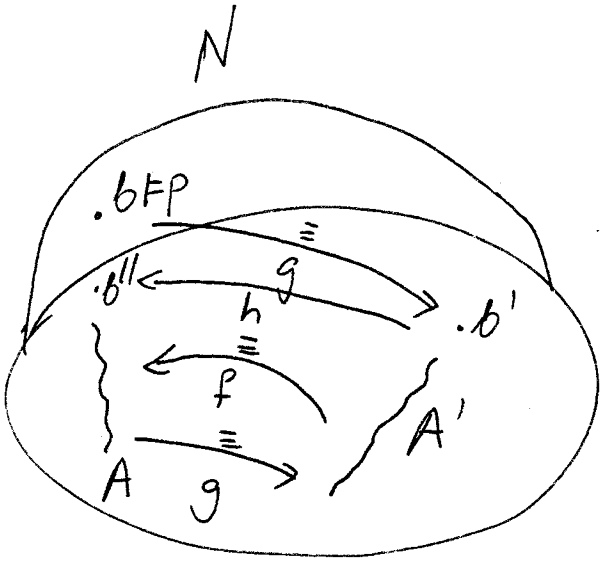
$\Downarrow$ $f := (g\restriction_A)^{-1} : A' \xrightarrow{\equiv} A$

$\cap|$

M: κ-homogeneous $\Rightarrow \exists\, h : A' \cup \{b'\} \xrightarrow{\equiv} A \cup \{b''\}$

for some $b'' \in M$.





$$\begin{tikzcd} Ab \arrow[rr, "\equiv", "h\circ g"'] \arrow[dr, "g"', "\equiv"] & & Ab'' \\ & A'b' \arrow[ur, "h"', "\equiv"] & \end{tikzcd}$$

Let $s = h \circ g$

$s\restriction A = \underbrace{(h\restriction_{A'})}_{= (g\restriction_A)^{-1}} \circ (g\restriction_A) = id_A$.

$\Downarrow$

$\cancel{s^*(tp(b''/A)) = tp(b}$

$s\restriction_A = id_A \Rightarrow \underset{id_{S(A)}}{\underset{\|}{s^*}} : S(A) \xrightarrow{\simeq} S(A)$

hence: $p = tp(b/A) \underset{\substack{\uparrow \\ \text{Lemma} \\ \text{and } b'' \models p}}{=} s^*(tp(b''/A)) \underset{\substack{\uparrow \\ s^* = id_{S(A)}}}{=} tp(b''/A)$

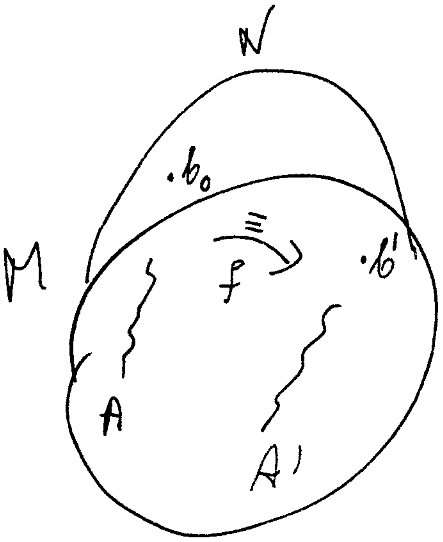
Case (b) $|A| = \mu$, $\aleph_0 \leq \mu < \kappa$.

$A = \{a_\alpha : \alpha < \mu\}$, $p \in S_1(A)$.

$p \restriction \emptyset \in S_1(\emptyset) \Longrightarrow \exists b' \in M \; b' \models p\restriction\emptyset$.

M: $(<\aleph_0)$-universal

$\exists N \succ M \; \exists b_0 \in N$, $b_0 \models p$



Will find $A' = \{a_\alpha' : \alpha < \mu\} \subseteq M$

s.t. $f : Ab_0 \longrightarrow A'b'$

given by $f(a_\alpha) = a_\alpha'$

$f(b_0) = b'$

is <u>elementary</u>!

We find a_α', $\alpha < \mu$ by induction on $\alpha < \mu$.

So suppose $\alpha < \mu$ and a_β' already defined for all $\beta < \alpha$

so that $\boxed{f\restriction_{\{a_\beta : \beta<\alpha\} b_0}} : \{a_\beta : \beta<\alpha\} b_0 \xrightarrow{\equiv} \{a_\beta' : \beta < \alpha\} b'$ $=: f_0$.

We look for a_α'.

Let $q = tp(a_\alpha / \{a_\beta : \beta < \alpha\} \cup \{b_0\})$

Then $(f_0^*)^{-1}(q) \in S(\underbrace{\{a_\beta' : \beta < \alpha\} \cup \{b'\}}_{\text{power } < \mu \leq |A|})$

By the lemma it is enough that $\underset{M}{a_\alpha'} \models (f_0^*)^{-1}(q)$.

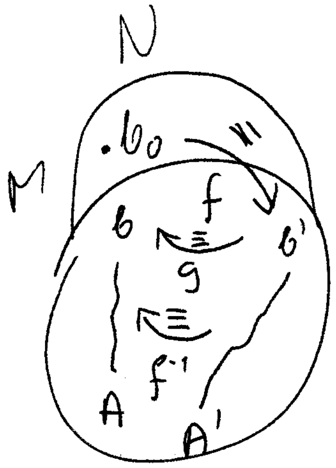
~~But M: κ-~~

~~Let A~~ By the inductive assumption on A:

$(f_0^*)^{-1}(q)$ is realized in M, so we are done with constructing A'.

Now: $f^{-1}: A' \xrightarrow{\equiv} A$ in $M \leftarrow \kappa$-homogeneous, so

$\exists\ \hat{g}^{\prime} : A'b' \xrightarrow{\equiv} Ab$ for some $b \in M$.



~~Let~~ Let $s = g \circ f$

$s : Ab_0^{\equiv} \xrightarrow{\equiv} Ab$

$$s\restriction_A = \underbrace{(g\restriction_{A'})}_{= (f\restriction_A)^{-1}} \circ (f\restriction_A) = id_A,$$

so $p = tp(b_0^{\equiv}/A) = tp(b/A)$

and p is realized in M.

<u>Corollary</u>

M is κ-saturated $\Leftrightarrow$ M is κ-homogeneous and ~~κ-universal~~ and κ-universal.

<u>Proof</u> $\Rightarrow$ by Thm (1).

$\Leftarrow$ κ-homogeneous + κ-universal $\Rightarrow$

κ-homogeneous + $(<\aleph_0)$-universal $\underset{\text{Thm (2)}}{\Rightarrow}$ κ-saturated

Properties of saturated models.

Thm. Assume $M, N \models T$ saturated models of the same power. Then $M \cong N$.

Proof $M = \{m_\alpha : \alpha < \varkappa\}$, $N = \{n_\alpha : \alpha < \varkappa\}$,

$\varkappa = \|M\| = \|N\|$. We find $f : M \xrightarrow{\cong} N$

back-and-forth method:

$$f = \bigcup^{\nearrow}_{\alpha<\varkappa} f_\alpha \qquad f_\alpha : M \xrightarrow[\circ]{\equiv} N \quad \text{s.t.:}$$

partial, elementary

(1) $m_\alpha \in \mathrm{Dom}\, f_{\alpha+1}$

$n_\alpha \in \mathrm{Rng}\, f_{\alpha+1}$, $|f_\alpha| \leq 2 \cdot |\alpha|$

(2) $f_0 = \emptyset$

(3) For $\delta \in \mathrm{Lim}$, $f_\delta = \bigcup^{\nearrow}_{\alpha<\delta} f_\alpha$.

(4) $f_{\alpha+1} = f_\alpha \cup \{\langle m_\alpha, \underset{\substack{\uparrow \\ N}}{n} \rangle, \langle \underset{\substack{\uparrow \\ M}}{m}, n_\alpha \rangle\}$

Inductive step:

Suppose we have f_α. Want: $f_{\alpha+1}$.

Let $A_\alpha = \mathrm{Dom}\, f_\alpha \subseteq M$, $B_\alpha = \mathrm{Rng}\, f_\alpha \subseteq N$.

$$f_\alpha : A_\alpha \xrightarrow{\equiv} B_\alpha$$

$$\Downarrow$$

$$f_\alpha^{\leftarrow} : S(B_\alpha) \xrightarrow{\cong} S(A_\alpha).$$

"forth": Find $n \in N$ st. $f_\alpha \cup \{\langle m_\alpha, n \rangle\}$ elementary

$$\Updownarrow$$

$$(f_\alpha^*)^{-1}(tp(m_\alpha/A_\alpha)) = tp(n/B_\alpha).$$

Let $p = tp(m_\alpha/A_\alpha)$.

So $(f_\alpha^*)^{-1}(p) \in S(B_\alpha)$ is realized in N by some n.

"~~forth~~ back": similarly.

<u>Thm</u> Assume $M, N \models T$ are homogeneous, of the same power and $\forall n<\omega\ \forall p \in S_n(\emptyset)\ (p(M) \neq \emptyset \Leftrightarrow p(N) \neq \emptyset)$. Then $M \cong N$.

<u>Lemma</u> Under the assumptions of the Thm,

$$\forall A \subseteq M\ \exists f : A \xrightarrow{\equiv} N.$$

<u>Proof</u>. Induction on $|A|$.

Case (a) $|A| < \aleph_0$. $A = \{a_1, \dots, a_n\}$.

Let $p = tp(\langle a_1, \dots, a_n \rangle) \in S_n(\emptyset)$, realized in M

$\Downarrow$

realized in N

by some $\langle b_1, \dots, b_n \rangle \subseteq N$.

$f(a_i) = b_i$ is <u>good</u>.

Case (b) $|A| = \mu \geq \aleph_0$, $A = \{a_\alpha : \alpha < \mu\}$

We find $f(a_\alpha)$ by induction on $\alpha < \mu$.

Inductive step.

Suppose $\alpha < \mu$ and for every $\beta < \alpha$ we have $f(a_\beta)$

s.t. $f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} N$.

We shall find $f(a_\alpha) \in N$ s.t. $f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} N$.

Let $a_{<\alpha} := \{a_\beta : \beta < \alpha\}$. Likewise $a_{\leq\alpha}$.

$|a_{\leq\alpha}| < \mu = |A|$

By inductive assumption: $\exists g : a_{\leq\alpha} \xrightarrow{\equiv} N$.

Then
$$\begin{array}{ccc} f\circ g^{-1} : g(a_{<\alpha}) & \xrightarrow{\equiv} & f(a_{<\alpha}) \\ \cap\!\!\mid & & \cap\!\!\mid \\ N & & N \end{array}$$

By homogeneity of N: $\exists f(a_\alpha) \in N$ s.t.
$$f\circ g^{-1} : \underbrace{g(a_{<\alpha})\, g(a_\alpha)}_{= g(a_{\leq\alpha})} \xrightarrow{\equiv} f(a_{<\alpha}) f(a_\alpha) = f(a_{\leq\alpha})$$

Then $f = (f\circ g^{-1})\circ g : a_{\leq\alpha} \xrightarrow{\equiv} N$.

Proof of the theorem $\quad \varkappa := \|M\| = \|N\|$

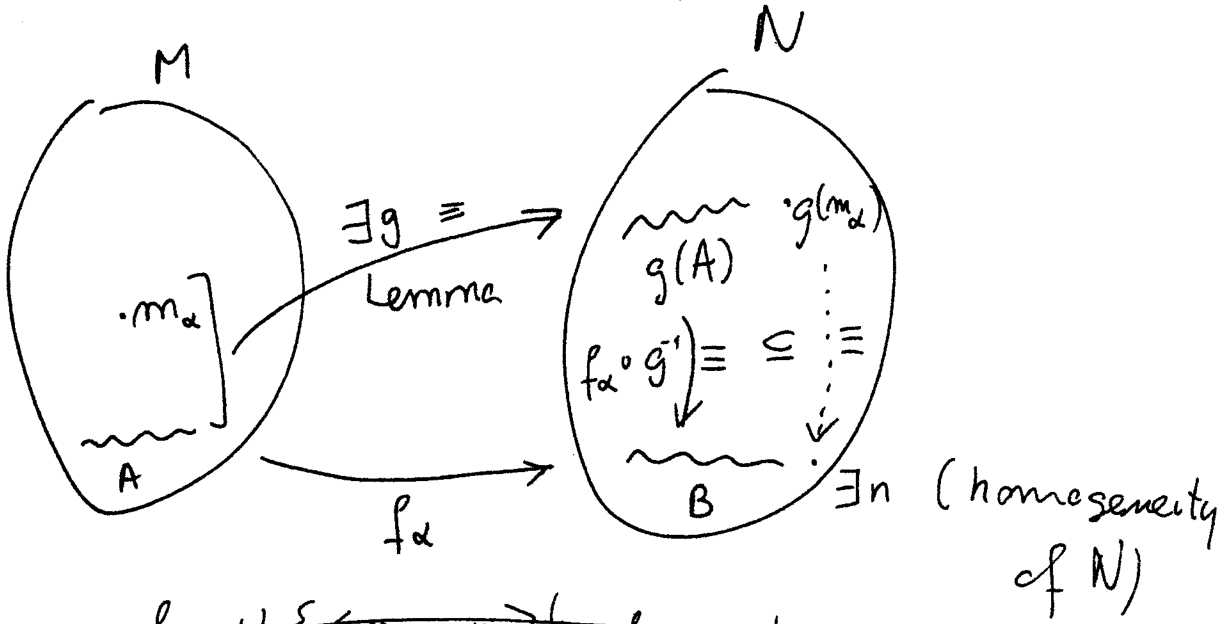
$f : M \xrightarrow{\cong} N$ constructed by back-and-forth method

$f = \bigcup\limits_{\alpha<\varkappa}^{\uparrow} f_\alpha$, $\; f_\alpha : M \xrightarrow[\circ]{\equiv} N$ (partial elementary), $\alpha < \varkappa$

$|f_\alpha| \leq 2\cdot|\alpha|$ + the same conditions as in the previous thm.

Inductive step $f_\alpha \longmapsto f_{\alpha+1}$

$A = \mathrm{Dom}\, f_\alpha$ $\quad B = \mathrm{Rng}\, f_\alpha$ $\quad$ "forth":



~~$f_\alpha \cup \{\langle m_\alpha, n\rangle\}$ elementary~~

$$h = (f_\alpha \circ g^{-1})\restriction_{g(A)} \cup \{\langle g(m_\alpha), n\rangle\} \text{ elementary}$$

$$h \circ g : A m_\alpha \xrightarrow{\equiv} B n \subseteq N$$

$$\cup\!\mid$$

$$f_\alpha$$

"back": similarly.

Constructions of models:

• saturated $\Longrightarrow$ • (strongly) homogeneous $\not\Leftarrow$

Thm. $\underbrace{\kappa = 2^{<\kappa},\ \kappa \in Reg}_{\kappa^{<\kappa}=\kappa}$, $\kappa > \aleph_0 \Longrightarrow \exists M \models T$ saturated, of power κ.

Proof

$(*)$ $|S_1(A)| \leq 2^{|A|+\aleph_0}$, because: $|L_1(A)| = |A| + \aleph_0$

Here: $|A| < \kappa \Rightarrow |S_1(A)| \leq \kappa$.

Lemma $N \models T$, $\|N\| \leq \kappa \Rightarrow X_N := \bigcup\{S_1(A): A \subseteq N \ \&\ |A| < \kappa\}$
the set has power $\leq \kappa$.

Pf • $|\{A \subset N : |A| < \kappa\}| \leq \kappa^{<\kappa} = \kappa$.

• $|S_1(A)| \leq \kappa$ for such A.

Proof of the thm.

M_α, $\alpha < \kappa$: elementary chain of models of T of power κ.

• M_0: whatever

• $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$, when $\delta < \kappa$ limit.

• $M_{\alpha+1} \succ M_\alpha$ such that $\forall p \in X_{M_\alpha}$ $p(M_{\alpha+1}) \neq \emptyset$:

$T' = Th(M_\alpha, m)_{m \in M_\alpha} \cup \bigcup_{\beta<\kappa} \{\varphi(c_\beta) : \varphi(x) \in p_\beta\}$,

new constant symbols,

where $X_{M_\alpha} = \{p_\beta : \beta < \kappa\}$

and T' in language $L(M_\alpha) \cup \{c_\beta : \beta < \kappa\}$.

T': consistent, has model of power $\varkappa$: $M_{\alpha+1}$
such that $M_\alpha \prec M_{\alpha+1}$.

$M = \bigcup\limits_{\alpha<\varkappa}^{\nearrow} M_\alpha$: of power $\varkappa$, saturated:

Let $A \subseteq M$, $|A| < \varkappa$ and $p \in S_1^M(A)$
$\varkappa \in Reg \Rightarrow A \subseteq M_\alpha$ for some $\alpha < \varkappa$.

proof: $A = \{a_\beta : \beta < \overset{\text{CN}}{\mu}\}$ for some $\mu < \varkappa$.
- $\forall \beta < \mu \ \exists \alpha_\beta < \varkappa \ a_\beta \in M_{\alpha_\beta}$
- $\{\alpha_\beta : \beta < \mu\} \subseteq \varkappa$, $\mu < cf(\varkappa) = \varkappa$

$\Rightarrow \exists \underset{\uparrow}{\alpha} < \varkappa \ \forall \beta < \mu \ \alpha_\beta < \alpha$
$A \subseteq M_\alpha$.

~~Let~~ $M_\alpha \prec M \Rightarrow p \in S_1^{M_\alpha}(A) = S_1^M(A)$

p realized in $M_{\alpha+1}$ by some $a \in M_{\alpha+1}$
$a \models p$ in $M_{\alpha+1} \underset{M_{\alpha+1} \prec M}{\Longrightarrow} a \models p$ in M.

Monster model:

Let $\bar{\varkappa}$: a large cardinal number.
"Ideal model" $\mathcal{M} \models T$: saturated of power $\bar{\varkappa}$
because: $\forall M \models T \ (\|M\| < \bar{\varkappa} \Rightarrow \exists M' \prec \mathcal{M} \ M \cong M')$.

Advantages of saturated model $\mathcal{M}$:

(i) universality

(ii) strong homogeneity

~~More~~ More Weakly (a bit):

(1) $\bar{\kappa}$-universality

(2) strong $\bar{\kappa}$-homogeneity

$Aut(M)$: the group of automorphisms of M

$Aut(M/A) = \{f \in Aut(M) : f\restriction_A = id_A\}$: automorphisms of M over A.

$A \subseteq M$

Lemma Assume M is strongly κ-homogeneous, κ-saturated, $A \subseteq M$, $|A| < \kappa$. Then:

(1) For $a, b \in M$ ($tp(a/A) = tp(b/A) \iff a, b$ are in the same orbit of $Aut(M/A)$ on M.

(2) $[\text{orbits } Aut(M/A) \text{ on } M^n] \xleftrightarrow[\text{onto}]{1\text{-}1} S_n(A)$

Proof (1) $\Leftarrow$: $f \in Aut(M/A)$, $f(a) = b$

$\Downarrow$

$tp(a/A) = tp(b/A)$.

$\Rightarrow$: $tp(a/A) = tp(b/A) \Rightarrow f : Aa \xrightarrow{\equiv} Ab$

$f\restriction_A = id_A$, $f(a) = b$

strong κ-homogeneity, $|A| < \kappa$ $\Rightarrow$ $f \subseteq g \in Aut(M)$, $g \in Aut(M/A)$

$g(a) = b$: a, b in the same orbit of $Aut(M/A)$

(2) $M^n \supseteq \mathcal{O} \xrightarrow[\varphi]{(1)} p_{\mathcal{O}} \in S_n(A)$

$\mathcal{O}$: orbit of $\mathrm{Aut}(M/A)$; $p_{\mathcal{O}}$ = common type $tp(a/A)$ for $a \in \mathcal{O}$.

Φ : {orbits of $\mathrm{Aut}(M/A)$ on M^n} $\xrightarrow{\varphi} S_n(A)$

$\mathcal{O}_1 \neq \mathcal{O}_2 \overset{(1)}{\Rightarrow} p_{\mathcal{O}_1} \neq p_{\mathcal{O}_2}$ $\boxed{\text{so } \varphi : 1\text{-}1}$

[if $p_{\mathcal{O}_1} = p_{\mathcal{O}_2}$ then let $a \in \mathcal{O}_1, b \in \mathcal{O}_2 \Rightarrow \exists g \in \mathrm{Aut}(M/A)$ $g(a) = b$ ↯.]

M: κ-saturated $\Rightarrow \varphi$: "onto".

<u>Def</u> Let $\bar{\kappa}$: a (large) cardinal number, $\mathcal{M} \models T$ <u>monster model</u> (w.r. to $\bar{\kappa}$), if $\mathcal{M}$: $\bar{\kappa}$-saturated, strongly $\bar{\kappa}$-homogeneous

Thm. Assume $\aleph_0 \leq \kappa \in CN$. Then $\exists M$: κ-saturated & strongly $\bar{\kappa}$-saturated.

<u>Proof</u> $M = \bigcup^{\nearrow}_{\alpha < \kappa^+} M_\alpha$: union of elementary chain s.t.:

(1) $M_0 \models T$ any

(2) $M_\delta = \bigcup_{\alpha < \delta} M_\alpha$ if $\delta \in Lim$,

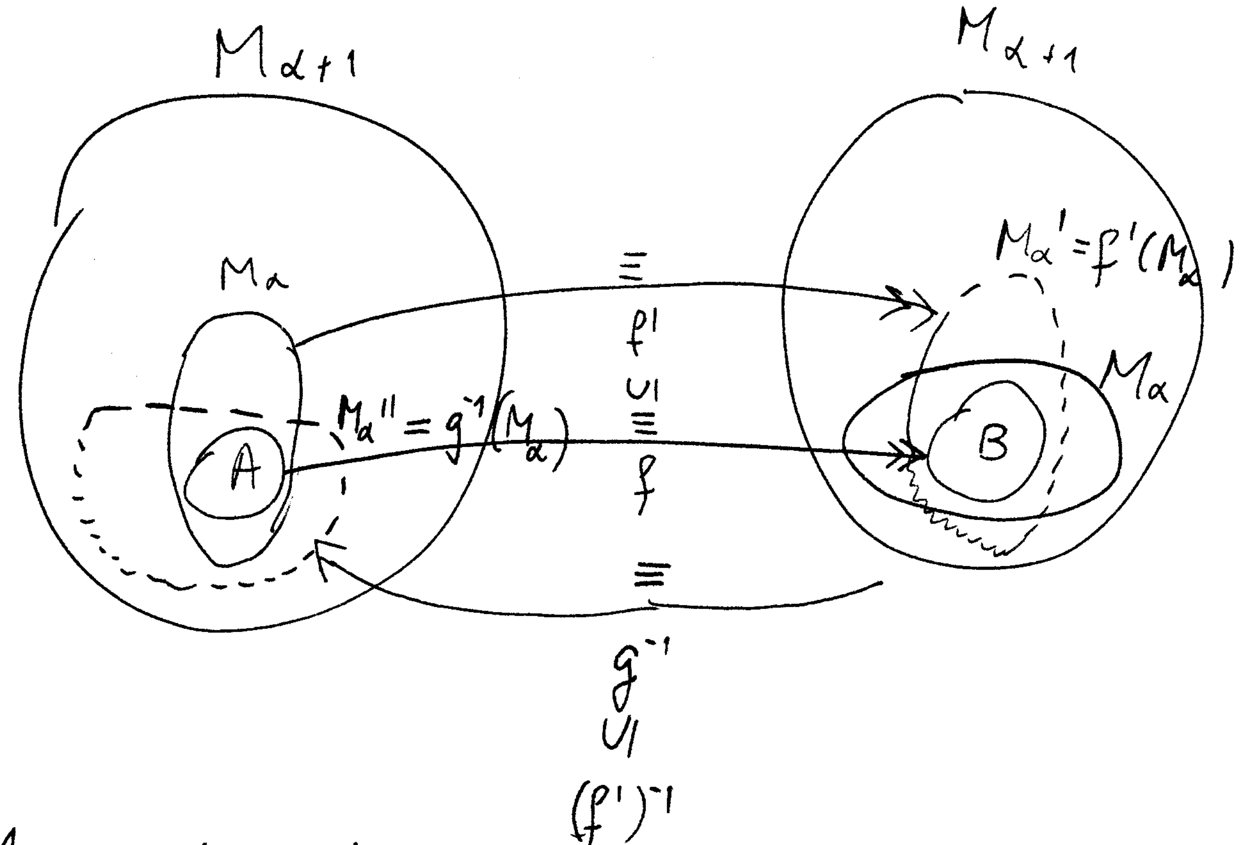
(3) $M_{\alpha+1} \succ M_\alpha$ s.t:

(a) $\forall p \in S_1(M_\alpha)$ p realized in $M_{\alpha+1}$

$$(b)\ \left(\forall f : \underset{M_\alpha}{\underset{\cap}{A}} \xrightarrow{\equiv} \underset{M_\alpha}{\underset{\cap}{B}}\right)(\exists\, g \supseteq f)\left(g : \underset{M_\alpha}{\underset{\cup}{A'}} \xrightarrow{\equiv} \underset{M_\alpha}{\underset{\cup}{B'}} \text{ in } M_{\alpha+1}\right)$$

It is enough that $M_{\alpha+1}$ is $\|M_\alpha\|^+$-saturated,
To satisfy (a), (b). $M_{\alpha+1} \succ M_\alpha$

Proof of (b) for such $M_{\alpha+1}$:



I. M: κ-saturated: <u>clear</u>

II M strongly κ-homogeneous:

Assume $A \subset M$, $|A| < \kappa$. Then $A \subseteq M_\alpha$, $B \subseteq M_\alpha$

$f : A \xrightarrow{\equiv} M$ for some $\alpha < \kappa$.

$B = f[A]$

- we construct

a sequence f_β, $\alpha \le \beta < \varkappa^+$:

- increasing $f_\beta : M \xrightarrow[\equiv]{\equiv} M$ partial elementary
- $f \subseteq f_\alpha$

(*) $M_\beta \subseteq \operatorname{dom} f_\beta \cap \operatorname{rng} f_\beta$.

- f_α constructed according to 3)(6)

$$f_\alpha : M_{\alpha+1} \xrightarrow[\equiv]{\equiv} M_{\alpha+1}$$

- $f_\beta : M_{\beta+1} \xrightarrow[\equiv]{\equiv} M_{\beta+1}$, as in 3.(6) when β : successor.
- $f_\delta = \bigcup_{\beta<\delta} f_\beta$ when δ limit. , still $f_\delta : M_\delta \xrightarrow[\equiv]{\equiv} M_\delta$.

$f_\infty = \bigcup^{\nearrow}_{\alpha \le \beta < \varkappa^+} f_\beta$, $f_\beta \in \operatorname{Aut}(M)$, $f \subseteq f_\beta$.

<u>Assumptions</u> Let $\bar{\varkappa}$: a cardinal number <u>large enough</u> so that:

(1) We consider only small models of T
of power $< \bar{\varkappa}$, or even $\ll \bar{\varkappa}$

(2) We work within a monster model $\mathcal{M} \models T$ (w.r.to $\bar{\varkappa}$)

(3) We consider only <u>small</u> models $M \prec \mathcal{M}$
of power $< \bar{\varkappa}$, or even $\ll \bar{\varkappa}$,

Consequences:

(1) For $M, N \prec \mathcal{M}$, $M \subseteq N \iff M \prec N$

(2) Convention: For $\bar{a} \subseteq \mathcal{M}$

$\models \varphi(\bar{a})$ means $\mathcal{M} \models \varphi(\bar{a})$

(3) For $A \subseteq M \prec \mathcal{M}$:

$S_n^M(A) = S_n^{\mathcal{M}}(A) =: S_n(A)$

<u>Notation</u> Assume $p(\bar{x})$, $q(\bar{x})$ types (small, over $\mathcal{M}$)

- $p(\bar{x}) \vdash q(\bar{x}) \iff p(\mathcal{M}) \subseteq q(\mathcal{M})$
  "p implies q"
- $p(\bar{x}) \underset{\substack{\uparrow \\ \text{equivalent}}}{\equiv} q(\bar{x}) \iff p \vdash q \ \&\ q \vdash p$

Special case: $p(\bar{x}) = \{\varphi(\bar{x})\}$.

$\varphi(\bar{x}) \vdash q(\bar{x})$: "φ isolates q".

<u>Remark</u>: Syntactically:

$p(\bar{x}) \vdash q(\bar{x}) \iff \forall \varphi(\bar{x}) \in q \ \exists \underset{\text{finite}}{p_0(\bar{x}) \subseteq p(\bar{x})}$

types over A

$T(A) \vdash \bigwedge p_0(x) \to \varphi(x)$

Remark (exercise)

$p(\bar{x}) \vdash q(\bar{x}) \iff \underset{|A|^+\text{-saturated}}{\forall M \models T} \ p(M) \subseteq q(M)$.

Def. (reminder)
Let $p(\bar{x})$: a type over A.
p is isolated over $A \iff \exists \varphi(\bar{x}) \in L(A)$ consistent (with T) $\varphi \vdash p$.

Thm (omitting types, Ehrenfeucht)
Assume $p_n(\bar{x}_n)$, $n<\omega$: a family of non-isolated types over $\emptyset$. Then $\exists M \models T$ $\forall n$ $\underbrace{p_n(M)=\emptyset}_{M \text{ omits } p_n}$.

Lemma
Assume A is ctble, $p_n(\bar{x}_n)$, $n<\omega$: a family of non-isolated types over A, $\varphi(x) \in L_1(A)$, $\underbrace{\varphi(\mathcal{M}) \neq \emptyset}_{\text{i.e. } \varphi \text{: consistent}}$.

Then $\exists c \in \varphi(\mathcal{M})$ $\forall n$ p_n non-isolated over $A \cup \{c\}$.

Proof [Lemma $\Rightarrow$ Thm]
By the Lemma: $\exists \underbrace{\{a_n : n<\omega\}}_{= A} \subseteq \mathcal{M}$ s.t.

(1) A satisfies the TV-test

(2) p_n, $n<\omega$: non-isolated over A.

Construction of a_n, $n<\omega$: recursion on n.
Let $\{\varphi_n(x, \bar{y}) : n<\omega\}$: all formulas of L of this form.

Suppose $n<\omega$ and $\{a_i : i<n\} = a_{<n}$ already constructed so that all p_k, $k<\omega$ still non-isolated over $a_{<n}$.

We consider a consistent formula $\varphi_n(x) \in L_1(a_{<n})$

By the Lemma we find $\underset{a_n}{\overset{\shortparallel}{c}} \in \varphi_n(\mathcal{M})$ so that

all p_k, $k<\omega$, still non-isolated over $a_{\leq n}$.

- the formulas $\varphi_n(x)$, $n<\omega$ may be chosen so that after ω steps:

  $\underset{\text{consistent}}{\forall \varphi(x) \in L(A)}\ \exists n\ \varphi = \varphi_n$.

Then $A = \{a_n : n<\omega\}$ satisfies TV-test

$A = M \prec \mathcal{M}$, every p_k still non-isolated over A.

$p_k(M) = \emptyset$ [if not, some $\underset{M}{\overset{\cap}{\overline{m}}} \models p_k$. then $(\overline{x}_k = \overline{m}) \vdash p_k(\overline{x}_k)$ ↯]

<u>Proof of the Lemma.</u>

Let $p(\overline{x})$: one of the types $p_n(\overline{x}_n)$.

Let $h(\overline{x}, y, \overline{a}) \in L(A$

$h(\overline{x}, c, \overline{a}) \vdash p(\overline{x}) \iff h(\mathcal{M}, c, \overline{a}) \subseteq p(\mathcal{M})$.

$\iff \forall \psi(\overline{x}) \in p(\overline{x})\ h(\mathcal{M}, c, \overline{a}) \subseteq \psi(\mathcal{M})$

$\iff \forall \psi \in p\ \ \mathcal{M} \models \forall \overline{x} \left(h(\overline{x}, c, \overline{a}) \to \psi(\overline{x}) \right)$

$\Leftrightarrow \forall \psi \in p \quad \psi_h(y) \in tp(c/A)(y)$

where $\psi_h(y) = \forall \bar{x}\,(h(\bar{x}, y, \bar{a}) \to \psi(\bar{x}))$

<u>hence</u>:

$$tp(c/A) = tp(c'/A) \Rightarrow [h(\bar{x}, c, \bar{a}) \vdash p \Leftrightarrow h(\bar{x}, c', \bar{a}) \vdash p]$$

$$h(\bar{x}, c, \bar{a}) \text{ consistent } \Leftrightarrow (\exists \bar{x}\, h(\bar{x}, y, \bar{a})) \in tp(c/A)(y).$$

Let $\underset{\uparrow}{X_{h,p}} = \{q \in S_1(A) : \text{For } c \models q,\ h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$ and $h(\bar{x}, c, \bar{a})$ consistent.$\}$

"bad types"

Let $q \in S_1(A)$ then

$$q \in X_{h,p} \Leftrightarrow q(y) \in \underbrace{S_1(A) \cap [\exists \bar{x}\, h(\bar{x}, y, \bar{a})]}_{\Updownarrow\ \text{For } c \models q \ h(\bar{x}, c, \bar{a}) \text{ consistent}} \cap \underbrace{\bigcap_{\psi \in p} [\psi_h(y)]}_{\Updownarrow\ \text{For } c \models q \ h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})}$$

(*) $X_{h,p}$: nowhere dense in $S_1(A)$.

Proof of (*) : (a.a.)

Suppose $\theta(y) \in L_1(A)$ and $\emptyset \neq S_1(A) \cap [\theta] \subseteq X_{h,p}$.

Let $\underset{\substack{\uparrow \\ L(A)}}{\alpha(\bar{x})} = \exists y (h(\bar{x}, y, \bar{a}) \wedge \theta(y))$

• $\alpha(\bar{x})$: consistent :

Let $c \in \theta(\mathcal{M})$

$\Downarrow$ $[\theta] \subseteq X_{h,p}$)

$\mathcal{M} \models \exists \bar{x}\ h(\bar{x}, c, \bar{a})$.

Let $\bar{d} \subseteq \mathcal{M}$ s.t. $\mathcal{M} \models h(\bar{d}, c, \bar{a})$.

$\bar{d}$ satisfies in $\mathcal{M}$: $\underbrace{\exists y\, h(\bar{x}, y, \bar{a})}_{\shortparallel}$

$\alpha(\bar{x})$.

• $\alpha(\bar{x}) \vdash p(\bar{x})$, i.e. $\alpha(\mathcal{M}) \subseteq p(\mathcal{M})$.

Let $\bar{d} \in \alpha(\mathcal{M})$. So there is $c \in \mathcal{M}$ st

$\models h(\bar{d}, c, \bar{a}) \wedge \theta(c)$

$\Downarrow$ $[\theta] \subseteq X_{h,d}$

$\forall \psi \in p \ \ \models \psi_h(c)$

$\Downarrow$

$\forall \psi \in p \ \ h(\bar{x}, c, \bar{a}) \vdash \psi(\bar{x})$ $\Rightarrow$ $h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$

$\overset{\#}{\bar{d}} \Rightarrow \overset{\#}{\bar{d}}$ ↯.

as : $p(\bar{x})$ non-isolated

Let $X = \bigcup_{h, p_n} X_{h, p_n} \subseteq S_1(A)$.

↑ meager. Let $q \in S_1(A) \cap [\varphi] \setminus X$

$c \models q$ good.

Pf (a.a) Suppose $p = p_n$ isolated over $A \cup \{c\}$.

$\exists\ h(\bar{x}, c, \bar{a})$ $\underset{\text{consistent}}{h(\bar{x}, c, \bar{a})} \vdash p(x)$ $\Rightarrow$ $\underset{tp(\bar{c}/A)}{\overset{\shortparallel}{q}} \in X_{h, p_n}$ ↯.

14.03.2022

Def T is quantifier eliminable if $\forall \varphi \in L\ \exists \underset{\text{open}}{\psi} \in L$ ($\equiv$ q.f.)

$T \vdash \varphi \leftrightarrow \psi$

Def. For $p(\bar{x}) \in S_n(\emptyset)$ let $p_0(\bar{x}) = \{\varphi(\bar{x}) \in p(\bar{x}) : \varphi \text{ open}\}$.

Remark T is q.e. $\iff \forall n\, \forall p \in S_n(\emptyset)\ p_0 \vdash p$

Proof „$\Rightarrow$" Obvious. „$\Leftarrow$" Let $\varphi(\bar{x}) \in L$.

- $\forall p \in [\varphi] \cap S_n(\emptyset)\ \exists \underset{\text{open}}{\psi} \in p\quad p \in [\psi] \subseteq [\varphi]$

Why? $p_0 \vdash p$

$\Downarrow$

$p_0 \vdash \varphi$

by compactness

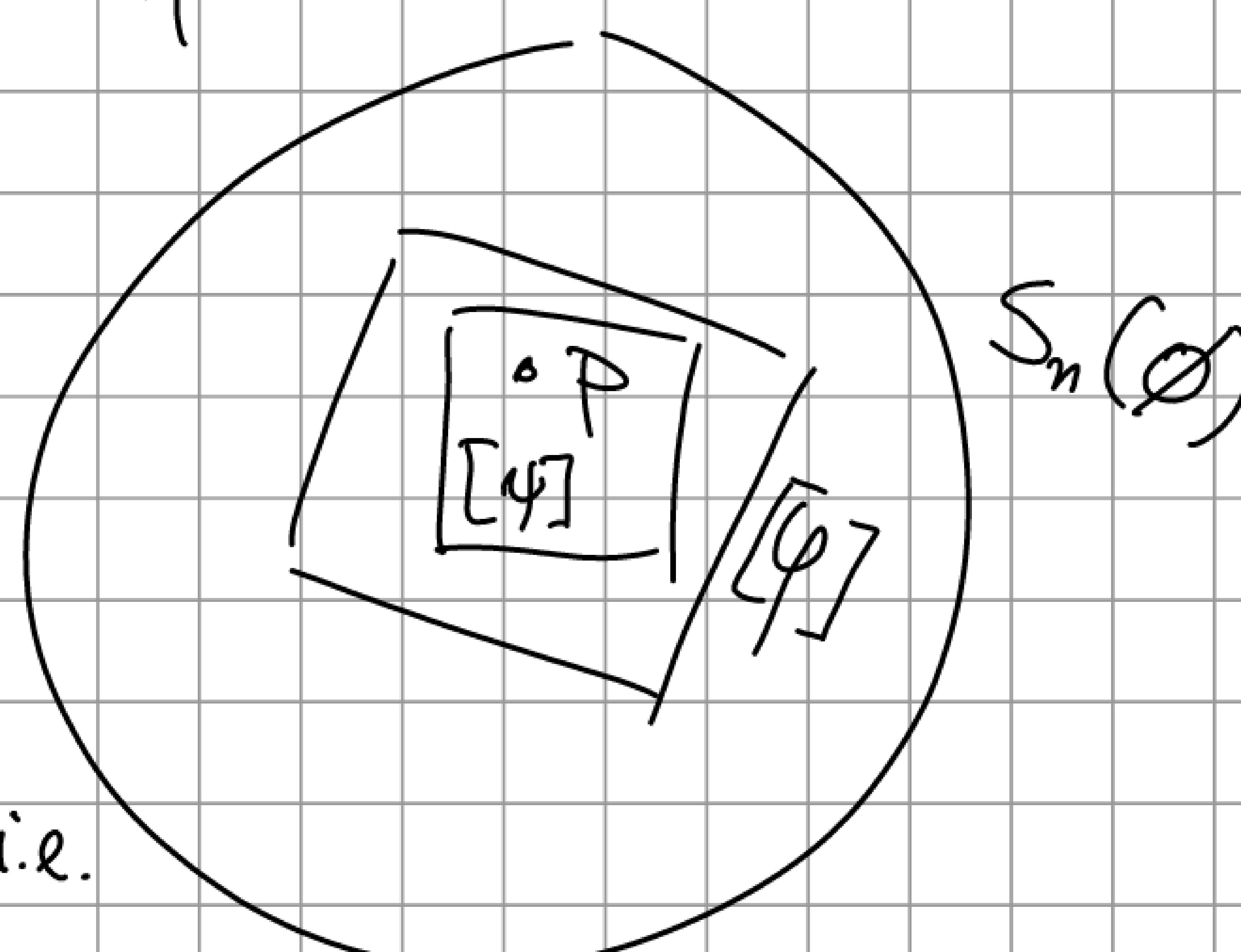
$\exists$ finite $p_0' \subseteq p_0$ s.t.

$p_0' \vdash \varphi$, i.e.

$p_0'(\mathcal{M}) \subseteq \varphi(\mathcal{M})$

$\wr$

$\psi(\mathcal{M}) = \left(\bigwedge_{\psi' \in p_0'} \psi'\right)(\mathcal{M}) \subseteq \varphi(\mathcal{M}) \rightsquigarrow \mathcal{M} \models \psi(\bar{x}) \to \varphi(\bar{x})$



$\psi \vdash \varphi \Rightarrow [\psi] \cap S_n(\emptyset) \subseteq [\varphi] \cap S_n(\emptyset)$

$\blacksquare$

Application $L = \{+, \cdot, 0, 1\}$: the language of rings.

ACF_p : the theory of algebraically closed fields of char p, in L.

Axioms:

1) field axioms

2) char $= p \neq 0$: $\underbrace{1 + \ldots + 1}_{p} = 0$

2') $p = 0$: $\underbrace{1 + \ldots + 1}_{n} \neq 0$ for $n \geq 1$

3) Every polynomial of deg n has a root:
$0 < n$

$$\forall y_{n-1}, y_{n-2}, \ldots, y_0 \, \exists x \; x^n + y_{n-1} x^{n-1} + \ldots + y_0 = 0.$$

Fact ACF_p is complete.

Proof Let $M, N \models ACF_p$. Enough to show that $M \equiv N$. Let $\varkappa > \|M\|, \|N\|$ and let $\underbrace{M' \succ M, \; N' \succ N}_{\text{power } \varkappa}$.

M', N': uncountable acl fields of the same power and char

$$\Downarrow \text{algebra}$$

$$M' \cong N' \Rightarrow M' \equiv N'$$

$$\begin{array}{ccc} \text{\rotatebox{90}{$\equiv$}} & \Downarrow & \text{\rotatebox{90}{$\equiv$}} \\ M & \equiv & N \end{array}$$

<u>Fact</u> ACF_p is q.e. (Chevalley, Tarski)

<u>Proof</u> (in $\mathcal{M}$) We will show that $\forall p \in S_n(\emptyset)$ $p_0 \vdash p$ ($\Leftrightarrow p_0(\mathcal{M}) \subseteq p(\mathcal{M})$). Let $\bar{a} \models p_0$, $\bar{b} \models p$, $\bar{a}, \bar{b} \subseteq \mathcal{M}$. It's enough to prove that $\exists f \in Aut(\mathcal{M})\ f(\bar{a}) = \bar{b}$.

$\bar{a} = (a_1, \dots, a_n)$
$\bar{b} = (b_1, \dots, b_n)$

Let $\langle \bar{a} \rangle, \langle \bar{b} \rangle$: the subrings with 1 of $\mathcal{M}$ generated by $\bar{a}, \bar{b}$. $\bar{a}, \bar{b} \models p_0 \Rightarrow \langle \bar{a} \rangle \cong \langle \bar{b} \rangle$

$a_i \overset{f}{\mapsto} b_i$

Some algebraic magic.

$$\langle \bar{a} \rangle \cong \langle \bar{b} \rangle$$
$$\cap \quad \Downarrow \text{unique} \quad \cap$$
$$\mathcal{M} \supseteq \langle \bar{a} \rangle_0 \cong \langle \bar{b} \rangle_0 \subseteq \mathcal{M}$$

(fraction field) (not unique) $\mathcal{M} \supseteq \underbrace{\langle \bar{a} \rangle_0^{alg}}_{F_a} \cong \underbrace{\langle \bar{b} \rangle_0^{alg}}_{F_b} \subseteq \mathcal{M}$

$$\operatorname{trdeg}(\mathcal{M}/F_a) = \|\mathcal{M}\| = \operatorname{trdeg}(\mathcal{M}/F_b)$$

$$\Downarrow$$

$$\begin{array}{ccc} \mathcal{M} \cong & F_a(X_\alpha,\ \alpha<\lambda)^{alg} & \\ f \ \cong\downarrow & & \cong \\ \mathcal{M} \cong & F_b(x_\beta,\ \beta<\lambda)^{alg} & \end{array}$$

$$f \in \operatorname{Aut}(\mathcal{M}),\ f(\bar a) = \bar b.$$

$\blacksquare$

Types in $T = ACF_p$

Let $\mathcal{M} \models T$: a monster model.

$\quad\quad\quad\quad\ \ \cup\!|$
subfield (small) K. We will describe $S_n(K)$.

Let $\bar a \subseteq \mathcal{M}$, $|\bar a| = n$.

$$K[\bar x] \rhd I(\bar a/K) = \{f \in K[\bar x] : f(\bar a) = 0\}$$

Remark 1) $tp(\bar a/K) = tp(\bar a'/K) \iff I(\bar a/K) = I(\bar a'/K)$

2) $\forall \underset{\text{prime}}{I} \lhd K[\bar x]\ \exists \bar a \subseteq \mathcal{M}\ \ I(\bar a/K) = I.$

Proof 1) "$\Rightarrow$" $tp(\bar a/K) = tp(\bar a'/K) \Rightarrow \exists f \in \operatorname{Aut}(\mathcal{M}/K)$

$$I(\bar a/K) = I(\bar a'/K) \qquad f(\bar a) = \bar a'$$

[alternatively: $f \in I(\bar a/K) \iff$ "$f(\bar x) = 0$" $\in tp(\bar a/K)$]

"$\Leftarrow$" Assume $I(\bar{a}/K) = I(\bar{a}'/K) = I$.

$$\Downarrow$$

$$K[\bar{a}] \underset{K}{\cong} K[\bar{x}]/I \cong K[\bar{a}']$$

$$\Downarrow$$

$$\exists f \in Aut(\mathcal{M}/K)\ f(\bar{a}) = \bar{a}'$$

$$\Downarrow$$

$$tp(\bar{a}/K) = tp(\bar{a}'/K).$$

2) $K \subseteq K[\bar{x}]/I = K[\bar{a}] \supseteq \bar{a} = \bar{x}/I$ and $I(\bar{a}/K) = I$.

$$\vdots$$

$\square$

$S_1(K) = \{tp(a/K) : a \in \mathcal{M}\}$: a top. space.

$\Downarrow$

$p(x) = tp(a/K)$, $I_p = I(a/K) \lhd K[x]$

a) $I_p \neq \{0\}$, i.e. a is algebraic $/K$, so

$0 \neq f$ $\quad$ "$f(x) = 0$" $\in p(x)$. In fact,

irreducible over K $\quad$ "$f(x) = 0$" $\vdash p(x)$ (isolates)

Let $a' \in \mathcal{M}$ s.t. $f(a') = 0 \Rightarrow I(a'/K) \ni f \Rightarrow$ f generates $I(a'/K)$

$p = tp(a/K) = tp(a'/K) \Leftarrow I(a/K) = I(a'/K) \Leftarrow$

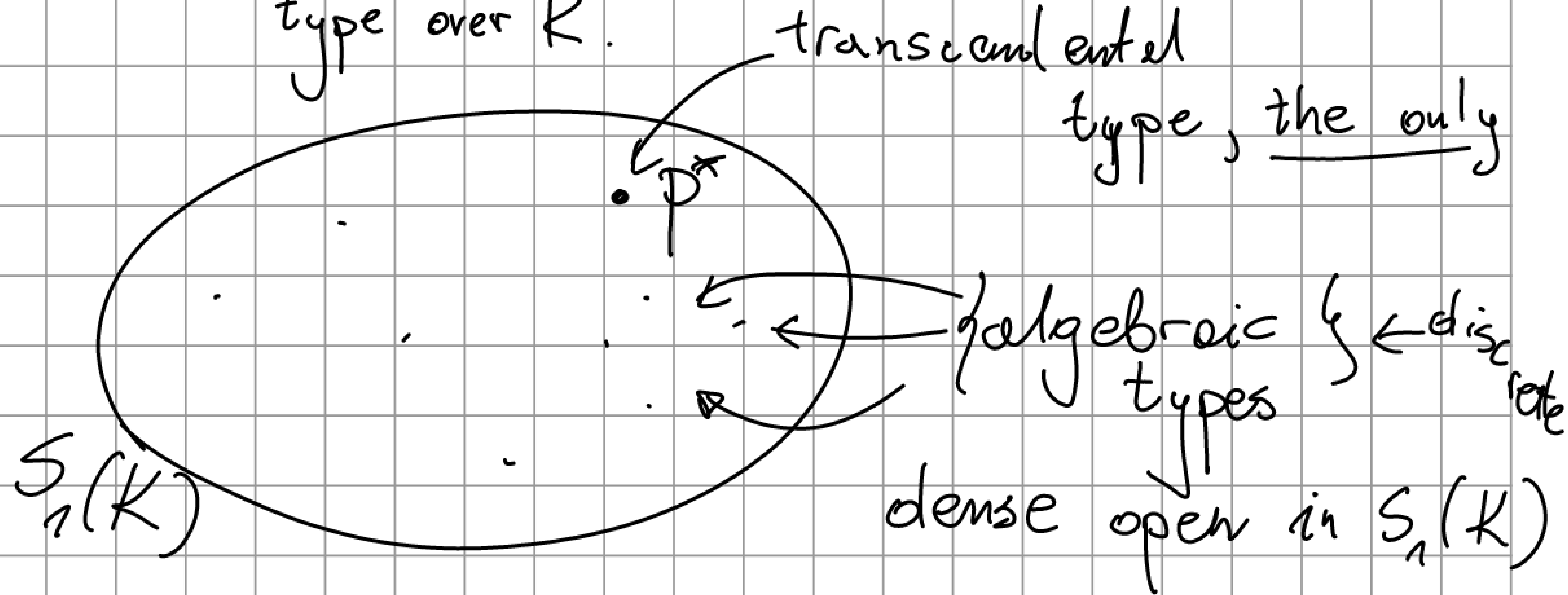
$p(x)$ here is called algebraic.

More generally (T: arbitrary)

Def $\varphi(\bar{x}) \in L(A)$ is called algebraic, if $0 < |\varphi(\mathcal{M})| < \aleph_0$, similarly for q: a type.

b) $I_p = \{0\}$: $p = tp(a/K)$ s.t. $a \in \mathcal{M}$ transcendental over K

the transcendental type over K.



If K ctble then $S_1(K)$ ctble

Corollary ACF_p is $\aleph_0$-stable [recall: T is $\varkappa$-stable $\iff \forall A \subseteq \mathcal{M}, |A| \leq \varkappa$ $|S_1(A)| \leq \varkappa$]

Proof Let $\underset{\text{ctble}}{A} \subseteq \mathcal{M}$.

$A \subseteq \underset{\substack{\uparrow \\ \text{ctble} \\ \text{subfield}}}{K} \subseteq \mathcal{M} \quad |S_1(A)| \leq |S_1(K)| = \aleph_0$

Remark T is $\underbrace{\text{totally transcendental}}$ $\iff T: \aleph_0$-stable

Proof "$\Rightarrow$" from def, "$\Leftarrow$": (A.a.) Let $\varkappa > \aleph_0$.

Suppose $|A| \leq \varkappa < |S_1(A)|$ for some $A \subseteq \mathcal{M}$.

Shall find $\underset{\text{ctble}}{A_0} \subseteq A$ with $|S_1(A_0)| \geq 2^{\aleph_0}$.

(def) $\varphi(x) \in L(A)$ is <u>large</u> iff $|S_1(A) \cap [\varphi]| > \varkappa$

otherwise: $\varphi(x)$ small.

(a) "$x = x$" is large

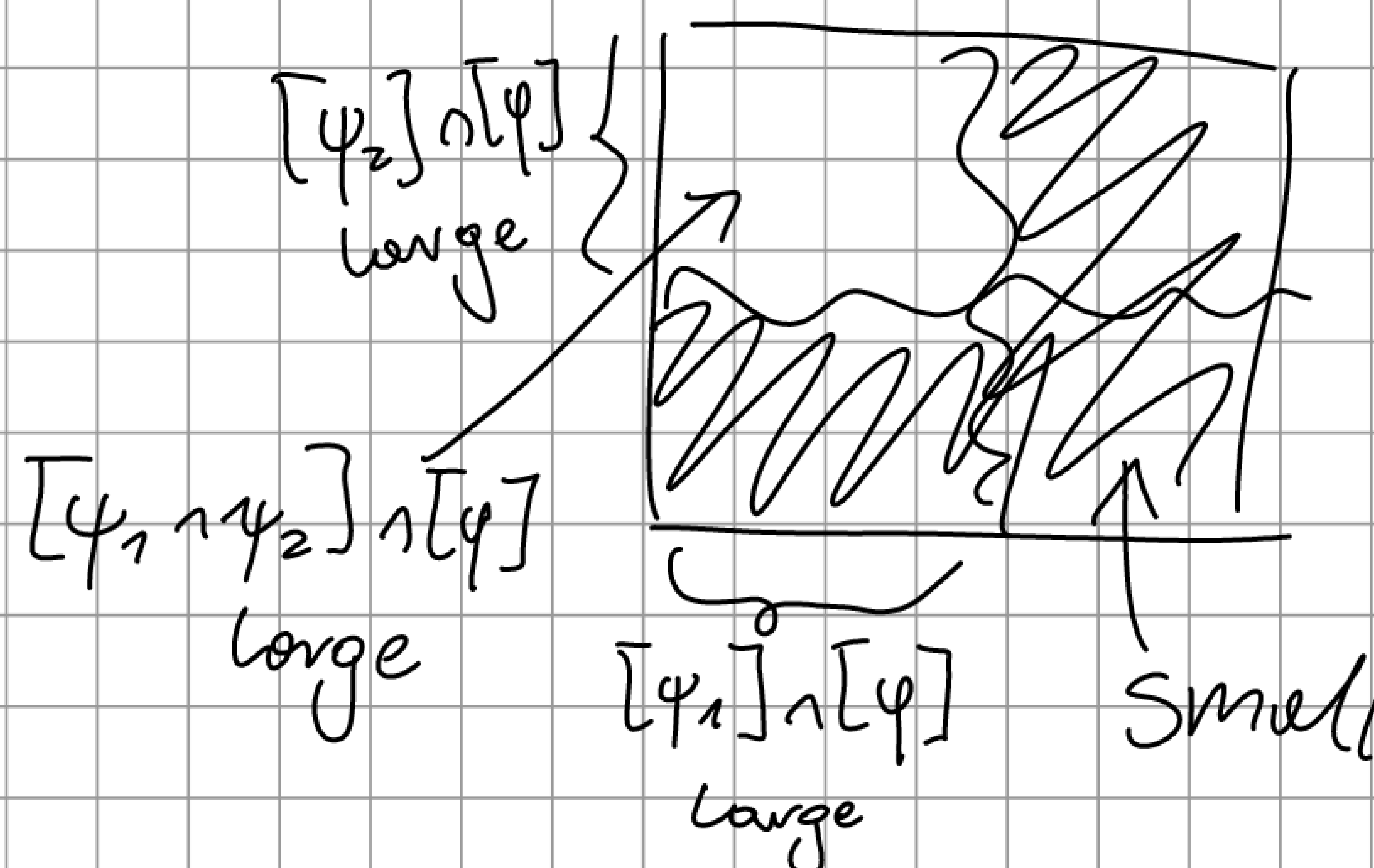
(b) if $\varphi(x)$: large, then $\exists \psi_1, \psi_2 \in L(A)$ large

s.t. $\varphi(\mathcal{M}) = \psi_1(\mathcal{M}) \,\dot\cup\, \psi_1(\mathcal{M})$

Pf. (b) <u>if not</u> then $p^* = \{\psi \in L_1(A) : \psi \wedge \varphi \text{ is large}\}$

is a complete type is $S_1(A) \cap [\varphi]$

1° p^*: consistent: $\psi_1, \psi_2 \in p^* \Rightarrow \psi_1 \wedge \psi_2 \in p^*$



2° p^*: complete OK

$p^* \in S_1(A) \cap [\varphi]$: the only large type.

$$\underbrace{S_1(A)\cap[\varphi]}_{>\varkappa} = \underbrace{\{p^*\} \cup \bigcup_{\substack{\psi \in L_1(A) \\ \text{small} \\ \underbrace{\psi \vdash \varphi}_{\leq \varkappa}}} \underbrace{(S_1(A)\cap[\varphi])}_{\leq \varkappa}}_{\leq \varkappa}$$

↯

c) a tree of large formulas in $L_1(A)$ $\varphi_\eta(x)$, $\eta \in 2^{<\omega}$ st. $\varphi_\eta(\mathcal{M}) = \varphi_{\eta 0}(\mathcal{M}) \dot\cup \varphi_{\eta 1}(\mathcal{M})$

by (b)

Let $A_0 \subseteq A$: the set of all params of $\varphi_\eta, \eta \in 2^{<\omega}$

Then $|A_0| \leq \aleph_0$.

For $\eta = 2^\omega$: $p_\eta^0 = \{\varphi_{\eta|n}(x) : x < \omega\}$: a consistent 1-type over A_0

When $\nu \neq \eta$ then $p_\eta \neq p_\nu$

$p_\eta \in S_1(A_0)$

$\Downarrow$

$|S_1(A_0)| \geqslant 2^{\aleph_0} > \aleph_0$ ↯

21.03.2022

CONSTRUCTION OF SPECIAL MODELS: $N, M \models T$

Def. (1) M is *atomic* if $\forall \bar{a} \subseteq M$ (finite) $tp(\bar{a}/\emptyset) = tp(\bar{a})$ is isolated.

(2) M is *prime* if $\forall N \models T \; \exists f : M \overset{\equiv}{\longrightarrow} N$

Example $T = ACF_p$, F_p: prime field of char p

(a) $\hat{F}_p$: atomic (exercise)

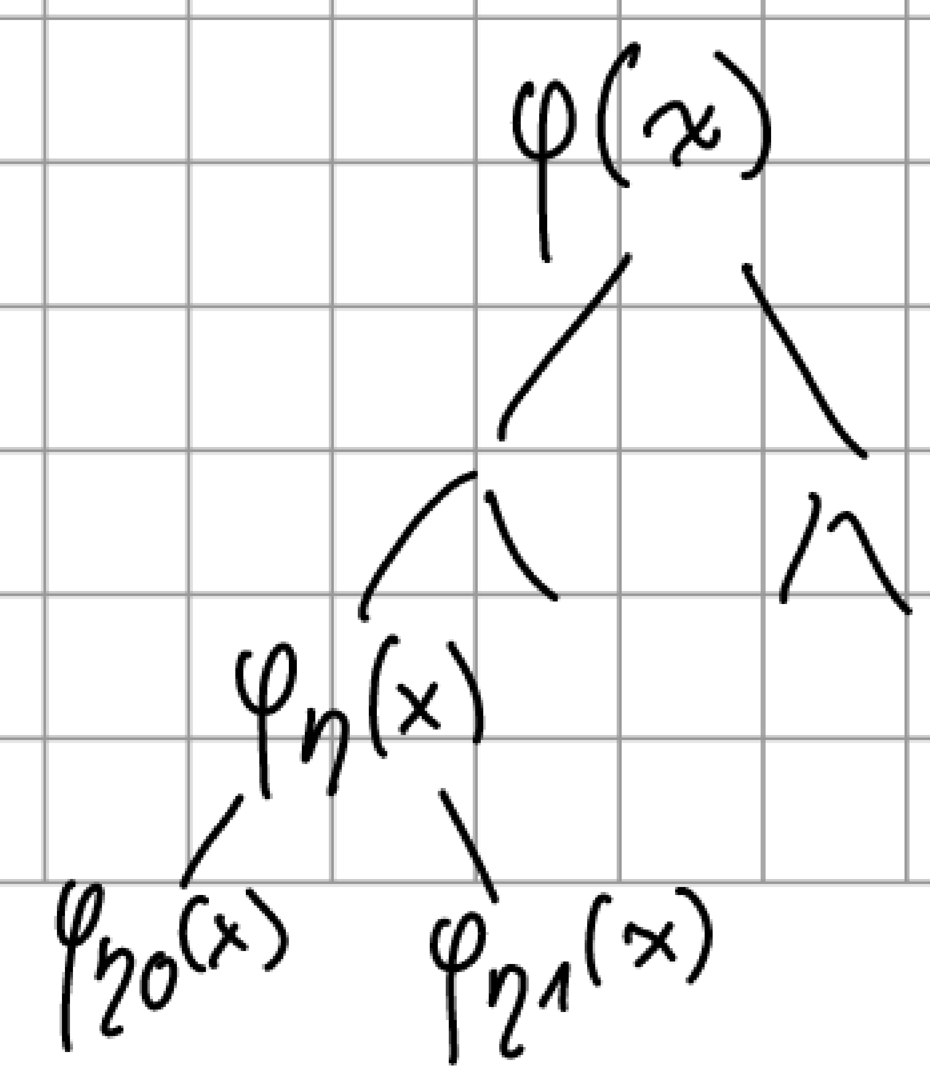
(b) $\hat{F}_p$: prime (exercise)

Thm. T: $\aleph_0$-stable $\Rightarrow$ T has a prime model.

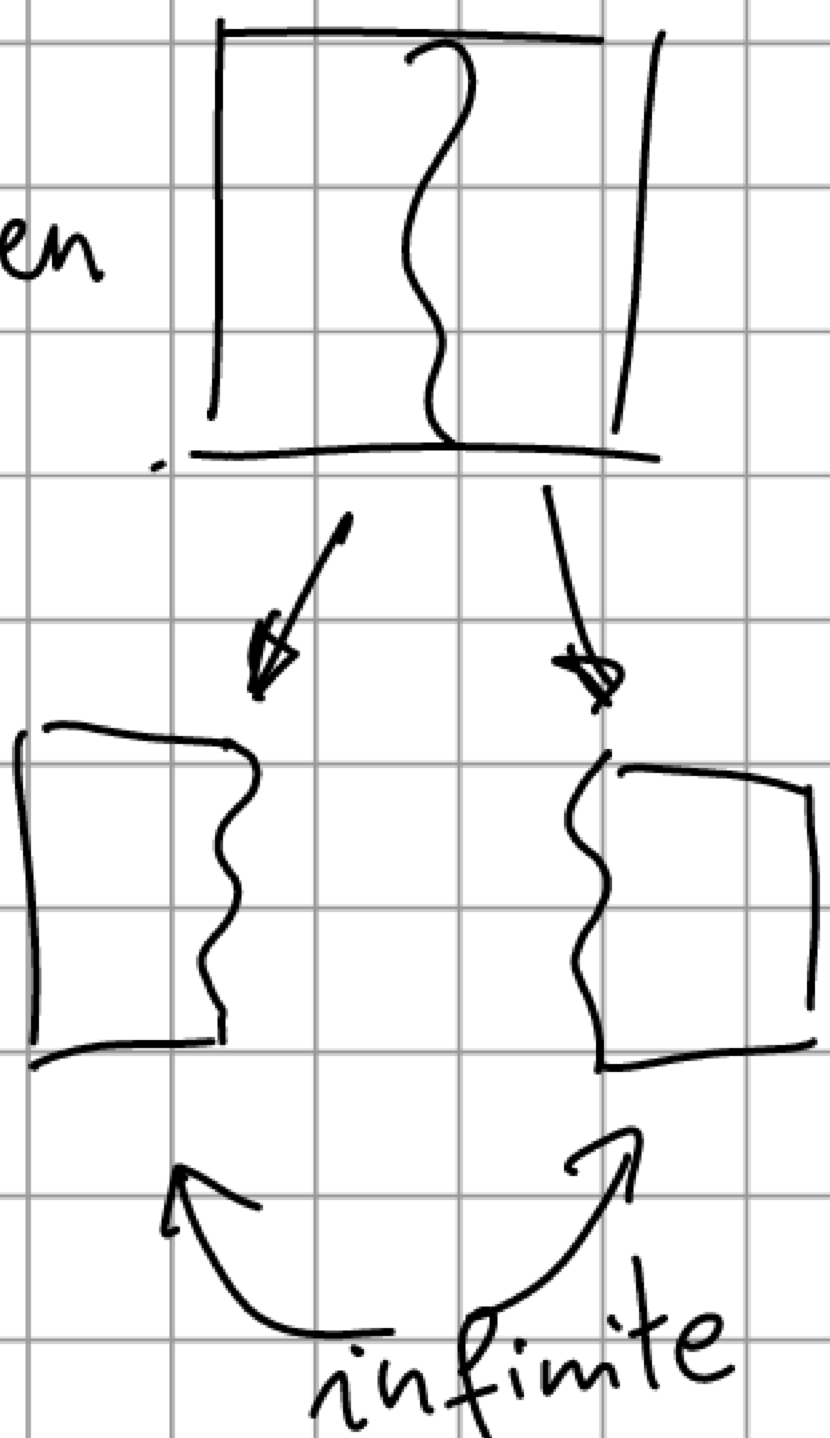
Lemma 1 T: $\aleph_0$-stable $\Rightarrow$ $\forall A \subseteq \mathcal{M}$ $\{$isolated types$\} \subseteq S_1(A)$ dense

Pf. Suppose $\varphi(x) \in L_1(A)$ (consistent with T) s.t. in $S_1(A) \cap [\varphi]$

there is no isolated types $\Rightarrow$ a tree of formulas $\varphi_\eta(x) \in L_1(A)$, $\eta \in 2^{<\omega}$



$\rightsquigarrow$ contradicts $\aleph_0$-stability of T

clopen  $\subseteq S_1(A) \cap [\varphi]$

$\square$

Lemma 2 ($a, b \in \mathcal{M}$) $tp(a)$ isolated and $tp(b/a)$ isolated
$\iff tp(ab)$ isolated

Pf "$\Rightarrow$": $\varphi(x) \vdash tp(a)$, $\psi(a,y) \vdash \underbrace{tp(b/a)(y)}_{= \, p_a(y) \subseteq S_y(a)}$

Then: $\varphi(x) \wedge \psi(x,y) \vdash tp(ab)$

Let $a', b' \in \mathcal{M}$ satisfy $\varphi(x) \wedge \psi(x,y)$

$\models \varphi(a') \Rightarrow tp(a') = tp(a) \Rightarrow \psi(a',y) \vdash \underset{\displaystyle S_y(a')}{\underset{\cap}{p_{a'}(y)}}$

$\models \psi(a',b') \Rightarrow \models p_{a'}(b')$

$$\Downarrow$$

$$ab \xrightarrow{\equiv} a'b'$$

and $tp(ab) = tp(a'b')$

"$\Leftarrow$": $\theta(x,y) \vdash tp(a,b)$.

(a) "$\exists y\, \theta(x,y)$" $\vdash tp(a)$.

$\underset{tp(a)}{\cap}$

<u>because</u> Let $a' \in \mathcal{M}$ satisfy "$\exists y\, \theta(x,y)$".

So there is b' s.t. $\models \theta(a',b')$

$\Rightarrow tp(ab) = tp(a',b') \Rightarrow tp(a) = tp(a')$.

(b) $\theta(a,y) \vdash tp(b/a)(y)$

<u>because</u>: similar to (a) ∎

<u>Proof of the thm.</u> Construction of a prime model of T:

$$A = \{a_n : n < \omega\} \subseteq \mathcal{M} \quad \text{so that:}$$

1) A satisfies TV-test

2) $\forall n \quad tp(a_n / a_{<n})$ is isolated

At step n choose a_n: $\quad [a_{<n} = \{a_k : k < n\}]$

Let $\varphi(x) \in L_1(a_{<n})$ consistent.

Let $a_n \in \varphi(\mathcal{M})$ s.t. $tp(a_n / a_{<n})$ is isolated (lemma 1)

Suitable choice of φ's ensures (1).

$M \models T$ is prime. Let $N \models T$ arbitrary. We find $f(a_n) \in N$ for $n < \omega$ s.t. $f: M \xrightarrow{\equiv} N$.

At step n $f[a_{<n}] \subseteq N$ with $f: a_{<n} \xrightarrow{\equiv} N$

Let $p(x) = tp(a_n / a_{<n})$ (isolated)

$$\Downarrow$$

$$f^*(p)(x) \in S(f(a_{<n}))$$

isolated too, hence realised by $f(a_n)$

$$\Downarrow$$

$$f: a_{\leq n} \xrightarrow{\equiv} N$$

■

<u>Remark</u> (1) A prime model $M \models T$ is atomic.

(2) If $M \models T$ ctble atomic, then M prime

<u>Corrolary</u> $\hat{\mathbb{F}}_p$ is atomic.

Proof of remark

(1) Let $p(\bar{x}) \in S_n(\emptyset)$ non-isolated. Will show $p(M) = \emptyset$. Let $N \models T$ be ommiting p.

$\exists f : M \xrightarrow{\equiv} N \Rightarrow p(M) = \emptyset$.

(2) Let $\underset{\overset{\rotatebox{90}{$\models$}}{T}}{M} = \{a_n : n < \omega\}$ atomic.

Then $\forall n \; tp(a_{\leq n})$ is isolated

$$\Downarrow \text{lemma 2}$$

$\forall n \; tp(a_n / a_{<n})$ is isolated

$$\Downarrow \text{pf of thm}$$

M prime. ■

Corollary A prime model of T is unique (up to $\cong$)

Proof Let $M, N \models T$ both prime $\underset{\text{remark}}{\Rightarrow}$ M, N are ctble and atomic, so we have embeddings in both direction, using back-and-forth we get the iso.

Def $M \models T$ is <u>minimal</u> if $\neg \exists N \precneqq M$

<u>Example</u> $\widehat{\mathbb{F}_p}$ is minimal.

<u>Fact</u> T has a prime model $\iff \forall n \ \{\text{isolated types}\} \subseteq_{\text{dense}} S_n(\emptyset)$

<u>Proof</u> "$\Rightarrow$": Let $\underset{\text{prime}}{M \models T} \Rightarrow M$: atomic

what we need

$N \models T$ any, then $\forall n \ \{p \in S_n(\emptyset) : p(N) \neq \emptyset\}$ is dense in $S_n(\emptyset)$

exercise

"$\Leftarrow$": <u>Claim</u> Assume $\bar{a} \subseteq_{\text{finite}} M$ and $tp(\bar{a})$ is isolated.

Then $\{\text{isolated types}\} \subseteq_{\text{dense}} S_1(\bar{a})$.

<u>Proof of claim</u> Let $n = |a|$, $\varphi(\bar{x}) \vdash tp(\bar{a})$.

Let $\psi(\bar{x}, y) \in L_{n+1}(\emptyset)$ s.t. $\psi(\bar{a}, y)$ is consistent.

We seek $q(y) \in S_1(A) \cap [\psi(\bar{a}, y)]$ isolated.

Let $\chi(\bar{x}, y) = \varphi(\bar{x}) \wedge \psi(\bar{x}, y)$.

By assumptions of $\Leftarrow$ $\exists \underset{\text{isolated}}{p(\bar{x}, y)} \in S_{\bar{x},y}(\emptyset) \cap [\chi(\bar{x}, y)]$.

Let $\bar{a}', b' \models p(\bar{x}, y)$. Then $\bar{a}' \models \underbrace{p(\bar{x}, y) \upharpoonright_{\bar{x}}}_{\Uparrow \ \varphi(\bar{x})} = tp(\bar{a})$

Let $f \in Aut(\mathcal{M}) : f(\bar{a}') = \bar{a}$
$b = f(b')$

Then $\bar{a}'b' \overset{\equiv}{\underset{f}{\longmapsto}} \bar{a}b \Rightarrow \bar{a}b \models p(\bar{x}, y)$

so $\underset{p}{\underset{\shortparallel}{tp(\bar{a}b)}}$ is isolated $\underset{\text{lemma 2}}{\Rightarrow}$ $\underset{\psi(\bar{a}, y)}{\underset{\psi}{tp(b/\bar{a})}}$ isolated

So $q(y) = tp(b/\bar{a})$

Claim ⇝ we construct a model $M = \{a_n : n < \omega\} \prec \mathcal{M}$

s.t. $\forall n$ $tp(a_n / a_{<n})$ is isolated

$\Downarrow$ lemma 2

M atomic ctble $\Rightarrow$ M prime.

<u>Corollary</u> If $\forall n$ $|S_n(\emptyset)| \leq \aleph_0$, then T has a prime model.

<u>Corollary</u> A prime model (of a ctble T) is homogeneous (exercise).

The number of countable models of T: $I(T, \aleph_0)$, $n(T)$.

<u>Remark</u> $1 \leqslant n(T) \leqslant 2^{\aleph_0}$

$$\underset{\text{ctble}}{M} \models T \Rightarrow M \cong \underbrace{(\mathbb{N}, \ldots)}_{\leqslant 2^{\aleph_0}}$$ L-structures like that

<u>Recall</u> $n(T) = 1 \Leftrightarrow \forall n \ |S_n(\emptyset)| < \aleph_0$

(T: $\aleph_0$-categorical)

<u>Vaught conjecture (1961)</u>

$$n(T) > \aleph_0 \Rightarrow n(T) = 2^{\aleph_0}$$

<u>Thm</u> (M. Morley, 1971) $\aleph_0 < n(T) < 2^{\aleph_0} \Rightarrow n(T) = \aleph_1$

<u>Thm</u> (Vaught, 1961) $n(T) \neq 2$

<u>Proof</u> (A.a) suppose $n(T) = 2$.

$n(T) < 2^{\aleph_0} \Rightarrow T$ small (i.e. $\forall n \ |S_n(\emptyset)| \leqslant \aleph_0$)

$\vdots$

25.03.2022

Example (Andrzej Ehrenfeucht) Theory with exactly 3 ctble theories.

$T_0 = Th(\mathbb{Q}, \leq)$. Look at $T = T_0(\mathbb{N}) = Th(\mathbb{Q}, \leq, n)_{n \in \mathbb{N}}$.

T_0 is q.e. $\Rightarrow$ T is also q.e.

$S_1^T(\emptyset) = S_1^{T_0}(\mathbb{N})$. The types in $S_1^T(\mathbb{N})$:

realised in $(\mathbb{Q}, \leq \mathbb{N})$

isolated

- $p_i(x) \equiv \{x = i\}$, $i \in \mathbb{N}$
- $r_i(x) \equiv \{i-1 \leq x \leq i\}$, $i \in \mathbb{N}$, $-1 \approx -\infty$
- $s(x) \equiv \{x > i : i \in \mathbb{N}\}$

ommited

We will point 3 ctble models $N \models T$.

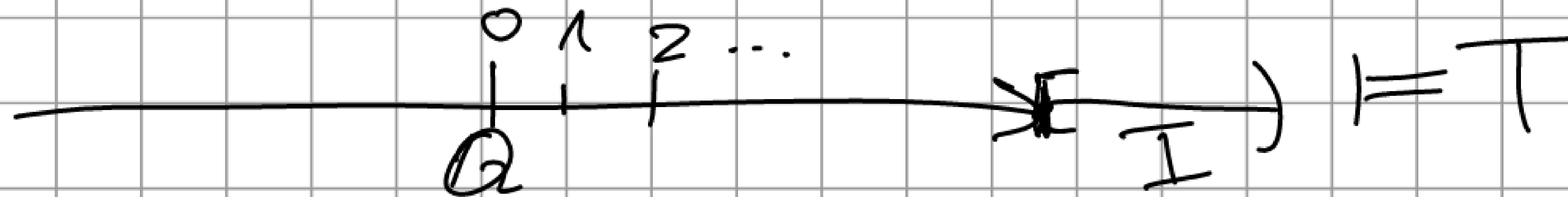
1° $N = M$: prime model of T.

2° $s(N)$ has the minimal element.

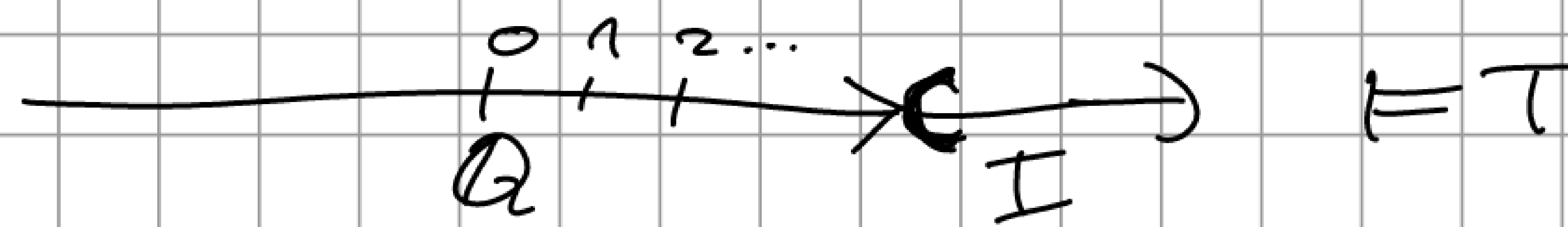
3° $s(N)$ hasn't minimal element.

2°: "$\mathbb{Q} \cup ([0,1)\cap \mathbb{Q})$" :

N:



3°:



Variants: 3, 4, 5, 6, ...

<u>Problem</u> Does there exist a stable T with

$1 < n(T) < \aleph_0$?

<u>Def</u> ($A \subseteq_{\text{small}} \mathfrak{M}$). $M \prec \mathfrak{M}$ is <u>prime over A</u> if:

(1) $A \subseteq M$

(2) $\forall \underset{A}{\underset{\cup\!\shortparallel}{N}} \prec \mathfrak{M} \ \exists f : M \xrightarrow{\equiv} N : f\restriction_A = id_A$

<u>Equivalently</u> M is a prime model of T(A).

- If A is ctble $\longrightarrow$ full description of prime models of T(A).
- If A is unctble $\longrightarrow$ in general not much can be set

Thm T $\aleph_0$-stable $\Longrightarrow \forall \underset{\mathcal{M}}{A} \exists M \models T(A)$ prime
$\hspace{7em}$ ($A \subseteq \mathcal{M}$)

Proof $M = A \cup \{a_\alpha : \alpha < ?\}$

Construction of a_α's s.t.:

(i) $A \cup \{a_\alpha : \alpha < ?\}$ satisfies the TV-test

(ii) $\forall \alpha < ?$ $tp(a_\alpha / A a_{<\alpha})$ is isolated.

At some point it has to terminate

(i.e. we cannot add more elements).

Claim Assume $\underset{A}{N} \prec \mathcal{M}$. Then $\forall \alpha \exists f : A \cup a_{<\alpha} \xrightarrow{\equiv} N$ ($N \supseteq A$)

s.t. $f\restriction_A = id_A$.

Proof We define $f(a_\beta)$ for all $\beta < \alpha$

by ind. on β so that $f\restriction_A = id_A$ and $f : A a_{\leq \beta} \xrightarrow{\equiv} N$.

Take $\beta < \alpha$ and suppose $\forall \beta' < \beta$ $f(a_{\beta'})$

so that the condition holds.

$p(x) = tp(a_\beta / A a_{<\beta})$ is isolated.

$f : A a_{<\beta} \xrightarrow{\equiv} f[A a_{<\beta}] \subseteq N$

$f(p)$ is realised by c

$\underset{\text{isolated}}{p(x)} \in S_1(A a_{<\beta}) \overset{}{\underset{f^*}{\rightsquigarrow}} \underset{\text{isolated}}{f(p)} \in S_1(f[A a_{<\beta}])$

Now we put $f(a_\beta)=c$.

Claim ■

By the claim after some time we cannot get any more elements.

Additional property of the construction:
At the step α we consider a consistent formula $\varphi(\alpha) \in L(A a_{<\alpha})$ with no realisation in $A a_{<\alpha}$, choose a_α s.t. $\models \varphi(a_\alpha)$.

Problems Is a prime model over A unique up to isomorphism over A?
Answer: not always. However the prime model M over A constructed by the previous construction is unique up to $\underset{A}{\cong}$ and it's called primary over A.

Thm M, N: primary over $A \Rightarrow M \underset{A}{\cong} N$.

Proof $M = A \cup \{a_\alpha : \alpha < \gamma\}$: an „isolated construction" of M over A, i.e. $tp(a_\alpha / Aa_{<\alpha})$ is isolated by a formula $\varphi_\alpha(x)$ over Aa_{C_α} s.t. $\underset{\text{finite}}{C_\alpha} \subseteq \gamma$

Def. $X \subseteq \gamma$ is closed if $\forall \alpha \in X$ $C_\alpha \subseteq X$

Remark (1) $\alpha \in \gamma \Rightarrow \exists$ minimal $\underset{\text{finite}}{X \subseteq \alpha}$ s.t. $\underbrace{X \cup \{\alpha\}}_{cl(\alpha)}$ is closed.

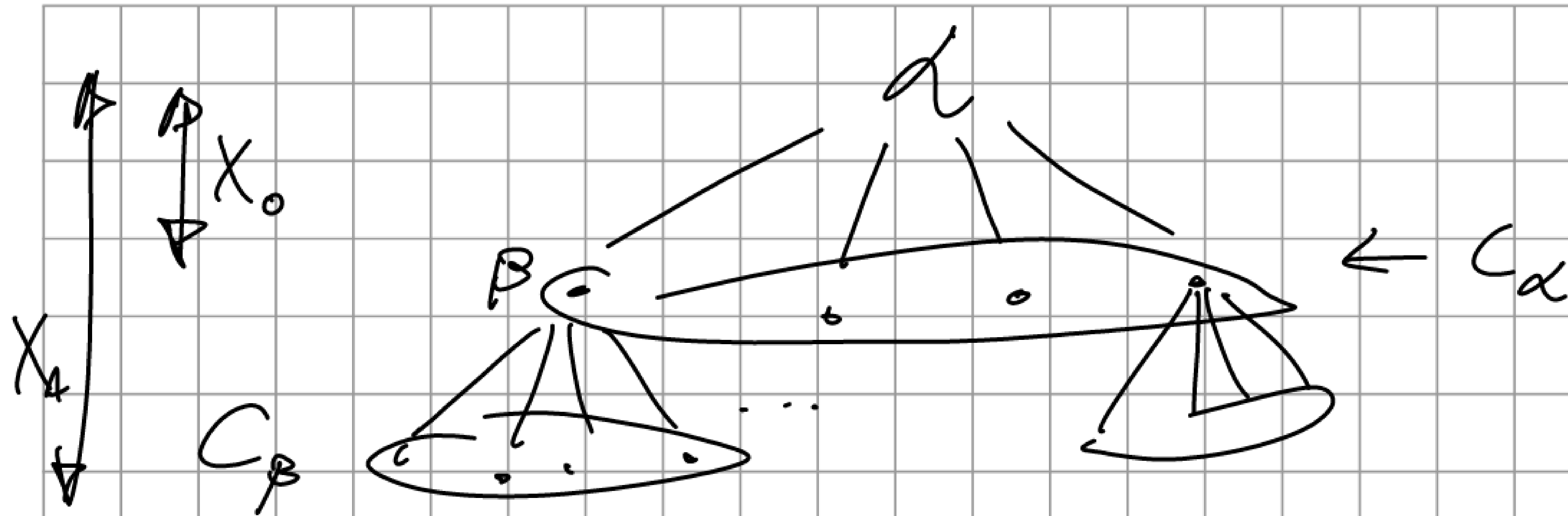
(2) A union of family of closed subsets of γ is closed.

Proof (remark) (2) is obvious.

(1): Take $X_0 = C_\alpha$, then $X_n = X_{n+1} \cup \bigcup_{\beta \in X_n} C_\beta$

finite because C_β finite

Then $X = \bigcup_n^{\nearrow} X_n$. Then $X \cup \{\alpha\}$ is finite and closed!



This tree has no infinite branch (because there are no infinite decreasing sequence of ordinals).

remark□

<u>Remark</u> Assume X: closed. Then $X \subseteq \gamma$

$$A \cup \langle a_\alpha : \alpha \in X \rangle ^\frown \langle a_\alpha : \alpha \in \gamma \setminus X \rangle$$

concatenation

is an isolated construction over A.

<u>Pf. (remark)</u> (1) $A \cup \langle a_\alpha : \alpha \in X \rangle$ is an i-construction over A (by the fact that X is closed)

(2) Suppose that $\alpha < \gamma$ and $\alpha \notin X$. Will show that $tp(a_\alpha / A a_X a_{X^c \cap (<\alpha)})$ is isolated.

$\varphi_\alpha(a) \vdash tp(a_\alpha / Aa_{<\alpha}) \vdash tp(a_\alpha / Aa_X a_{X \cap \gamma < \alpha})$

will show

Let $X_0 \subseteq X$ finite s.t. $X_0 \cap \alpha \neq \emptyset$.

Enough to show that $tp(a_\alpha / Aa_{<\alpha}) \vdash tp(a_\alpha / Aa_{<\alpha} a_{X_0})$

<u>Wlog</u> By the remark $X_0 \cup \alpha$ is closed.

So: $A \cup \langle a_\beta : \beta < \alpha \rangle ^\frown \langle a_\beta : \beta \in X_0 \rangle$ is

an i-construction over A, but for $\beta \in X_0$

$tp(a_\beta / Aa_{<\alpha} a_{(<\beta) \cap X_0}) \vdash tp(a_\beta / Aa_{<\beta})$

Because $\varphi_\beta(x)^{\vdash}$ and $\varphi_\beta{}^{\in}$.

So it implies also $tp(a_\beta / Aa_{\leq \alpha} a_{(<\beta) \cap X_0})$.

$\Rightarrow$ $tp(a_\alpha / Aa_{<\alpha} a_{(<\beta) \cap X_0}) \vdash tp(a_\alpha / Aa_{<\alpha} a_{(\leq \beta) \cap X})$

the proof of e1.3

(we switch a_α with a_β)

We just continue with induction on β

(start with $\beta = \min X_0$).

$\boxed{}$

<u>Claim</u> M: primary $/A \Rightarrow$ atomic $/A$

<u>Pf.</u> Let $\bar{m} \subseteq M = A \cup \{a_\alpha : \alpha < \gamma\}$.

$\bar{m} \subseteq A \cup a_X$, $\underset{\text{finite closed}}{X \subseteq \gamma}$

$A \cup a_X$: a partial i-construction.

$\Downarrow$

$tp(a_X/A)$ is isolated

$\Downarrow$

$tp(\bar{m}/A)$ —— $''$ ——

claim $\square$

<u>Pf (of thm)</u> $M = A \cup \{a_\alpha : \alpha < \gamma\}$,

$N = A \cup \{b_\alpha : \alpha < \delta\}$: i-constructions $/A$.

We construct $f : M \xrightarrow{\cong} N$, $f = \bigcup_\alpha f_\alpha$: elementary.

(i) $\mathrm{Dom}\, f_\alpha \supseteq A$, $\mathrm{Rng}\, f_\alpha \supseteq A$, $f_\alpha \restriction_A = \mathrm{id}_A$

(ii) $|\mathrm{Dom}\, f_\alpha \setminus A|$, $|\mathrm{Rng}\, f_\alpha \setminus A| \leq |\alpha| \cdot \aleph_0$

(iii) $\beta \in \mathrm{Lim}$ $f_\beta = \bigcup_{\alpha < \beta} f_\alpha$

(iv) $a_\alpha \in \mathrm{Dom}\, f_{\alpha+1}$, $b_{\alpha+1} \in \mathrm{Rng}\, f_{\alpha+1}$.

(v) $\mathrm{Dom}\, f_\alpha \setminus A = a_X$, $\mathrm{Rng}\, f_\alpha \setminus A = b_Y$, where $X \subseteq \gamma$, $Y \subseteq \delta$ are closed.

The recursive step from f_α to $f_{\alpha+1}$.

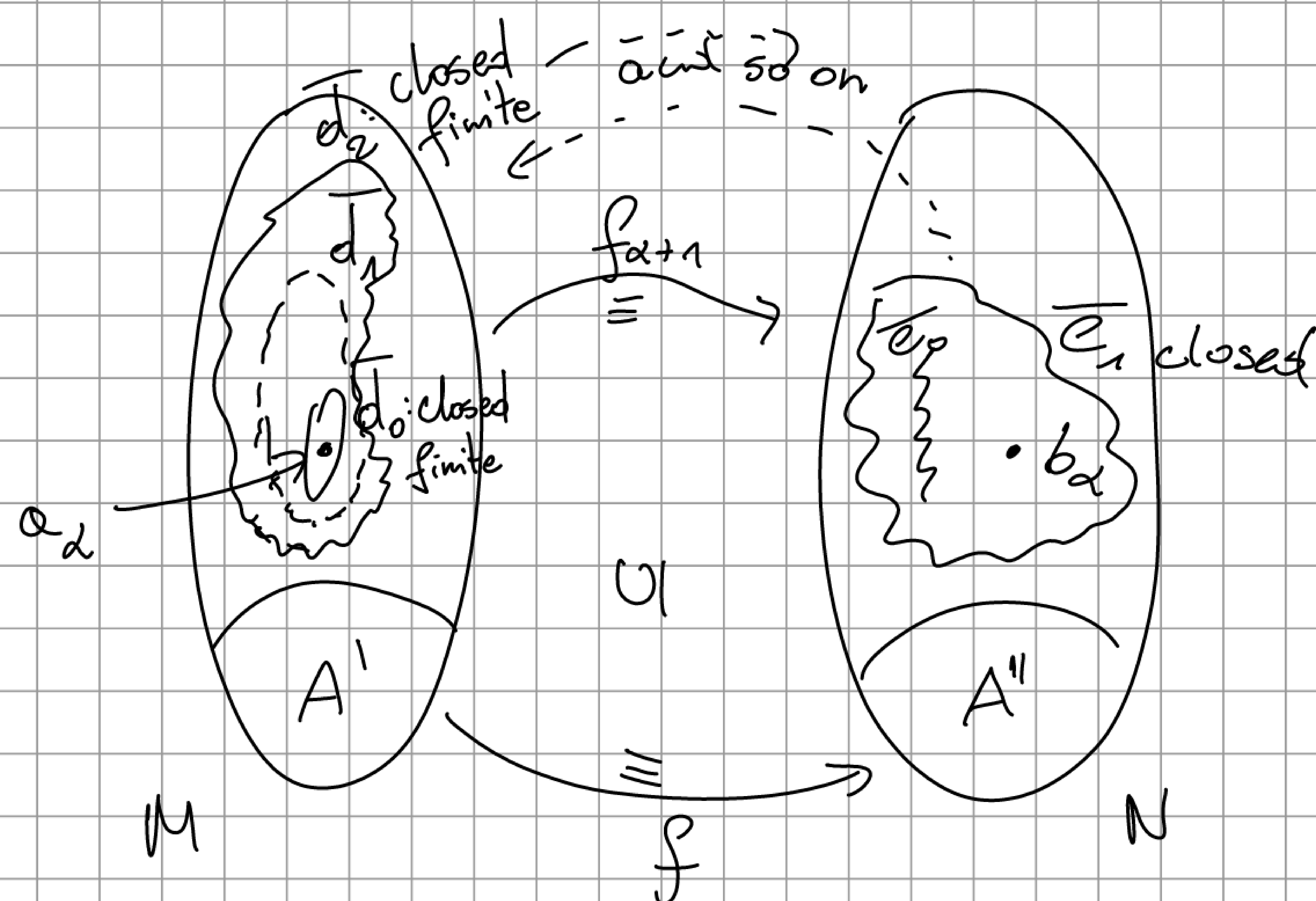
Let $A' = A \cup \mathrm{Dom}\, f_\alpha$: an iconstruction over A,

likewise $A'' = A \cup \mathrm{Rng}\, f_\alpha$: —— " ——

and M is primary over A' (by the remark)

and N is primary over A''.

$p(x) = tp(a_\alpha / A')$ is isolated, so $f_\alpha(p)$ is isolated too, therefore $\exists b \in N$ s.t. $f_\alpha(p) = tp(b/A'')$.



4.04.22

Def. (1) $\varphi(\bar{x}, \bar{a}) \in L_n(\bar{a})$ is algebraic if $0 < |\varphi(\mathcal{M})| < \aleph_0$

(2) a type $p(\bar{x})$ (over $\mathcal{M}$) is algebraic if $0 < |p(\mathcal{M})| < \aleph_0$

(3) $a \in \underset{\text{algebraic closure}}{acl}(A)$, if $tp(a/A)$ is algebraic

(4) $a \in \underset{\text{definable closure}}{dcl}(A)$, if $a \in \mathcal{M}$ is the only realisation of $tp(a/A)$

Remark (1) $p(\bar{x})$: an algebraic type $\Leftrightarrow p(\bar{x}) \vdash \varphi(\bar{x})$ for some algebraic formula $\varphi(\bar{x})$

(2) $|p(\mathcal{M})| = 1 \Leftrightarrow \exists \varphi \, (p \vdash \varphi$ and $|\varphi(\mathcal{M})| = 1)$

Proof. 1 ($\Leftarrow$) $p(\mathcal{M}) \subseteq \varphi(\mathcal{M})$

($\Rightarrow$) (a.a.) let $n \in \mathbb{N}$ arbitrary. Will show: $|p(\mathcal{M})| \geq n$.

Let $\bar{x}_1, \dots, \bar{x}_n$: disjoint tuples of variables, $|\bar{x}_i| = |\bar{x}|$.

$\{\varphi(\bar{x}_i) : \varphi(\bar{x}) \in p, i = 1, \dots, n\} \cup \{``\bar{x}_i \neq \bar{x}_j" : 1 \leq i < j \leq n\}$:

a consistent type.

It is realised in $\mathcal{M}$ so it has $\geq n$ realisations. ↯

<u>Fact</u> $acl(A) = \bigcup\{\varphi(\mathcal{M}) : \varphi(x) \in L_1(A) \text{ algebraic}\}$

$dcl(A) = \bigcup\{\varphi(\mathcal{M}) : \varphi(x) \in L_1(A) \wedge |\varphi(\mathcal{M})| = 1\}$

<u>Remark</u> Let $\varphi(\bar{x}) \in L_n(M)$. Then $\varphi(\bar{x})$ algebraic $\iff 0 < |\varphi(M)| < \aleph_0$

<u>Proof</u> $M \prec \mathcal{M}$, $|\varphi(M)| = k \iff M \models (\exists!^k \bar{x})\varphi(\bar{x})$

$\iff \mathcal{M} \models (\exists!^k \bar{x})\varphi(\bar{x}) \iff |\varphi(\mathcal{M})| = k$

<u>Remark</u> Let $A \subseteq \mathcal{M}$, $a, b \in \mathcal{M}$, then: $tp(^{ab}/_A)$ is algebraic

$\iff tp(^a/_A)$ is algebraic and $tp(^b/_{Aa})$ is algebraic

<u>Pf.</u> $(\Rightarrow)$ $p(x,y) = tp(^{ab}/_A)$. Let $q(x) = p\restriction_x$

$= tp(^a/_A)$. Let $f : \mathcal{M}^2 \to \mathcal{M}$ projection to the first coord. Then $f : p(\mathcal{M}) \twoheadrightarrow q(\mathcal{M})$

Why?

Take $a' \in q$, choose $g \in Aut(\mathcal{M}/_A)$, $g(a) = a'$,

then $b' = g(b) \Rightarrow tp(^{ab}/_A) = tp(^{a'b'}/_A)$.

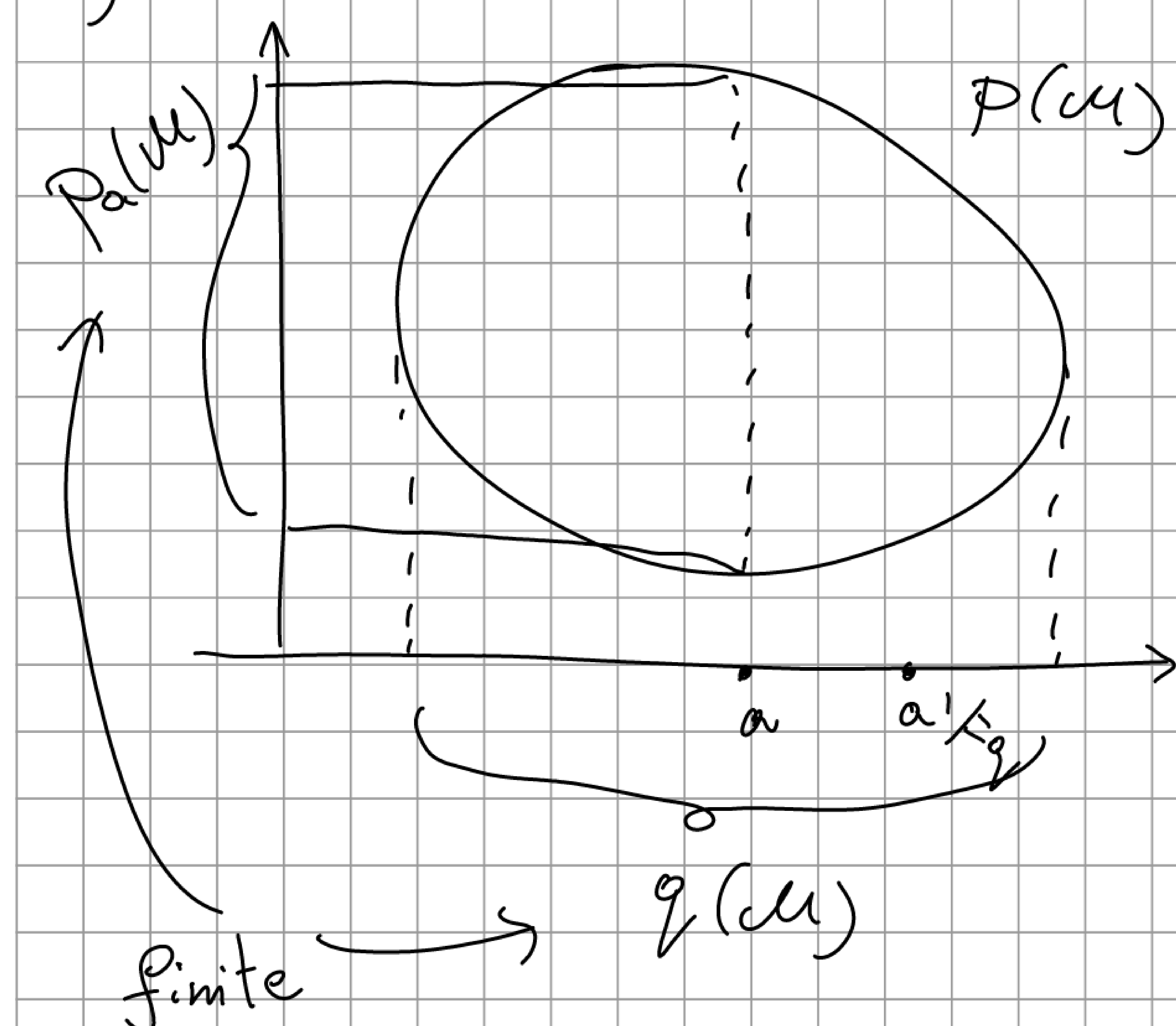
Then $|p(\mathcal{M})| < \aleph_0 \Rightarrow |q(\mathcal{M})| < \aleph_0$ and $|p_a(\mathcal{M})| < \aleph_0$

Let $p_a(y) = tp(b/Aa) = \{\varphi(a,y) : \varphi(x,y) \in p\}$

Then $p_a(\mathcal{M}) = p(\mathcal{M})_a$ ← "verticalization"

this is finite

($\Leftarrow$)



$p(\mathcal{M})_{a'} = g(p(\mathcal{M})_a)$ for any $g \in Aut(\mathcal{M}/A)$ with $g(a) = a'$.

$p(\mathcal{M}) = \bigcup_{a' \models q} (\{a'\} \times p_{a'}(\mathcal{M}))$: finite $\square$

Remark (1) (i) $A \subseteq \operatorname{acl}(A)$, (ii) $\operatorname{acl}(\operatorname{acl}(A)) = \operatorname{acl}(A)$,
(iii) $A \subseteq B \Rightarrow \operatorname{acl}(A) \subseteq \operatorname{acl}(B)$ (a closure operator)
(iv) $\operatorname{acl}(A) = \bigcup_{\substack{A_0 \subseteq A \\ \text{finite}}} \operatorname{acl}(A_0)$ (finite character)

(2) The same for dcl

Pf. 1 • $a \in \operatorname{acl}(A) \Leftrightarrow a \in \underset{\text{finite}}{\varphi(\mathcal{M})}$, $\varphi(x) \in L_1(A)$, so $\varphi \in L_1(A_0)$ for some $\underset{\text{finite}}{A_0 \subseteq A}$, so (iv).

(ii): $a \in \operatorname{acl}(\operatorname{acl}(A)) \Rightarrow a \in \operatorname{acl}(A \cup \bar{b})$, for some $\bar{b} \subseteq \operatorname{acl}(A) \Rightarrow tp(\bar{b}/A)$ algebraic and $tp(a/A\bar{b})$ algebraic $\overset{\text{remark}}{\Rightarrow} tp(a\bar{b}/A)$ is algebraic

$\Rightarrow tp(a/A)$ algebraic ∎

Example $T = ACF_p$, $A \subseteq \mathcal{M} \models T$. $\operatorname{acl}(A) =$ the algebraic closure (in $\mathcal{M}$) of the field generated by A.

Measuring definable sets and types.

The Cantor–Bendixson rank.

Def. Let X: compact T_2 space.

$$\underbrace{X'}_{\text{CB-derivative}} = X \setminus \underbrace{\{\text{isolated points}\}}_{\text{open in } X} \Rightarrow \underbrace{X' \subseteq X}_{\text{closed}}$$

Iteration: $X^{(\alpha+1)} = (X^{(\alpha)})'$, $X^{(\delta)} = \bigcap_{\alpha<\delta} X^{(\alpha)}$ where $\delta \in Lim$,

$$\underbrace{X^{(\infty)}}_{\text{the perfect core of } X} = \bigcap_{\alpha \in Ord} X^{(\alpha)} = X^{(\beta)} \text{ for some } \beta < \underbrace{|w(X)|}_{\text{minimal cardinality of basis of } X}{}^{+}$$

Def $\underline{CB}: X \to Ord \cup \{\infty\}$

$$\underset{\text{(CB-rank)}}{CB(p)} = \begin{cases} \min\{\alpha \in Ord : p \notin X^{(\alpha+1)}\} & \text{if } p \notin X^{(\infty)} \\ \infty & p \in X^{(\infty)} \end{cases}$$

Now: $X = S(A)$: 0-dimensional (extremly disconnected).

Let: $Clopen(X) = \{V \subseteq X : V \text{ clopen}\}$.

Def. $CB : \text{Clopen}(X) \to \{-1\} \cup \text{Ord} \cup \{\infty\}$: the smallest function (value-wise) s.t.: $CB(V) \geq \alpha+1 \iff$ $\forall n < \omega \ \exists \underset{\substack{\text{clopen} \\ \text{disjoint}}}{V_1, \dots, V_n} \subseteq V \quad CB(V_i) \geq \alpha$.

(also we define "$\geq \delta$")

Then $CB(V) := \min\{\alpha : \neg CB(V) \geq \alpha+1\}$

Properties Let $U, V \subseteq X$ clopen.

(0) $CB(U) = -1 \iff U = \emptyset$

(1) $CB(U) = 0 \iff 0 < |U| < \aleph_0$

(2) $U \subseteq V \Rightarrow CB(U) \leq CB(V)$

Very easy

(3) $CB(U \cup V) = \max\{CB(U), CB(V)\}$

Pf. 3 Obv. $CB(U \cup V) \geq \max\{CB(U), CB(V)\}$.

Now assume $CB(U \cup V) \geq \alpha \Rightarrow \max\{CB(U), CB(V)\} \geq \alpha$.

Pf by ind on α. Base and limit easy.

Succesor step $\alpha \to \alpha+1$. Assume $CB(U \cup V) \geq \alpha+1$.

So $\forall n \ \exists \underset{\substack{\text{clopen} \\ \text{disjoint}}}{W_1, \dots, W_n} \subseteq U \cup V \quad \bigwedge_{i=1}^{n} CB(W_i) \geq \alpha$.

By ind hyp.: $\max\{CB(W_i \cap U), CB(W_i \cap V)\} \geq \alpha$.

So $\forall n \left(\exists W_1, \dots, W_n \subseteq U \ \bigwedge_{i=1}^{n} CB(W_i) \geq \alpha\right)$

(or the same for V)

$\Downarrow$

$CB(U) \geq \alpha+1$ or $CB(V) \geq \alpha+1$.

(4) $CB(V) \geq \alpha+1 \iff \left(\exists V_n \underset{\substack{\text{clopen}\\ \text{disjoint}}}{\subseteq} V, n<\omega\right) \bigwedge_n CB(V_n) \geq \alpha$

(5) For $p \in X$

$$CB(p) = \min\{CB(U) : p \in U \underset{\text{clopen}}{\subseteq} X\}$$

<u>Notation</u>

In model theory: $X = S(A)$.

$CB(a/A) := CB(tp(a/A))$

$CB(a/A) = 0 \iff tp(a/A)$ is isolated

$p \in S(A) \rightsquigarrow CB(p) = CB_A(p)$

$\varphi \in L(A) \rightsquigarrow [\varphi] \underset{\text{clopen}}{\subseteq} S(A) \rightsquigarrow CB_A(\varphi)$

$= CB_A([\varphi] \cap S(A))$

Morley rank "CB on $L(\mathcal{M})$, in $S(\mathcal{M})$".

Def. RM : $L(\mathcal{M}) \longrightarrow \{-1\} \cup \mathrm{Ord} \cup \{\infty\}$:

the minimal function s.t. For $\mathcal{U} \subseteq \mathcal{M}^n$ definable

$[\varphi(\bar{x}) \in L_n(\mathcal{M})$ identified with $\mathcal{U} = \varphi(\mathcal{M}) \subseteq \mathcal{M}^n]$

(1) $\mathcal{U} = \emptyset \Rightarrow \mathrm{RM}(\mathcal{U}) = -1$

(2) If $\mathcal{U} \neq \emptyset$, then $\mathrm{RM}(\mathcal{U}) \geqslant \alpha + 1$

$\Longleftrightarrow \forall n < \omega \; \exists V_1, \ldots, V_n \underset{\substack{\text{def.} \\ \text{disjoint}}}{\subseteq} \mathcal{U} \; \bigwedge_{i \leq n} \mathrm{RM}(V_i) \geqslant \alpha$

Def. • For $\varphi(\bar{x}) \in L(\mathcal{M})$, $\mathrm{RM}(\varphi) = \mathrm{RM}(\varphi(\mathcal{M}))$

• For a type $p(\bar{x})$ over $\mathcal{M}$ (not necesserly completed)

$\mathrm{RM}(p) = \min\{\mathrm{RM}(\varphi) : p \vdash \varphi\}$

$= \min\{\mathrm{RM}(\mathcal{U}) : p(\mathcal{M}) \subseteq \mathcal{U}\}$

Remark (1) $\mathrm{RM}(\varphi) = \mathrm{CB}_{\mathcal{M}}(\varphi)$

(2) For $p \in S(\mathcal{M})$ $\mathrm{RM}(p) = \mathrm{CB}_{\mathcal{M}}(p)$

Here $\mathcal{M}$ may be replaced with any $\aleph_0$-saturated $M \prec \mathcal{M}$.

Properties (1) $\varphi \vdash \psi \Rightarrow RM(\varphi) \leq RM(\psi)$

(2) $p \vdash q \Rightarrow RM(p) \leq RM(q)$

(3) $RM(\varphi \vee \psi) = \max\{RM(\varphi), RM(\psi)\}$

(4) $RM(\varphi) \geq \alpha+1 \iff (\exists \varphi_n \vdash \varphi,\ n<\omega) \bigwedge_n RM(\varphi_n) \geq \alpha$
pairwise contradictory

(5) If $\delta \in Lim$, then $RM(\varphi) \geq \delta \iff (\forall \alpha<\delta) RM(\varphi) \geq \alpha$

Thm T is $\aleph_0$-stable $\iff RM(\text{"}x=x\text{"}) < \infty$.

Pf. $(\Rightarrow)$ (a.a.) If $tp(\bar{a}) = tp(\bar{b})$, then

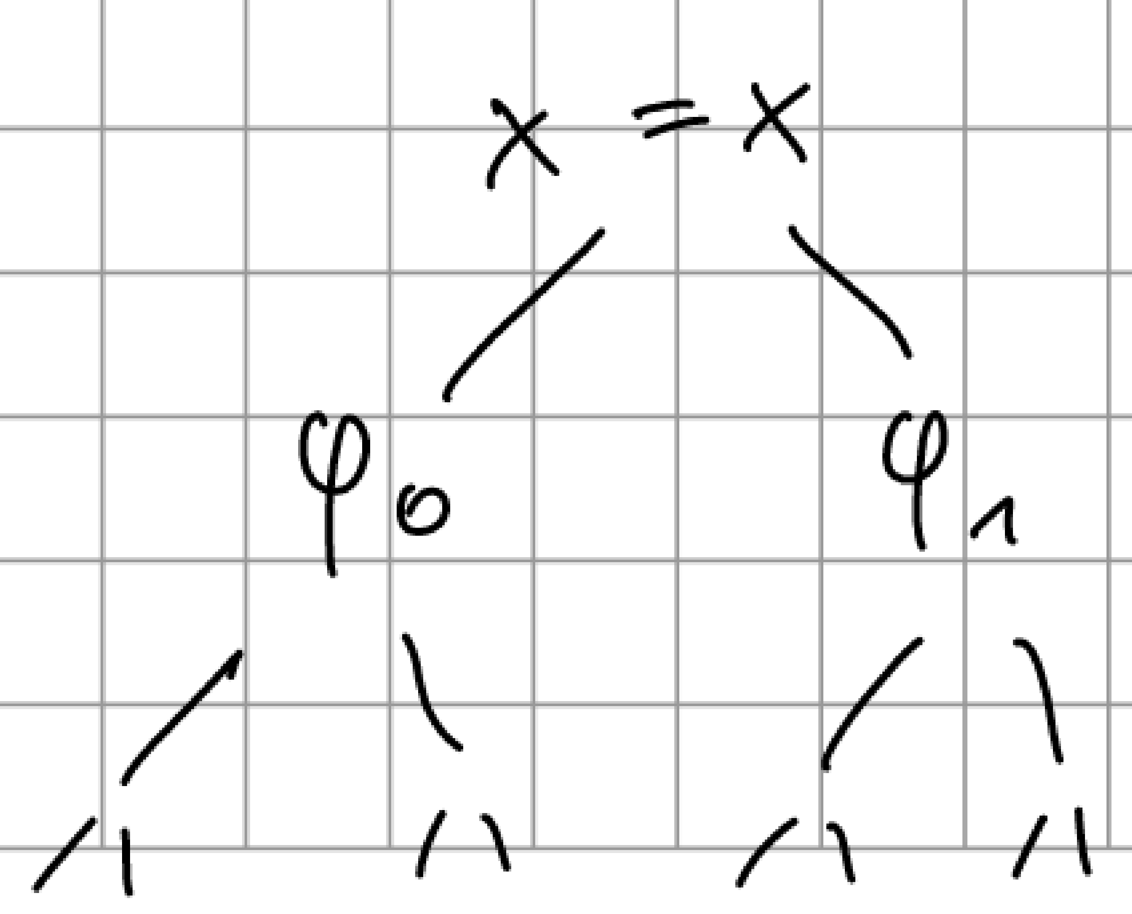
$RM(\varphi(\bar{x},\bar{a})) = RM(\varphi(\bar{x},\bar{b}))$

$\exists f \in Aut(\mathcal{M})\ f(\bar{a}) = \bar{b} \Rightarrow f(\varphi(\mathcal{M},\bar{a})) = \varphi(\mathcal{M},\bar{b})$

So $|Rng(RM)| \leq 2^{|T|} \Rightarrow \exists \alpha \in Ord\ \forall \varphi [RM(\varphi) \geq \alpha$

$\Rightarrow RM(\varphi) = \infty]$.

Suppose $RM(\text{"}x=x\text{"}) = \infty \Rightarrow \geq \alpha+1$



: pairwise contradictory, $RM \geq \alpha \Rightarrow \geq \alpha+1$

proceed like that

We get $2^{\aleph_0}$ many types over ctble set A

$\Rightarrow T$ is not $\aleph_0$-stable.

11.04.2022

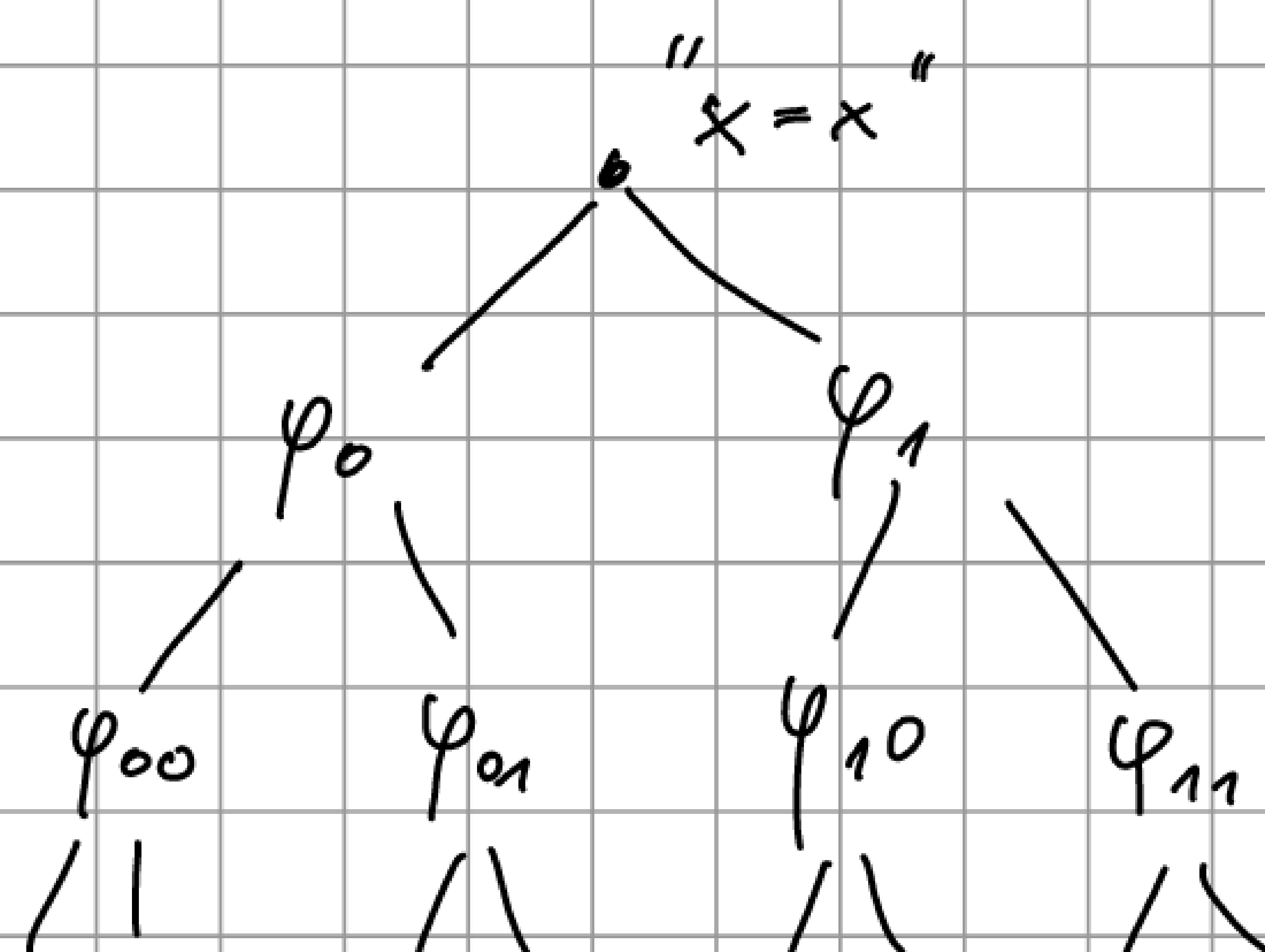
Thm T is $\aleph_0$-stable $\iff$ $RM(x=x) \in Ord$

Pf. "$\Rightarrow$" last time. "$\Leftarrow$" Suppose T is not $\aleph_0$-stable.

$|S(A)| = 2^{\aleph_0}$, $S(A)$: a Polish space

$\Rightarrow S(A)^{(\infty)} \neq \emptyset \Rightarrow$ get a binary tree of formulas

$$\{\overbrace{\varphi_\eta(x)}^{\in L_1(A)} : \eta \in 2^{<\omega}\}$$



Let $\alpha = \min\{RM(\varphi_\eta) : \eta \in 2^{<\omega}\}$.

If $\eta \subseteq \nu \in 2^{<\omega}$, then $RM(\varphi_\nu) = \alpha$.

$RM(\varphi_\eta) \not\geq \alpha+1$, so:

$$(*) \ \exists m \in \mathbb{N} \ \neg\exists \psi_1 \ldots \psi_m \vdash \varphi_\eta \ \bigwedge_{i=1}^{m} RM(\psi_i) \geq \alpha$$

Take n s.t. $2^n > m$. φ_ν, $\eta \subseteq \nu \in 2^{<\omega}$, $|\nu| = |\eta| + n$

contradicts $(*)$. $\blacksquare$

Def. Multiplicity of $\varphi(\bar{x}) \in L(\mathcal{M})$ s.t. $\underbrace{RM(\varphi(\bar{x}))}_{\alpha \in Ord} < \infty$:

$Mlt(\varphi)$ = the largest $m \in \mathbb{N}$ s.t.

$$\exists \psi_1 \dots \psi_m \ \bigwedge_{i=1}^{m} RM(\psi_i) \geq \alpha$$

Properties • $RM(\varphi_1) = RM(\varphi_2)$ and $\varphi_1(\mathcal{M}) \cap \varphi_2(\mathcal{M}) = \phi$, then $Mlt(\varphi_1 \vee \varphi_2) = Mlt(\varphi_1) + Mlt(\varphi_2)$

• If $RM(\varphi_1) < RM(\varphi_2) < \infty$, then $Mlt(\varphi_1 \vee \varphi_2) = Mlt(\varphi_2)$

Example If $\underbrace{\varphi(\bar{x}) \text{ is algebraic}}_{\Updownarrow \; RM(\varphi) = 0}$, then $Mlt(\varphi) = |\varphi(\mathcal{M})|$

Def. Assume $p(\bar{x})$: a type with $RM(p(\bar{x})) < \infty$.

$$Mlt(p(\bar{x})) = \min\{Mlt(\varphi(\bar{x})) : p \vdash \varphi \text{ and } RM(\varphi) = RM(p)\}$$

Def. $p(\bar{x})$ is stationary, if $Mlt(p(\bar{x})) = 1$

Remark Assume $p(\bar{x})$: a type over A. Then $\exists q \in S(A)$, $p(\bar{x}) \subseteq q(\bar{x})$ s.t. $RM(p) = RM(q)$

Pf. Let $q_0 = \{\varphi(\bar{x}) \in L(A) : RM(p \cup \{\neg\varphi\}) < RM(p)\}$

• $q_0 \supseteq p$, if $\varphi \in p$, then $RM(p \cup \{\neg\varphi\})$
$= -1 < RM(p)$
consistent

• q_0: a consistent type: $\varphi_1, \dots, \varphi_n \in q_0$, $RM(p \cup \{\neg\varphi_i\}) < \alpha = RM(p)$

Choose ψ with $p \vdash \psi$ and $RM(\psi) = \alpha$;

ψ_i with $p \vdash \psi_i$ and $RM(\psi_i \wedge \neg\varphi) < \alpha$.

Wlog $\psi = \bigwedge_{i=1}^{n} \psi_i$

⋮

Let $q_0 \subseteq q \in S(A)$, then $RM(q) = \alpha$. Clearly, $RM(q) \leq \alpha$. If $RM(q) < \alpha \Rightarrow q \vdash \varphi$ with some $RM(\varphi) < \alpha$. By compactness $\exists$ finite $q' \subseteq q$ $q' \vdash \varphi$ $\Leftrightarrow \underbrace{\bigwedge q'}_{\chi(\bar{x}) \in q(\bar{x})} \vdash \varphi$, $\chi(\bar{x}) \vdash \varphi(\bar{x})$ so $RM(\chi(\bar{x})) < \alpha$,

then $RM(p \cup \{\chi(\bar{x})\}) < \alpha$

$$\Downarrow$$

$$\begin{cases} \neg\chi(\bar{x}) \in q_0(\bar{x}) \subseteq q(\bar{x}) \\ \chi(\bar{x}) \in q(\bar{x}) \end{cases} \quad \lightning$$

$\Leftrightarrow p$: algebraic

<u>Example</u> $RM(p) = 0$, then $Mlt(p) = |p(\mathcal{M})|$

<u>Example</u> $T = ACF_p$, $\underset{\text{subfield}}{K} \subseteq \mathcal{M}$, $p \in S_n(K)$, $p = tp(a/K)$, $a \in \mathcal{M}$

a) p algebraic $\Leftrightarrow RM(p) = 0$,

$\underset{K[x]}{(W)} = I(a/K) \neq \{0\}$, "$W(x) = 0$" $\in p(x)$,

$Mlt(p) = Mlt(W(x) = 0) = \#\{\text{roots of } W \text{ in } \mathcal{M}\}$.

b) p: transcendental, then $I(a/K) = \{0\}$,

$RM(p) = 1 = RM(x = x)$

$Mlt(p) = Mlt(x = x) = 1$

So p: stationary.

c) $p = tp(\bar{a}/K)$, $\bar{a} = \langle a_1, \dots, a_n \rangle \subseteq \mathcal{M}$. By q.e.

it is determined by $I(\bar{a}/K) \trianglelefteq K[X_1, \dots, X_n]$

$\{W(\bar{x}) \in K[\bar{x}] : W(\bar{a}) = 0\}$.

Let $V_p = V(I(\bar{a}/K)) = \bigcap_{W \in I(\bar{a}/K)} Z(W) =$

$Z(W) = \{\bar{b} \in \mathcal{M}^n : W(\bar{b}) = 0\}$

$= \bigcap_{i=1}^{l} Z(W_i)$ definable on $\mathcal{M}$ over K,

$I(\bar{a}/K) = (W_1, \dots, W_l)$

$\bar{a} \subseteq V_p \Rightarrow p(\bar{x}) \vdash$ "$\bar{x} \in V_p$"

$RM(p) = RM(V_p) = \dim V_p$

$Mlt(p) = Mlt(V_p) =$ The number of irreducible components of V_p of $\dim = \dim V_p$

In $T = Th(ACF_p)$: $RM(\bar{x}=\bar{x}) < \omega$

(Order) indiscernible sets:

Let $(I, \leq)$: a linear ordered set of indicies.

<u>Def.</u> $\{\bar{a}_i : i \in I\} \subseteq \mathcal{M}$: <u>order indiscernible over $A \subseteq \mathcal{M}$</u>, if $\forall k\, \forall i_1 < \dots < i_k \in I$, $j_1 < \dots < j_k \in I$: $tp(\bar{a}_{i_1} \dots \bar{a}_{i_k}/A) = tp(\bar{a}_{j_1} \dots \bar{a}_{j_k}/A)$

<u>Recall</u>: 1) Assume $p(\bar{x})$: a non-algebraic type over $A \subseteq \mathcal{M}$. Then $\exists \{\bar{a}_n : n < \omega\} \subseteq p(\mathcal{M})$ infinite order indiscernible

2) (stretching) Assume $\{a_i : i \in I\}$ order indiscernible $/A$, I: infinite, $(J, \leq)$: a linear ordering. Then $\exists \{b_j : j \in J\}$: order ind. $/A$ s.t. $\forall k\, \forall i_1 < \dots < i_k \in I$ $\forall j_1 < \dots < j_k \in I$

$$tp(a_{i_1} \dots a_{i_k}/A) = tp(b_{j_1} \dots b_{j_k}/A)$$

Pf. (1) by Ramsey thm.

(2) Let $b_j, j \in J$: new constant symbols.

$T^* = T(A) \cup \{\varphi(b_{j_1}, \dots, b_{j_k}) : \varphi(\bar{x}) \in L_k(A), j_1 < \dots < j_k \in J$

and $\forall i_1 < \dots < i_k \in I \models \varphi(a_{i_1}, \dots, a_{i_k})\}$.

T^* : consistent. $b_{j_1}, \dots, b_{j_k} \xrightarrow{\text{interpret as}} a_{i_1}, \dots, a_{i_k}$

has a model M for any $i_1 < \dots < i_k \in I$

$M = (\underbrace{M, a^M}_{\Pi}, b_j^M)_{\substack{a \in A \\ j \in J}} \models T^*$

$T(A) \Rightarrow \exists f : (M, a^M)_{a \in A} \xrightarrow{\equiv} (\mathcal{M}, a)_{a \in A}$

$a^M \overset{f}{\longmapsto} a$

Let $b_j = f(b_j^M)$. $\{b_j : j \in J\} \subseteq \mathcal{M}$ is good.

Example (1) $M = (\mathbb{Q}, \leq) \leftarrow \mathbb{Q}$ is order indisc (indexed by itself)

(2) $T = ACF_p$, $\mathcal{M} \models T$, $\{a_i : i \in I\} \subseteq \mathcal{M}$, acl-indepen. over $K \subseteq \mathcal{M}$ (subfield),

then it is indisc $/K$ in $\mathcal{M}$

$\Updownarrow$

$I(a_{i_1} \dots a_{i_k}/K) = \{0\}$

Def. $\{\bar{a}_i : i \in I\} \subseteq \mathcal{M}$ is <u>indiscernible over $A \subseteq \mathcal{M}$</u>

if $\forall k\ \forall i_1, \ldots, i_k \subseteq I$ $tp(a_{i_1} \ldots a_{i_k}/A) = tp(b_{i_1} \ldots b_{i_k}/A)$
$j_1, \ldots, j_k \subseteq I$

(3) $T = Th(V, +, 0, k)_{k \in K}$: infinite vector space.

$V \supseteq \{a_i : i \in I\}$ indiscernible $/\emptyset \iff$ linear independ.

Thm. T : stable, $\{a_i\}_{i \in I}$ order indisc., then
$\{a_i\}_{i \in I}$ indiscernible. (I infinite)

Pf. (A.a.) T : $\varkappa$-stable. <u>Wlog</u> (by stretching) $(I, \leq)$ is
a dense linear ordering with $J \underset{\text{dense}}{\subseteq} I$, $|J| = \varkappa$
s.t. every nonempty interval in I has power $> \varkappa$.

$\{a_i\}_{i \in I}$: ord-indisc. but not indisc.:

$\exists k\ \exists i_1 < \ldots < i_k \in I\ \exists j_1, \ldots, j_k \in I$ $tp(a_{i_1} \ldots a_{i_k}) \neq tp(a_{j_1} \ldots a_{j_k})$

$\Rightarrow$ for some φ: $\models \varphi(a_{i_1} \ldots a_{i_k}) \wedge \neg\varphi(a_{i_{\sigma(1)}} \ldots a_{i_{\sigma(k)}})$
for some $\sigma \in Sym(k)$

25.04.22 Pf. c.d. $\sigma = \tau_1 \circ \tau_2 \circ \dots \circ \tau_l$: product of transpositions of consecutive numbers.

Let $\sigma_t = \tau_1 \circ \dots \circ \tau_t$, $t = 1, \dots, l$, $\sigma_0 = id$. Then

$\models \varphi(\bar{a}_{i_{\sigma_0}})$ and $\models \varphi(\bar{a}_{i_{\sigma_l}})$. For some

$0 \le t < l$:

$$\models \varphi(\bar{a}_{i_{\sigma_t}}) \wedge \neg \varphi(\bar{a}_{i_{\sigma_{t+1}}})$$

Let $\sigma' = \sigma_t$, $\sigma'' = \sigma_{t+1} = \sigma' \circ \overset{\tau}{\overset{\shortparallel}{\tau_{t+1}}}$

Then $\models \underbrace{\varphi(a_{i_{\sigma'(1)}}, \dots, a_{i_{\sigma'(k)}})}_{\psi(a_{i_1}, \dots, a_{i_k})}$ (by renaming)

But $\models \neg \varphi(a_{i_{\sigma'(\tau(1))}}, \dots, a_{i_{\sigma'(\tau(k))}})$,

so $\models \neg \psi(a_{i_{\tau(1)}}, \dots, a_{i_{\tau(k)}}) \wedge \psi(a_{i_1}, \dots, a_{i_k})$

$(\forall i_1 < i_2 < \dots < i_k \in I)$

e.g. $\tau = (3,4)$ and $k > 4$. Choose $i_1 < i_2 < i_5 < \dots < i_k$.

Let $\chi(x_3, x_4) = \varphi(a_{i_1}, a_{i_2}, x_3, x_4, a_{i_5}, \dots, a_{i_k}) \in L(\bar{a}_J)$ $J \overset{\text{dense}}{\cap} I$

Will show: $|S(\bar{a}_J)| > \varkappa$, $|\bar{a}_J| \leq \varkappa$.

Namely: let $i < i' \in (i_2, i_5)_I$. Then

$$tp(a_i/\bar{a}_J) \neq tp(a_{i'}/\bar{a}_J) \leftarrow \text{enough}$$

$\exists$ $i_1 < i_2 < i < j < i' < i_5 < \ldots < i_k$.

Then $\models \varphi(a_i, a_j)$, $\models \neg\varphi(a_{i'}, a_j)$
because $i < j$ because $i' > j$

Then $\varphi(x, a_j) \in tp(a_i/\bar{a}_J)$,
$\neg\varphi(x, a_j) \in tp(a_{i'}/\bar{a}_J)$. $\lightning$

<u>Remark</u> (T: stable, $\varphi(x, \bar{y}) \in L$). There is $n < \omega$ $\forall I \subseteq \mathcal{M}$ $\forall \bar{a} \subseteq \mathcal{M}$ one of the
indiscernible
infinite

sets $I^+_{\bar{a}} = \{c \in I : \models \varphi(c, \bar{a})\}$

$I^-_{\bar{a}} = \{c \in I : \models \neg\varphi(c, \bar{a})\}$

has $\leq n$ elements.

Pf. (a.c.) If there's no such n, then $\forall n \exists I_n \exists \bar{a}_n \ |I^+_{n,\bar{a}_n}|, |I^-_{n,\bar{a}_n}| > n$. Let

$\{c_i, i<\omega\}$: new constant symbols.

$\bar{d}$, $|\bar{d}| = |\bar{y}|$.

Let $T' = T \cup \{$"$\underbrace{\{c_i : i<\omega\}}_{I}$ is indescernible in T"$\}$

$\cup \{$"$I^+_{\bar{d}} = \{c_{2i} : i<\omega\}$"$\} \cup \{$"$I^-_{\bar{d}} = \{c_{2i+1} : i<\omega\}$"$\}$.

wlog $|I| = \varkappa$

T' is consistent, so it has a model M'.

Wlog $M' \restriction_L \prec \mathcal{M}$. So $I \subseteq \mathcal{M}$ indiscernible,

$\underset{I,\bar{d}}{\cup}$ $I^+_{\bar{d}}, I^-_{\bar{d}} \subseteq \mathcal{M} \Rightarrow |S(I)| = 2^{\varkappa} > \varkappa$.

Pf. $|I^+_{\bar{d}}|, |I^-_{\bar{d}}| = \varkappa$. For any $I' \subseteq I$ with $|I'| = |I \setminus I'| = \varkappa$ $\exists f \in Aut(\mathcal{M}) \left[f[I] = I, f(I') = I^+_{\bar{d}}, f(I \setminus I') = I^-_{\bar{d}} \right]$. Let $\bar{a}_{I'} = f^{-1}(\bar{d})$.

If $I' \neq I''$, then $tp(\bar{a}_{I'}/I) \neq tp(\bar{a}_{I''}/I)$.

$\aleph_1$-categorical theories

Examples. 1. ACF_p,

2. $Th(V, +, k)_{k \in K}$ ← ctble field
   (V: inf. vec. space)

3. $Th(\mathbb{N}, S)$, $Th(\mathbb{Z}, S)$, $Th(\mathbb{N}, =)$ ← "no structure"

4. $Th(G, +)$, G: torsion free divisible abelian

5. $Th(\mathbb{Z}_p^{\aleph_0}, +)$

6. $Th(G)$ where G: an algebraic group

Theorem If $\kappa > \aleph_0$ and T is κ-categorical, then T is $\aleph_0$-stable.

Lemma $\forall \kappa \geq \aleph_0\ \exists M \models T,\ \|M\| = \kappa\ \forall A \subseteq M,\ |A| < \aleph_0$

$$|\{p \in S(A) : p(M) \neq \emptyset\}| \leq \aleph_0 .$$

Pf. thm (Lemma $\Rightarrow$ thm) (A.c.) Suppose T is not $\aleph_0$-stable. $\exists_{\text{ctble}} N \models T$ $|S(N)| > \aleph_0$

But T: κ-categorical. $N \prec N_1$ s.t. $\|N_1\| = \aleph_1$,

$$|\{p \in S(N) : p(N_1) \neq \emptyset\}| = \aleph_1 .$$

$N_1 \prec N_\kappa \leftarrow$ of power κ. Let M_κ : a model from the lemma. Then $M_\kappa \cong N_\kappa$ ↯.

Pf. (lemma) $T \subseteq T^S$: the skolemization in $L^S \supseteq L$.

Let $I = \{a_n : n < \omega\}$: an infinite order indisc. set in T^S. $I \subseteq J = \{a_\alpha : \alpha < \kappa\}$ (stretching).

Then $J \subseteq N^S \models T^S$. Let $M^S = \mathcal{H}(J) \prec N^S$, i.e.

$M^S = \{ t^{N^S}(\vec{j}) : t(\bar{x})$: a term in L^S, $\vec{j} \subseteq J$ increasing $\}$

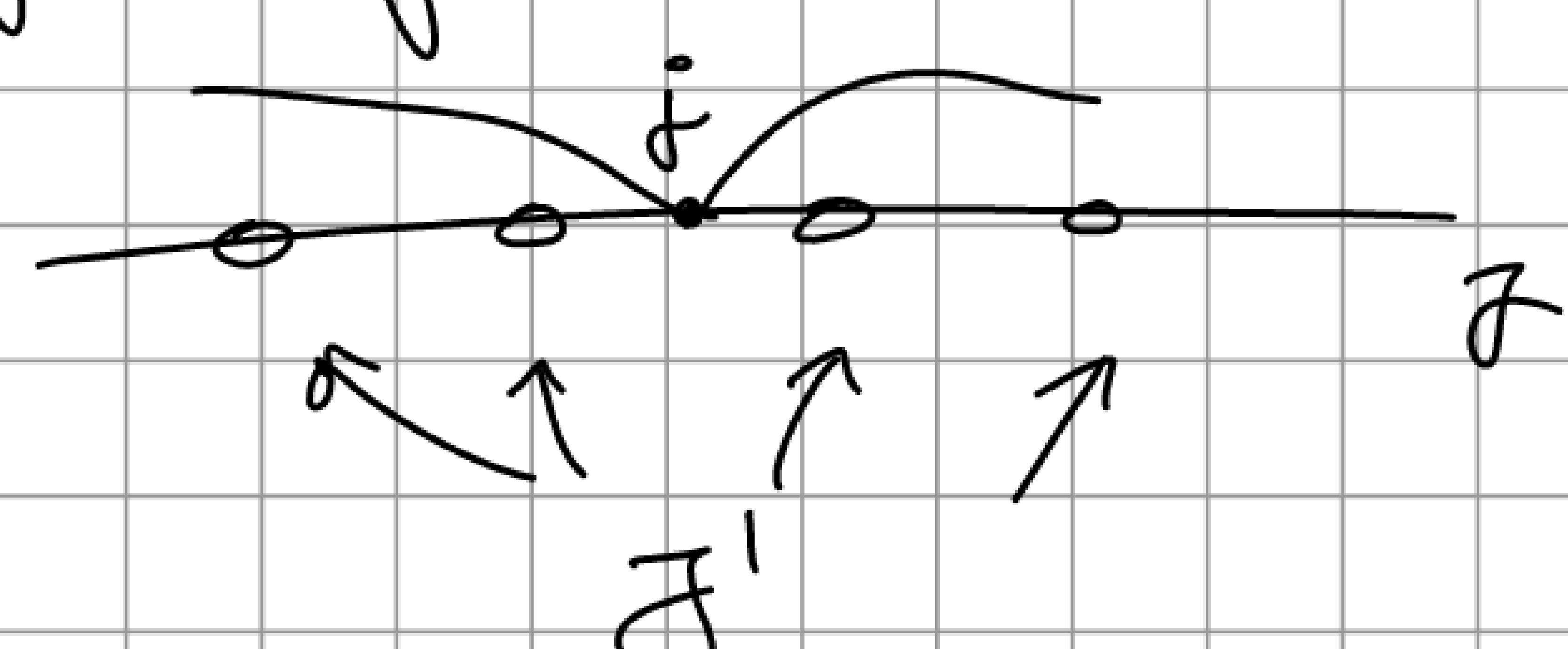
Will show that M^S satisfies the conditions on the lemma for T^S (then $M := M^S \restriction_L$ satisfies conditions for T)

Let $A \subseteq \mathcal{H}(J)$ ctble. Wlog $A = \mathcal{H}(J')$ for some $J' \subseteq J$.

$M^S \ni a = t(j_1, \dots, j_k)$ for some $j_1 < \dots < j_k \in J$.

$\sim$ eq. rel. on J:

$j \sim j' \overset{def.}{\iff} j$ and j' determine the same cut in J'.



$|\mathcal{J}/\sim|$: ctble as $|\mathcal{J}'|$ ctble. Now

$$(\underset{<\ <}{j_1, \dots, j_k}) \sim (\underset{<\ <}{j_1', \dots, j_k'}) \overset{\text{def}}{\iff} \bigwedge_{1 \le i \le k} j_i \sim j_i'$$

again ctbly many classes

Let $\underset{\substack{\parallel \\ t(j_1,\dots,j_k)}}{a}, \underset{\substack{\parallel \\ t'(j_1',\dots,j_k')}}{a'} \in M^s$. $(*)$ If $t = t'$ and $(j_1, \dots, j_k) \sim (j_1', \dots, j_k')$ then $tp^{M^s}(a/A) = tp^{M^s}(a'/A)$.

(it obvs. implies the lemma).

Pf $(*)$: $\underset{\substack{\parallel \\ \varphi(x, b),\ b \subseteq A \\ \varphi(x, \bar{y}) \in L}}{\varphi(x)} \in L(A)$, $b = \bar{t}(\bar{j}'')$, $\bar{j}'' \subseteq \mathcal{J}'$.

$\models \varphi(a, b) \iff \models \varphi(t(\bar{j}), \bar{t}(\bar{j}'')) \overset{\bar{j} \sim \bar{j}' + \text{order indiscernibility of } \mathcal{J}}{\iff} \models \varphi(t(\bar{j}'), \bar{t}(\bar{j}''))$

$\iff \models \varphi(a', b)$ ∎

<u>Remark</u>

Let T: $\aleph_0$-stable, $p \in S(A)$, $RM(p) < \infty$, $Mlt(p) = 1$.

Then $\forall \underset{\substack{\cap\!\shortparallel \\ \mathfrak{M}}}{B} \supseteq A$ $\exists! p_B \in S(B)$ $RM(p_B) = RM(p)$

Def. $I = \{a_\alpha : \alpha < \beta\} \subseteq M$ is a Morley sequence in $p \in S(A)$ if $\forall \alpha < \beta$ $a_\alpha \models p_{A a_{<\alpha}} \in S(A a_{<\alpha})$

$$p_{A a_{<\alpha}} \sqsupseteq p, \quad M\ell t(1) \text{ in } S(A)$$

Remark A Morley sequence in p is indiscernible over A.

Pf. Enough to show order-indiscernibility. Wlog $I = \{a_\alpha : \alpha < \beta\}$, $\beta = \kappa \geq \aleph_0$, $\kappa \in N$. Induction on $k < \omega$:

$$\forall \alpha_1 < \ldots < \alpha_k < \kappa \quad tp(a_{\alpha_1} \ldots a_{\alpha_k} / A) = tp(a_{\beta_1} \ldots a_{\beta_k} / A)$$
$$\beta_1 < \ldots < \beta_k$$

Step $k \to k+1$, $\alpha_{k+1} > \alpha_k$, $\beta_{k+1} > \beta_k$.

$$p \subseteq tp(a_{\alpha_{k+1}} / A a_{\alpha_1} \ldots a_{\alpha_k}) \subseteq p_{A a_{<\alpha_{k+1}}}$$

the same RM, $M\ell t = 1$ $\implies tp(a_{\alpha_{k+1}} / A a_{\alpha_1} \ldots a_{\alpha_k}) = p_{A a_{\alpha_1} \ldots a_{\alpha_k}}$

Likewise $tp(a_{\beta_{k+1}} / A a_{\beta_1} \ldots a_{\beta_k}) = p_{A a_{\beta_1} \ldots a_{\beta_k}}$

Consider $f: Aa_{\alpha_1 \dots \alpha_k} \longrightarrow Aa_{\beta_1} \dots a_{\beta_k}$ s.t.

$f|_A = id$, $f(a_{\alpha_i}) = a_{\beta_i}$, $i = 1, \dots, k$. Then f: element.

and $f(\underbrace{p_{Aa_{\alpha_1} \dots A_{\alpha_k}}}_{\substack{\cup \\ p}}) = p_{Aa_{\beta_1} \dots a_{\beta_k}}$

and has the same RM as $RM(p)$

$$\Downarrow$$

$f \cup \{\langle a_{\alpha_{k+1}}, a_{\beta_{k+1}} \rangle\}$ elementary

So $tp(a_{\alpha_1} \dots a_{\alpha_{k+1}} / A) = tp(a_{\beta_1} \dots \beta_{k+1} / A)$. $\blacksquare$

<u>Thm</u> (Morley, Shelah) If T: $\aleph_0$-stable and $\varkappa \geq \aleph_0$ then T has a saturated model of power $\varkappa$.

9.05.2021 **Thm (Morley, Shelah)** T: $\aleph_0$-stable $\Rightarrow$

T has a saturated model of power $\varkappa$

Pf. • $M = \bigcup\limits_{\alpha<\varkappa}^{\uparrow} M_\alpha$ ← elementary chain of models of T of power $\varkappa$ $\|M_\alpha\| = \varkappa$

• $M_\delta = \bigcup\limits_{\alpha<\delta}^{\uparrow} M_\alpha$ when $\delta \in Lim$

• $\alpha \to \alpha+1$: $M_{\alpha+1} \succ M_\alpha$ s.t.

(a) $\forall \underset{\text{finite}}{A \subseteq M_\alpha}\ \forall p \in S(A)\ p(M_{\alpha+1}) \neq \emptyset$

(b) $\forall \underset{\text{finite}}{A \subseteq M_\alpha}\ \forall \underset{\text{stationary}}{p \in S(A)}\ \exists I \subseteq M_{\alpha+1} \big(|I| = \varkappa \wedge$ I is a Morley sequence in $p\big)$

Claim M is saturated

• M is $\aleph_0$-saturated (easy)

• M is $\varkappa$-saturated: $\underline{(\text{a.c.})}$ let $A \subseteq M$, $|A| < \varkappa$, $p \in S(A)$, $p(M) = \emptyset$. Choose A and p so that $(RM(p), Mlt(p))$ is lexicographically minimal. Then $Mlt(p) = 1$.

pf. let choose $\varphi \in p$ with $RM(p) = RM(\varphi)$, $Mlt(p) = Mlt(\varphi)$.

- If $Mlt(\varphi) > 1$, then

(*) $\exists \psi(x) \in L(\mathcal{M})$ $RM(\varphi \wedge \psi) = RM(\varphi \wedge \neg\psi) = RM(\varphi)$

So $\psi(x) = \psi(x, \underset{\mathcal{M}}{\underset{\cap}{\bar{c}}})$, $\varphi(x) = \varphi(x, \underset{M}{\underset{\cap}{\bar{a}}})$. Choose $\bar{c}' \subseteq M$ s.t. $tp(\bar{c}'/\bar{a}) = tp(\bar{c}/\bar{a}) \in S_k(\bar{a})$, $(k = |\bar{c}|)$

(we can choose it by (a))

(*) holds for $\psi'(x, \bar{c}')$ in place of ψ

Let $A' = A \cup \bar{c}' \subseteq M$, $|A'| < \kappa$

$$\begin{array}{ccc} S(A) & & S(A') \\ \cup\!\!\!\!\,\shortmid & & \\ & & p_1 \\ p & & p_2 \\ & & \vdots \\ & & p_l \end{array}$$

s.t. $RM(p_i) = RM(p)$

$(l \leq Mlt(p))$

Also (*) $\Rightarrow$ $Mlt(\varphi) = Mlt(\varphi \wedge \psi') + Mlt(\varphi \wedge \neg\psi')$

Look at p_1: either $\varphi \wedge \psi \in p_1$ or $\varphi \wedge \neg\psi \in p_1$

$\Downarrow$

$(RM(p_1), Mlt(p_1)) <_{lex} (RM(p), Mlt(p))$

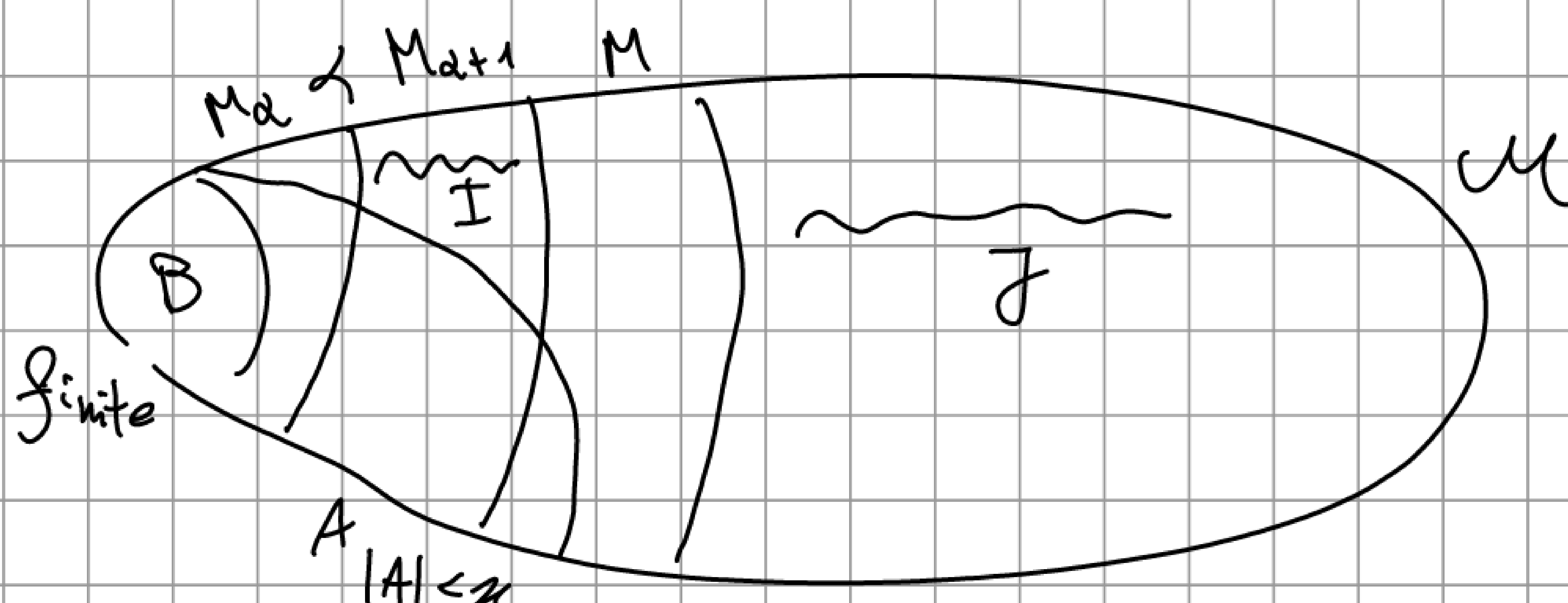
$p_1(M) \subseteq p(M) = \emptyset \Rightarrow p_1(M) = \emptyset$ ↯

Therefore $\mathrm{Mlt}(\varphi)=1$. Choose a finite $B\subseteq A$ s.t. $RM(p)=RM(p|B)$, $\mathrm{Mlt}(p)=\mathrm{Mlt}(\underbrace{p|B}_{p'})$

(enough that $\varphi\in L(B)$)

By (b) $\exists I\subseteq M$: a Morley sequence in p', $|I|=\varkappa$. Let $\underset{\mathcal{M}}{\underset{\cap\!\!/}{J}}=\{a_\alpha:\alpha<\varkappa\}$: a M. sequence in $\underset{p}{\underset{\cup\!\!/}{p'_{AI}}}\in S(AI)$

Then $I\cup J$ is a Morley sequence in p'_{AI}.



Let $\chi(x)\in p\subseteq p'_{AI}\Rightarrow \underset{\text{infinite}}{J}\subseteq\chi(\mathcal{M})$,

However $I\cup J$ indiscernible $\Rightarrow\{i\in I:\models\chi(i)\}$ is also cofinite in I (by the lemma from prev. lecture)

$I\cup J=(I\cup J)^+_\chi\cup\underset{\substack{\uparrow\\ \text{finite}}}{(I\cup J)^-_\chi}$

$$\Big|\bigcup_{\chi\in p} I_\chi^-\Big| < \varkappa \Rightarrow \Big|\bigcap_{\chi\in p} I_\chi^+\Big| = \varkappa \Rightarrow \Big|\bigcap_{\chi\in p} I_\chi^+\Big| \neq \emptyset.$$

$(|p|<\varkappa)$ Any $c \in \bigcap_{\chi\in p} I_\chi^+$ realises p in M ∎

So: if $\varkappa > \aleph_0$, T: $\varkappa$-categorical

$$\Downarrow$$

T: $\aleph_0$-stable

$$\Downarrow$$

$\exists \underset{\text{saturated}}{M} \models T \quad \|M\| = \varkappa$

S-isolation (s = "set")

<u>Def.</u> (1) $p \in S(A)$ is <u>s-isolated</u> if $\exists B \underset{\text{finite}}{\subseteq} A$

$p|_B \vdash p$.

(2) M is <u>s-atomic over A</u> if

$\forall \underset{\text{finite}}{\bar{a} \subseteq M}$ $tp(\bar{a}/A)$ is s-isolated

(3) M is an <u>s-model</u> if $\forall \underset{\text{finite}}{A \subseteq} M$ $\forall p \in S(A)$ $p(M) \neq \emptyset$

$\equiv$ $\aleph_0$-saturated

(4) M is <u>s-prime over A</u> if M is s-model

and $\forall N \supseteq A$ $\exists f : M \xrightarrow[A]{\equiv} N$ $(f|_A = id_A)$

$\hat{M}$ $\aleph_0$-saturated

<u>Remark</u> T: $\aleph_0$-stable, $p \in S(B)$, $\underset{\text{finite}}{B} \subseteq A$

$\Rightarrow \underset{p \subseteq}{\exists q} \in S(A)$ q: s-isolated

<u>Proof</u> (A.c.) Suppose there's no such q.

(1) $p \vdash\!\!\!/$ a type in $S(A)$. So there's $B \subseteq B_0 \subseteq A$

and $p_0 \neq p_1 \in S(B_0)$ extending p.

(2) $\underset{i=0,1}{p_i} \vdash\!\!\!/$ a type in $S(A)$. So: there is $B_0 \subseteq B_1 \subseteq A$

and $\underset{\text{pairwise distinct}}{p_{00}, p_{01}, p_{10}, p_{11}} \in S(B_1)$, $p_0 \subseteq p_{00}, p_{01}$, $p_1 \subseteq p_{10}, p_{11}$

(3) ....

$B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots$

$$\begin{tikzcd}[column sep=small]
 & & & p \in S(B) \arrow[dll, no head] \arrow[drr, no head] & & & \\
 & p_0 \arrow[dl, no head] \arrow[d, no head] & & & & p_1 \arrow[d, no head] \arrow[dr, no head] & \in S(B_0) \\
p_{00} & p_{01} & & & & p_{10} & p_{11} \in S(B_1)
\end{tikzcd}$$

Let $B_\omega = \bigcup^{\uparrow}_n B_n \subseteq A$: ctble set, but $|S(B_\omega)| \geq \aleph_0$ ■

Property of s-isolation:

$tp(ab/A)$ s-isolated $\iff$ $tp(a/A)$ is s-isolated and $tp(b/Aa)$ is s-isolated.

(exercise)

Corollary T : $\aleph_0$-stable $\implies \forall A \subseteq \mathcal{M}\ \exists M \supseteq A$

M s-prime over A

Proof (sketch) $M = A \cup \underbrace{\{a_\alpha : \alpha < ?\}}_{\text{s-construction}}$ s.t.:

(1) $tp(a_\alpha / Aa_{<\alpha})$ is s-isolated

(2) M is $\aleph_0$-saturated

it terminates at some point

Then M is s-prime

The model M constructed this way is s-constructible$/A$ and s-primary$/A$, it's unique up to $\cong_A$.

Corollary T: $\aleph_0$-stable, M: s-prime $/A$

$\Rightarrow M$: s-atomic

Proof Let N: s-primary $/A$

$\Downarrow$ property of s-isolation

N: s-atomic

$$M: \text{s-prime}/A \Longrightarrow \exists f: M \xrightarrow[A]{\equiv} N$$

$\Downarrow$

M: s-atomic

Def (M, N) is a Vaughtian pair for T, if

$N \precneqq M \models T$ and for some $\varphi(x) \in L_1(N)$ non-algebraic consistent

$$\varphi(N) = \varphi(M)$$

(here (M, N, φ): Vaughtian triple)

Lemma 1 Assume T: $\aleph_0$-stable (enough T: small). If T has a Vaughtian pair, then there's $Vp\ (M, N)$ s.t. M, N: ctble and saturated.

Proof Let (M_0, N_0, φ): Vaughtian triple.

$$\varphi(\bar{x}, \bar{a}), \quad \bar{a} \subseteq N_0$$

Let $L' = L \cup \{\bar{a}\} \cup \{P(x)\}$

$\bar{a}$: new constant symbols; $P(x)$: new predicate symbol

T': a complete theory in L' s.t.

(0) $T' \supseteq Th(N_0, \bar{a}) = Th(M_0, \bar{a})$ and for any $M' \models T'$:

(1) $N := P^{M'}(M')\restriction_L \prec M := M'\restriction_L$

(2) $\bar{a}^{M'} \subseteq P(M')$: $\bigwedge_i P(a_i) \in T'$

(3) (M, N, φ): a V. triple : $\varphi = \varphi(x, \bar{a}^{M'})$

- $[\forall x (\varphi(x) \longrightarrow P(x))] \in T'$
- $[\exists x \neg P(x)] \in T$

Ad(1): Let $\psi(\bar{x}, y) \in L$.

$T' \ni \left[\forall \bar{x} \left[\bigwedge_i P(x_i) \wedge \exists y\, \psi(\bar{x}, y) \longrightarrow \exists y (P(y) \wedge \psi(\bar{x}, y)\right]\right]$

__Fact__ $\exists_{\text{ctble}} M' \models T'$ ($M := M'\restriction_L$ and $N : P(M')\restriction_L$ are both ctble and saturated)

__Pf__ (fact) $M' = \bigcup_{n<\omega} M'_n$: models of T' elem. chain

- M'_0: arbitrary
- $n \rightsquigarrow n+1$: $\underset{\text{ctble}}{M'_{n+1}} \succ M_n$ such that:

(i) $\underset{\text{finite}}{\forall A \subseteq M'_n} \forall p \in S^L(A)\ p(M'_{n+1}) \neq \emptyset$

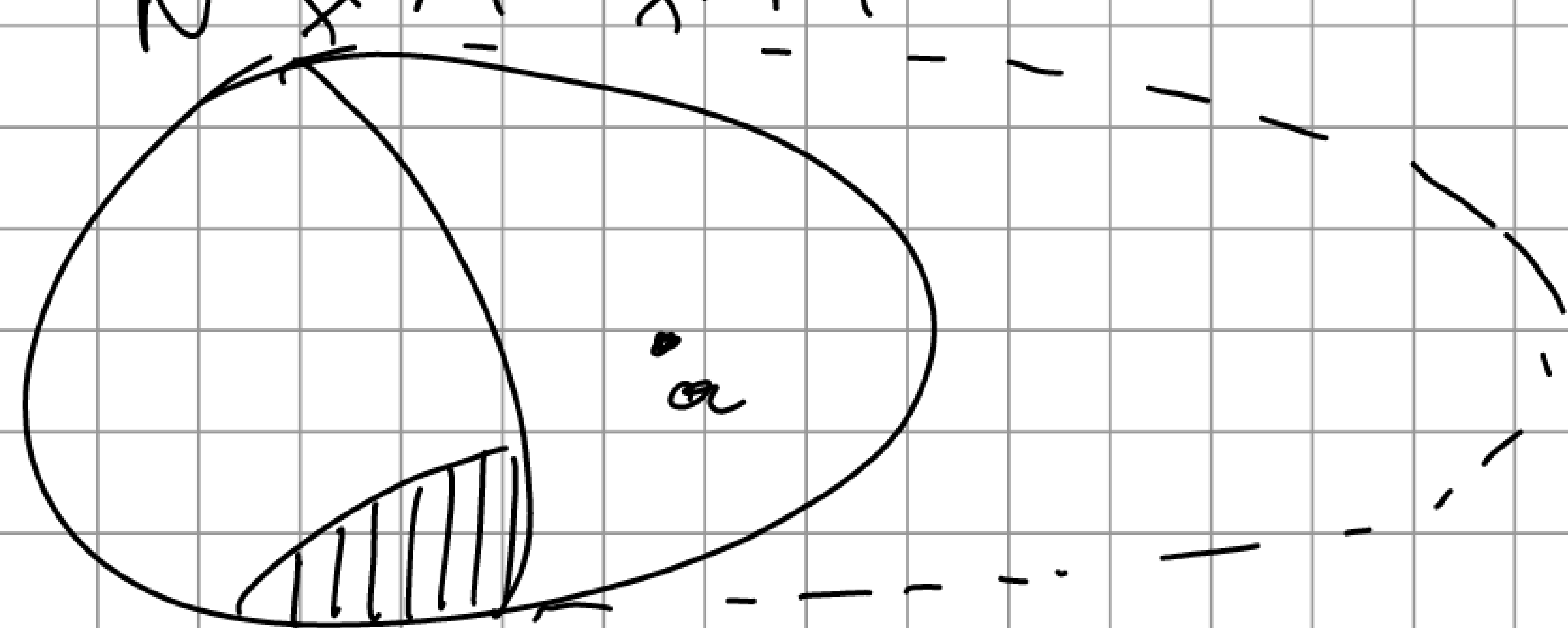
(ii) $\underset{\text{finite}}{\forall A \subseteq P(M'_n)} \forall p \in S^L(A)\ p(x) \cup \{P(x)\}$ is realised in M'_{n+1}.

Fact, Lemma 1 ■

Lemma 2 (stretching Vaughtian pair)

Let (M, N, φ): V. triple, M, N: $\aleph_0$-saturated, T: $\aleph_0$-stable. Then $\exists M' \succneqq M$ ($\aleph_0$-saturated) (M', N, φ) is a V. triple

$N \precneqq M \precneqq M'$



$\varphi(N) = \varphi(M) = \varphi(M')$

Proof Let $a \in M \setminus N$, $p = tp(a/N)$, $MH(p) = 1$.

So $p \subseteq q \in S(M)$, $RM(q) = RM(p)$ (unique)

⋮

16.05.2022

Lemma 2 (Stretching Vaughtian pair) <- able

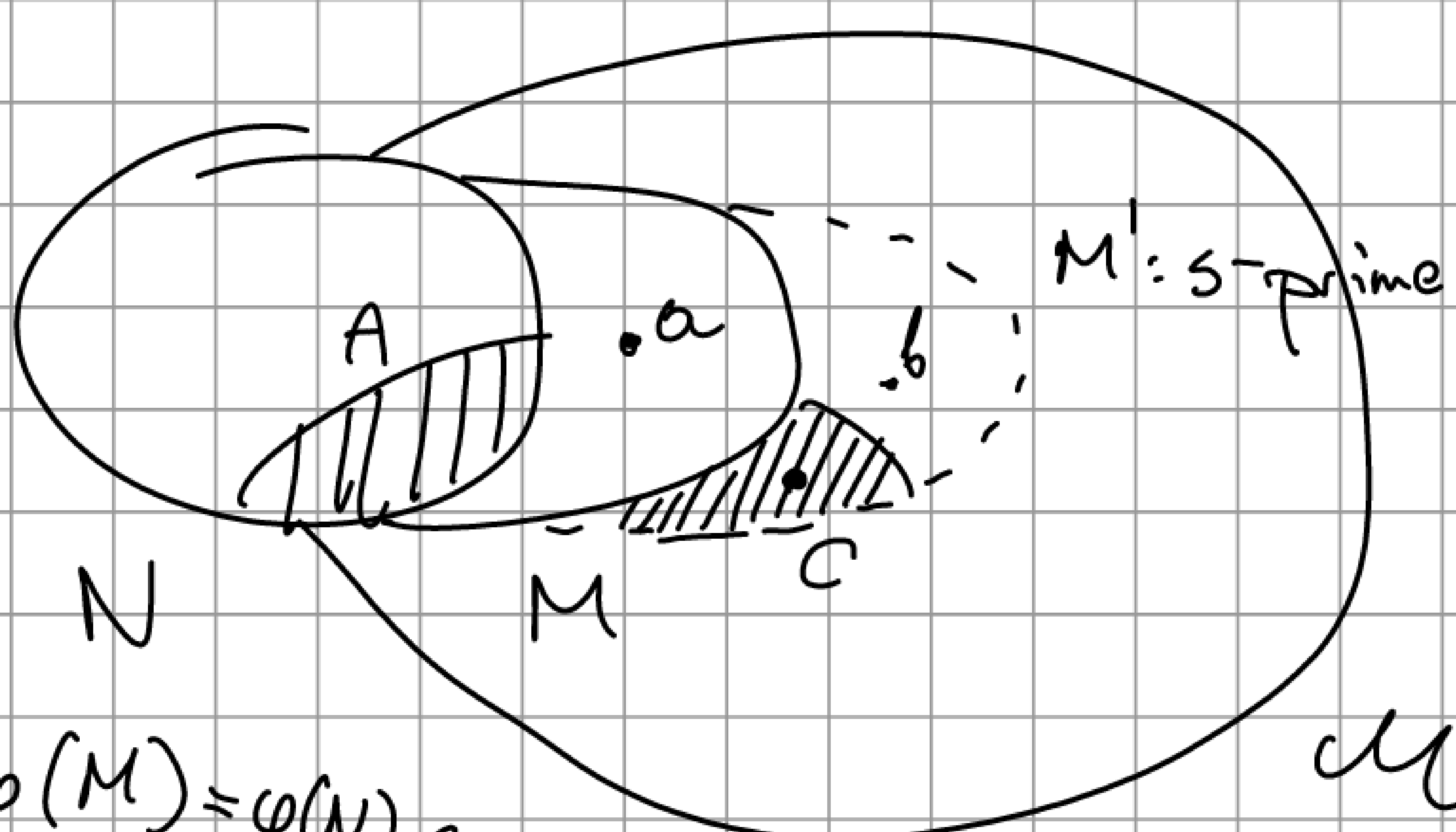
T: $\aleph_0$-stable, (M, N, φ): Vaughtian triple

M, N: $\aleph_0$-saturated

$\Rightarrow \exists M' \succnsim M$ (M', N, φ): Vt.

M': $\aleph_0$-saturated

Proof



$A = \varphi(M) = \varphi(N) \subseteq \varphi(M')$

$tp(b/M) \supseteq tp(a/M)$, $RM(b/A) = RM(b/M) = RM(a/N)$

Claim: $\varphi(M) = \varphi(M')$

∎

Corollary Suppose $\kappa > \aleph_0$, T: κ-categorical.

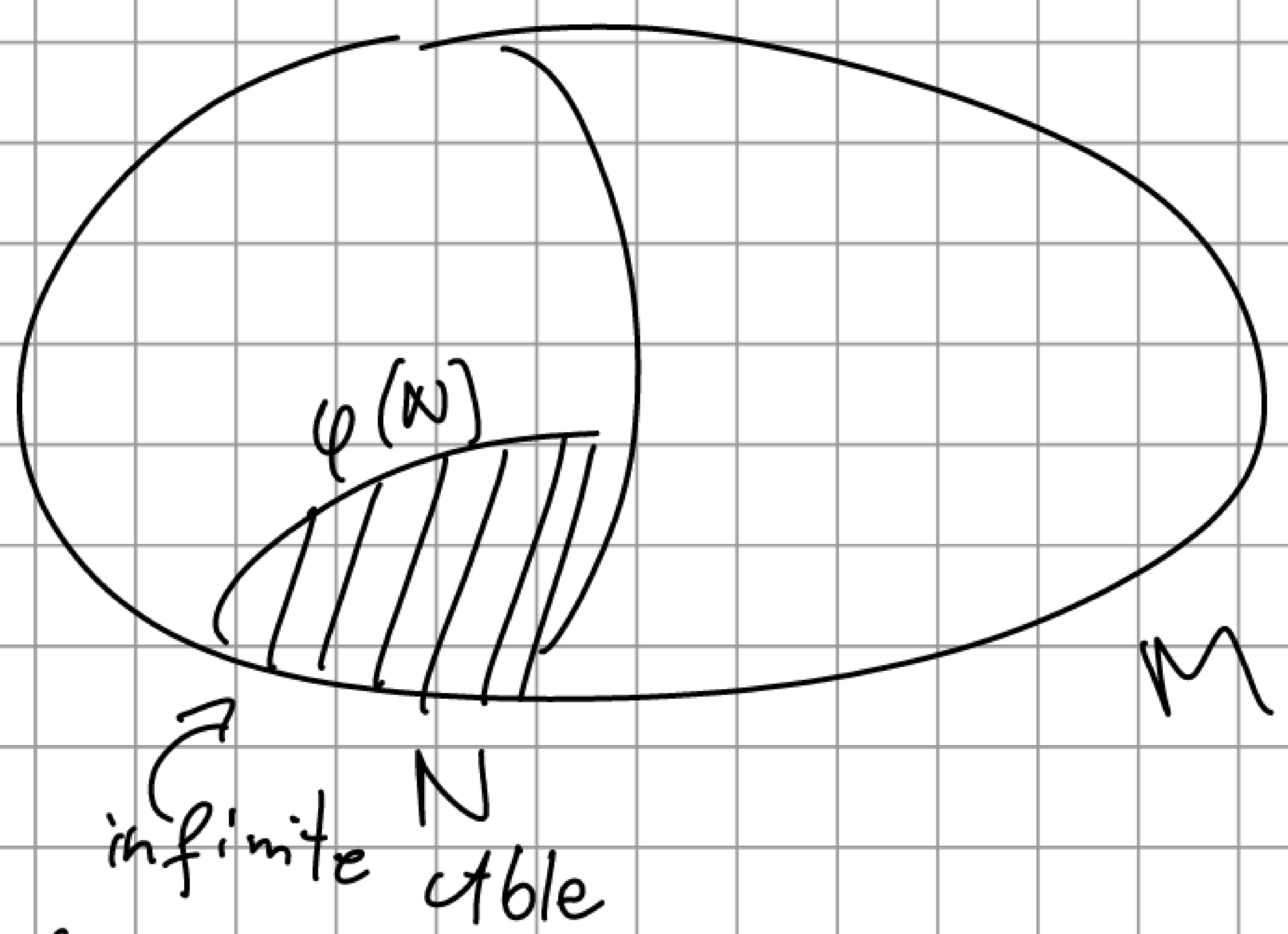
T has no Vaughtian pair and T: $\aleph_0$-stable.

Proof T: $\aleph_0$-stable: proved earlier.

No Vaughtian pair: (A.c.) suppose (M, N, φ): Vt.

By lemma 1 wlog M, N are $\aleph_0$-saturated, N: ctble

by lemma 2 wlog $\|M\| = \kappa$, so M is not saturated



$p(x) = \{\varphi(x)\} \cup \{x \neq a\}_{a \in \varphi(N)}$ $\Rightarrow$ not realised in M.

But T: $\aleph_0$-stable $\rightsquigarrow$ $\exists M'$ $\|M'\| = \kappa$ saturated

$$M \not\cong M' \quad \text{↯}$$

<u>Theorem</u> If T: $\aleph_0$-stable without a Vaughtian pair. Then $\forall \kappa > \aleph_0$ T: κ-categorical.

<u>Corollary</u> (Morley thm, 1964) Let $\kappa > \aleph_0$, D

(1) T: κ-categorical

(2) T: $\aleph_1$-categorical

Lemma 3 (T: $\aleph_0$-stable with no Vaughtian pair).

$\exists$ a $\underbrace{\text{strongly minimal}}_{RM(\varphi)=Mlt(\varphi)=1}$ formula $\varphi(x,\bar{c})$, $\bar{c} \subseteq \underset{\text{prime}}{M} \models T$

Proof Problem from list 6 $\blacksquare$

Lemma 4 (T: $\aleph_0$-stable, no V. pair) Assume

$M \models T$, $\underbrace{\varphi(x)}_{s.m.} = \varphi(x, \underset{\underset{M}{\cap}}{\bar{a}})$, then:

(a) M is prime and minimal $/\varphi(M) \cup \{\bar{a}\}$

(b) If $\|M\| > \aleph_0$, then $\dim(\varphi(M)/\bar{a}) = \|M\|$

Explanation 1. $A \subseteq M \prec \mathcal{M}$, M: minimal $/A$

$\iff \neg\exists \underset{\underset{A}{\cup}}{M'} \precneq M$

2. acl-dimension: Assume $\mathcal{U} = \varphi(M)$, where

$\varphi(x,\bar{a})$: s.m., $acl_{\bar{a}}: P(\mathcal{U}) \to P(\mathcal{U})$,

$acl_{\bar{a}}(\underset{\underset{\mathcal{U}}{\cap}}{A}) = acl(A\bar{a}) \cap \mathcal{U}$

3. $(\mathcal{U}, \underbrace{acl_{\bar{a}}}_{cl})$ is a pregeometry, i.e.

• $cl(cl(A)) = cl(A) \supseteq A$

- $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$
- $cl(A) = \bigcup_{\substack{A_0 \subseteq A \\ \text{finite}}} cl(A_0)$
- $a \in cl(Ab) \setminus cl(A) \Rightarrow b \in cl(Aa)$

$\rightsquigarrow$ Basis $\mathcal{B} \subseteq \mathcal{U}$ cl-independent set, $\dim(\mathcal{U}) = |\mathcal{B}|$.

In the case of $acl_{\bar{a}}$ on $\varphi(M)$ additionally:

Let $\underset{\text{s.m.}}{p(x)} \in S(\bar{a}) \cap [\varphi(x)]$ i.e. $RM(p) = MH(p) = 1$

Then an $acl_{\bar{a}}$-independent set $I \subseteq \varphi(M)$ is a Morley sequence in $p \Rightarrow$ indiscernible over $\bar{a}$.

Pf (lemma 4)

(a) Suppose M prime and is not minimal $/ \varphi(M) \cup \bar{a}$.

So $\varphi(M) \cup \bar{a} \subseteq N \not\cong M$

N: prime $/ \varphi(M) \cup \bar{a}$

But then (M, N, φ): a V. triple $\lightning$

(b) Let $I \subseteq \varphi(M)$ an $acl_{\bar{a}}$-basis of $\varphi(M)$. Then $\bar{a} \cup \varphi(M) \subseteq acl_{\bar{a}}(I) \cup \bar{a} \subseteq M$

$$\|M\| \overset{(a)}{=} |\bar{a} \cup \varphi(M)| \underset{\substack{\uparrow \\ \varphi(M) \\ \text{uncountable}}}{=} |I| = \dim(\varphi(M)/\bar{a})$$

<u>Proof of Morley thm</u>

Let $\varkappa > \aleph_0$, $\underset{\curlyvee}{M}, \underset{\curlyvee}{N} \models T$ of power $\varkappa$.

$\underset{\cup}{M_0}$: prime

$\bar{a}$ $\quad \varphi(x, \bar{a})$: s.m. (lemma 3)

By lemma 4: M prime & minimal $/ \varphi(M, \bar{a}) \cup \bar{a}$

N : —— $\|$ —— $/ \varphi(N, \bar{a}) \cup \bar{a}$

Let $I \subseteq \varphi(M)$ bases.
$\quad J \subseteq \varphi(N)$

Let $p \in S(\bar{a}) \cap [\varphi(x)]$: a s.m. type

I, J : Morley sequences in p. By L46

$|I| = |J| = \varkappa$. $f : I\bar{a} \to J\bar{a}$ s.t.

$$f = \begin{cases} f\restriction\bar{a} = id_{\bar{a}} \\ f : I \xrightarrow[\text{onto}]{1-1} J \end{cases} \qquad f \text{ is elementory}$$

$$\Downarrow$$

$$\underset{\cap}{f} : \underset{\substack{\cup \\ \varphi(M)_{\bar{a}}}}{ad(I\bar{a})} \xrightarrow{\cong} \underset{\substack{\cup \\ \varphi(N)_{\bar{a}}}}{ad(J\bar{a})}$$

$$f\restriction_{\varphi(M)_{\bar a}} : \underset{\substack{\cap\!\!|\\ M}}{\varphi(M)_{\bar a}} \xrightarrow{\cong} \underset{\substack{\cap\!\!|\\ N}}{\varphi(N)_{\bar a}}$$

$$\cap\!\!|$$

$$f' : M \xrightarrow{\cong} N$$

(lemma 4a) ∎

<u>Thm</u> (Baldwin, Lachlan, 1971) Assume T: $\aleph_1$-categorical not $\aleph_0$-categorical. Then:

(1) T has $\aleph_0$ many ctble models

(2) Every model of T is homogenous.

$\aleph_1$-categorical theories:

- $\aleph_0$-stable
- have prime model
- no V. pairs
- s.m. sets
- M. sequences, "bases"

Stable formulas

T: fixed complete theory with infinite models

$$\underset{\text{ctble}}{L} \ni \varphi = \varphi(\overset{x}{\bar{x}}, \overset{y}{\bar{y}})$$

distinguished variables

parameters variables

<u>Def</u> Let $\delta(x,y) \in L$.

(1) $\delta(x,c)$, $c \subseteq \mathcal{M}$: an <u>instance</u> of δ

(2) $\varphi(x)$ is <u>positive δ-formula</u> if

$\models \varphi \leftrightarrow$ [positive] boolean combination of instances of δ

(3) $(A \subseteq \mathcal{M})$ <u>δ-type</u> = a type consisting of δ-formulas,

$L_\delta(A) = \{\delta\text{-formulas over } A\}$

(4) $S_\delta(A) = \{\text{complete } \delta\text{-types over } A\}$ (ultrafilters in $L_\delta(A)$)

<u>Remark</u> Let $\varphi \in L(A)$, $A \subseteq M$. Then

φ is a δ-formula $\Leftrightarrow \models \varphi \leftrightarrow$ boolean comb. of instances of δ over M

<u>Pf</u> $(\Rightarrow)$ $\mathcal{M} \models \varphi(\bar{x}) \leftrightarrow \bigwedge_i \bigvee_j \delta(x, c_{ij})^{\varepsilon_{ij}}$ $\varepsilon_{ij} \in \{0,1\}$

$c_{ij} \in \mathcal{M}$

$\mathcal{M} \models (\exists y_{ij})\left(\varphi(\bar{x}) \leftrightarrow \bigwedge_i \bigvee_j \delta(x, y_{ij})\right)$

$\succ$

$M \models$ ——— " ———

So c_{ij} may be taken from M.

<u>Example</u> T: the theory of a simple equivalence relation E with two infinite classes. Let $\delta(x,y) = E(x,y)$.

$\models \underbrace{x = x}_{\sim} \leftrightarrow E(x,a) \vee E(x,b)$

is a δ-formula $/ \emptyset$, where $a, b \in \mathcal{M} \models T$, $\models \neg E(a,b)$

<u>Def</u> (1) $\delta(x,y)$ has <u>order property</u>, if

$\exists_{i<\omega} a_i, b_i \subseteq \mathcal{M} \quad \forall i,j < \omega \models \delta(a_i, b_j) \Leftrightarrow i \leq j$

(2) $\delta(x,y)$ is <u>stable</u> if it does not have the order property

<u>Lemma 1</u> (1) $\varphi(x,y), \psi(x,z)$: stable

$\Longrightarrow \neg\varphi(x,y), (\varphi \vee \psi)(x, yz), (\varphi \wedge \psi)(x, yz)$

are stable

(2) Let $\psi(y,x) = \varphi(x,y)$, then

φ stable $\Longleftrightarrow$ ψ stable

(3) φ : stable $\Longleftrightarrow \exists n < \omega \; \neg \exists a_i b_i \; (i \leq n)$

$\bigwedge_{i,j} \models \varphi(a_i, b_j) \leftrightarrow i \leq j$

Pf. excercise

Def Let $p \in S(M)$ or $p \in S_\delta(M)$.
A δ-definition of p: a formula $\psi(y) \in L(M)$ s.t. $\forall c \in M$ $(\delta(x,c) \in p \iff \models \psi(c))$
(i.e. $\{c \in M : \delta(x,c) \in p\} = \psi(M)$)

Lemma 2 Assume $\delta(x,y)$ is stable, $p \in S(M)$ or $p \in S_\delta(M)$. Then:

(1) $p(x)$ has a δ-definition $\psi(y)$ that is a positive δ^*-formula, where $\delta^*(y,x) = \delta(x,y)$

(2) $A \subseteq M$ and M is $|A|^+$-saturated, then $\exists c_1, c_2, \ldots \in M$ $c_i \models p|_{Ac_{<i}}$ and the δ-definition of p is equivalent to a positive boolean combination of formulas $\delta(c_i, y)$, $i < \omega$.

23.05.2022

Lemma 2 Assume δ is stable, ~~if~~ $p \in S(M)$ or $p \in S_\delta(M)$. Then

(1) $p(x)$ has a δ-definition $\psi(y)$, that is a positive δ-formula

(2) If $A \subset M \leftarrow |A|^+$-saturated, then $\exists c_1, c_2, \ldots \in M$ $\bigwedge_i c_i \models p\restriction_{Ac_{<i}}$ and ψ from (1) is a positive Boolean combination of formulas $\delta(c_i, x)$, $i<\omega$

Proof Let $N<\omega$ s.t.

(i) $\neg(\exists c_i, a_i, i\leq N)\left(\bigwedge_{i,j} \models \delta(c_i, a_j) \Leftrightarrow i\leq j\right)$

(ii) $\neg(\exists c_i, b_i, i\leq N)\left(\bigwedge_{i,j} \models \neg\delta(c_i, b_j) \Leftrightarrow i<j\right)$

Let $c^* \models p$. So: $\delta(x,a) \in p \Leftrightarrow \models \delta(c^*, a)$ for $a \in M$.

We shall define $c_i \subseteq M$, $\underbrace{A_i, B_i}_{\text{finite}} \subseteq M$, $i<\omega$ as follows:

- $c_0 \subseteq M$ arbitrary $(c_0 \models p\restriction_A)$
- Suppose we have $c_0, \ldots, c_n, A_0, \ldots, A_n, B_0, \ldots, B_n$. We shall find $c_{n+1}, A_{n+1}, B_{n+1}$.
- $A_{n+1} \subseteq M$ s.t. $\forall W \subseteq \{0, \ldots, n\}$ <u>if</u> $(\exists a \in M)\left(\bigwedge_{j\in W} \models \delta(c_j, a) \text{ and } \underbrace{\models \neg\delta(c^*, a)}^{\text{i.e. } \delta(x,a)\notin p}\right)$ <u>then</u> $\exists$ such an $a_W \in A_{n+1}$
- $B_{n+1} \subseteq M$ s.t. $\forall W \subseteq \{0, \ldots, n\}$ <u>if</u> $(\exists b \in M)\left(\bigwedge_{j\in W} \models \neg\delta(c_j, b) \text{ and } \models \delta(c^*, b)\right)$ <u>then</u> $\exists$ such an $b_W \in B_{n+1}$.
- for $b \in B_{\leq n+1}$ $\models \delta(c^*, b)$, for $a \in A_{\leq n+1}$ $\models \neg\delta(c^*, a)$

  we may assume
- $c_{n+1} \in M$ s.t. $(\forall a \in A_{\leq n+1})(\forall b \in B_{\leq n+1})(\models \delta(c_{n+1}, b) \text{ and } \models \neg\delta(c_{n+1}, a))$

  [for (2): $c_{n+1} \models p\restriction_{A A_{\leq n+1} B_{\leq n+1}}$ is enough]
- So for $a \in A_{\leq n}$, $b \in B_{\leq n}$ $\models \neg\delta(c_n, a) \wedge \delta(c_n, b)$
- <u>if</u> $0 \leq i_0 < \ldots < i_n < \omega$ and $\exists a \in M$

  $\models \delta(c_{i_0}, a) \wedge \delta(c_{i_1}, a) \wedge \ldots \wedge \delta(c_{i_n}, a) \wedge \neg\delta(c^*, a)$

  <u>then</u> $\exists d_0, \ldots, d_n \in M$ $(\forall 0 \leq j, r \leq m)(\models \delta(c_{i_j}, d_r) \Leftrightarrow j \leq r)$

pf Let $W_r = \{i_0, \dots, i_{r-1}\} \subseteq \{i_0, \dots, i_n\}$. Let $d_r = a_{W_r} \in A_{(i_{r-1})+1}$ s.t.

$\models \sigma(c_{i_0}, d_r) \wedge \dots \wedge \sigma(c_{i_{r-1}}, d_r) \wedge \neg\sigma(c^*, d_r)$, $\models \neg\sigma(c_j, d_r)$ for all $j > i_{r-1}$ ■

- simmetrically: if $0 \leq i_0 < \dots < i_n < \omega$ and $\exists b \in M$

  $\models \neg\sigma(c_{i_0}, b) \wedge \dots \wedge \neg\sigma(c_{i_n}, b) \wedge \sigma(c^*, b)$

  then $\exists d_0, \dots, d_n \in M$ $\forall 0 \leq j, r \leq n$ $(\models \neg\sigma(c_{i_j}, d_r) \Leftrightarrow j < r)$

- Therefore:

  (a) If $a \in M$ and $\exists W \subseteq \{0, \dots, 2N\}$, $|W| = N+1$ s.t. $\bigwedge_{i \in W} \models \sigma(c_i, a)$, then $\models \sigma(c^*, a)$

  (b) If $a \in M$ and $\exists W \subseteq \{0, \dots, 2N\}$, $|W| = N+1$ s.t. $\bigwedge_{i \in W} \models \neg\sigma(c_i, a)$, then $\models \neg\sigma(c^*, a)$

- Let $a \in M$. $\sigma(x, a) \in p \Leftrightarrow \models \sigma(c^*, a) \Leftrightarrow \models \underbrace{\bigvee_{\substack{W \subseteq \{0, \dots, 2N\} \\ |W| = N+1}} \bigwedge_{i \in W} \sigma(c_i, a)}_{\text{put } y:\ \psi(c_{\leq 2N}, y)}$

- So $\forall a \in M$ $(\sigma(x, a) \in p \Leftrightarrow \models \psi(c_{\leq 2N}, a))$: a delta definition of p.

Lemma ■

Corollary Let $\sigma \in L$. TFAE:

(1) $\sigma(x, y)$: stable

(2) $\forall M \forall p \in S_\sigma(M)$ p has a σ-definition

(3) $(\forall \lambda \geq \aleph_0)(\forall A \subseteq \mathcal{M}, |A| \leq \lambda)(|S_\sigma(A)| \leq \lambda)$

(4) $(\exists \lambda \geq \aleph_0)$ —— " ——

Proof (1) ⇒ (2): by lemma 2

- (2) ⇒ (3) $A \subseteq M \prec \mathcal{M}$, $\|M\| \leq \lambda$, $|S_\sigma(A)| \leq |S_\sigma(M)| \underset{\uparrow}{\leq} \|M\| \leq \lambda$

  there are $\leq \|M\|$ σ-definitions$/M$

- (3) ⇒ (4) trivial

(4) $\Rightarrow$ (1) by contradiction: suppose δ has order property. Let $\lambda \geq \aleph_0$.

We shall find A s.t. $|A| \leq \lambda$ and $|S_\delta(A)| > \lambda$

- $(\exists a_i, b_i, i<\omega)(\models \delta(a_i, b_j) \Leftrightarrow i<j)$, choose $(\underset{\text{dense}}{I, \leq}) \models DLO_0$, $|I| > \lambda$ where $\underset{|J|\leq\lambda}{\exists J \subseteq I}$: J dense. By compactness

$$(\exists a_i, b_i, i \in I) \bigwedge_{i,j\in I} (\models \delta(a_i, b_j) \Leftrightarrow i<j)$$

- δ defines an order:

$$0 < \underset{i \in I \supseteq J}{\overset{a_i}{b_i}} < 0 \underset{\delta}{\leq} 0 < \ldots$$

Let $A = \{ b_j : j \in J \}$, then $|A| \leq \lambda$ and $|S_\delta(A)| > \lambda$:

when $i_1 < i_2 \in I$ then $tp_\delta(a_{i_1}/A) \neq tp_\delta(a_{i_2}/A)$, just choose $j \in J$ s.t. $i_1 < j < i_2$. ■

Corollary (1) T: stable $\Leftrightarrow (\forall \delta \in L)(\delta$: stable in $T)$

(2) T stable and $\lambda^{|T|} = \lambda$, then T: λ-stable

Proof (1) ($\Rightarrow$) T: λ-stable. Then $\overbrace{\forall A \subseteq M}^{\forall \delta \in L}_{|A|\leq\lambda}$ $|S_\delta(A)| \leq |S(A)| \leq \lambda$, so by previous corollary δ: stable.

($\Leftarrow$) later...

(2) $|T| = \aleph_0$, assume $\lambda = \lambda^{\aleph_0}$. Let $|A| \leq \lambda$, ~~$S(A)$~~

$$S(A) \ni p \xmapsto{1-1} \langle p|_\delta : \delta \in L \rangle \in \prod_{\delta \in L} S_\delta(A)$$

$$\left|\prod_{\delta\in L} S_\delta(A)\right| \leq \lambda^{\aleph_0} = \lambda \rightsquigarrow |S(A)| \leq \lambda$$

($\Leftarrow$) Assume $\forall \delta \in L$ δ: stable in T. Then $\underset{|A|\leq\lambda}{\wedge}$ $|S(A)| \leq \left|\prod_{\delta\in L} S_\delta(A)\right| \leq \lambda^{\aleph_0}$

$\Downarrow$

T stable in λ if $\lambda^{\aleph_0} = \lambda$ ■

Def (1) $\delta(x,y) \in L$ has <u>strict order property</u> (SOP)

if $(\exists b_i, i<\omega)(\delta(\mathcal{M}, b_0) \supsetneq \delta(\mathcal{M}, b_1) \supsetneq \dots)$

$[\Leftrightarrow \ \delta(b_0, \mathcal{M}) \subsetneq \delta(b_1, \mathcal{M}) \subsetneq \dots]$

alternative definition

T has SOP $\Leftrightarrow (\exists \delta \in L)(\delta$ has SOP$)$

(2) δ has independence property (IP)

if $(\exists b_i, i<\omega)$ $\{\delta(\mathcal{M}, b_i) : i<\omega\}$ is independent

T has IP $\Leftrightarrow (\exists \delta \in L)(\delta$ has IP$)$

<u>Theorem</u> (Shelah)

T has OP $\Leftrightarrow$ T has SOP or T has IP

(is unstable)

<u>Examples</u> • $DLO_0 = Th(\mathbb{Q}, \leq)$ has SOP : $\varphi(x,y) = x \leq y$ has SOP,

but DLO_0 has NIP

• Random graph has IP and NSOP

~~TYPES AND DEFINABL~~

<u>TYPES & TYPE-DEFINABLE SETS.</u>

$p(\bar{x})$: a type $\rightsquigarrow$ $p(\mathcal{M}) = \{a \in \mathcal{M}^n : a \models p\}$: a type-definable set (over A)

$|\bar{x}| = n$ (over A)

$$= \bigcap_{\varphi \in p} \varphi(\mathcal{M})$$

$$p = \{\varphi_i : i \in I\} \longrightarrow = \left(\bigwedge_{i \in I} \varphi_i\right)(\mathcal{M})$$

Let $p(\bar{x}), q(\bar{x})$: types, $(p \wedge q)(\bar{x}) = (p \cup q)(\bar{x})$,

$(p \vee q)(\bar{x}) = \{((\bigwedge p_0) \vee (\bigwedge q_0))(\bar{x}) : p_0 \subseteq p, \ q_0 \subseteq q \text{ finite}\}$

<u>Fact</u> (1) $(p \wedge q)(\mathcal{M}) = p(\mathcal{M}) \cap q(\mathcal{M})$

(2) $(p \vee q)(\mathcal{M}) = p(\mathcal{M}) \cup q(\mathcal{M})$

$$\left(\exists \bar{y}\, p\right)(\bar{x},\bar{y}) = \exists \bar{y} \bigwedge_i \varphi_i(\bar{x},\bar{y}) \stackrel{\text{def}}{=} \{\exists \bar{y} \bigwedge p_0(\bar{x},\bar{y}) : p_0 \subseteq_{\text{finite}} p\}$$

where $p = \{\varphi_i : i \in I\}$

(3) $\left(\exists \bar{y}\, p\right)(\mathcal{M}) = \pi_{\bar{x}}\big(\underset{\mathcal{M}^{\bar{x}} \times \mathcal{M}^{\bar{y}}}{\underset{\cap}{p(\mathcal{M})}}\big)$

(4) $(\forall \bar{y} p)(\mathcal{M}) = \mathcal{M}^{\bar{x}} \setminus \pi_{\bar{x}}(\mathcal{M}^{\bar{x}\bar{y}} \setminus p(\mathcal{M}))$

<u>Remark</u> $\mathcal{M} \setminus p(\mathcal{M})$ is usually <u>not</u> type-definible

Let $\Delta = \{\varphi_0(x,y) \ldots \varphi_n(x,y)\} \subseteq L$ — tuples

Δ-formula, Δ-type

<u>Shelah trick</u> There's formula $\delta(x,\bar{y})$ s.t. "instances of formulas Δ" = "instances of δ" over any set A of power ≥ 2.

<u>Proof</u> $\delta = \delta(x, y, z_0, \ldots, z_n) =$

$$\begin{cases} \varphi_0(x,y), & \text{when } z_0 = z_1 \\ \varphi_1(x,y), & \text{when } z_0 \neq z_1 \wedge z_0 = z_2 \\ \varphi_2(x,y), & \text{when } z_0 \neq z_1, z_2 \wedge z_0 = z_3 \\ \vdots & \\ \varphi_n(x,y), & \text{when } z_0 \neq z_1, \ldots, z_n \end{cases}$$

- $\delta(\mathcal{M}, a, \bar{c}) = \varphi_i(\mathcal{M}, a)$ for some $i \in \{0, \ldots, n\}$
- For every $a \subseteq A$, every $i \in \{0, \ldots, n\}$ $\exists \bar{c} \subseteq A$ $\varphi_i(\mathcal{M}, a) = \delta(\mathcal{M}, a, \bar{c})$

∎

$p \in S_\delta(A)$ $[p \in S_\Delta(A)]$ (Δ possibly infinite)

- $CB_\delta(p) = CB$ in $S_\delta(A)$ (CB-delta rank, CB-local rank)
- $\varphi \in L_\delta(A) \rightsquigarrow CB_{\delta,A}(\varphi) = CB(S_\delta(A) \cap [\varphi])$
- $CB_\delta \rightsquigarrow \underset{\substack{\text{local } \delta\text{-rank} \\ \text{local } \Delta\text{-rank}}}{R_\delta} : L(\mathcal{M}) \longrightarrow \{-1\} \cup \mathrm{Ord} \cup \{\infty\}$

The smallest function s.t. $\forall \alpha \in \mathrm{Ord} \cup \{-1\}$

$$R_\delta(\varphi) \geq \alpha+1 \iff \forall n<\omega \ \exists \underset{\substack{\delta\text{-formulas} \\ \text{pairwise contr.}}}{\psi_1, \dots, \psi_n} \ \bigwedge_i R_\delta(\varphi \wedge \psi_i) \geq \alpha$$

[Alternatively]

$$R_\delta(\varphi) = CB_{\delta,\mathcal{M}}(\{p \in S_\delta(\mathcal{M}) : \underset{\text{consistent}}{p \cup \{\varphi\}}\})$$

$$R_\delta(p) = \min\{R_\delta(\varphi) : p \vdash \varphi\}$$

[Alternatively]

$$R_\delta(p) = CB_{\delta,\mathcal{M}}(\{q(x) \in S_\delta(\mathcal{M}) : p \cup q \text{ consistent}\})$$

<u>Remark</u> (a) $\underset{\delta=\delta(x,y)}{R_\delta} : L_x(\mathcal{M}) \xrightarrow{\text{onto}} (\{-1\} \cup \text{proper initial segment of ordinals} \cup \{\infty\})$ possibly

(b) $R_\delta(\varphi) \geq \alpha+1 \iff \exists \underset{\substack{\text{pairwise contr.} \\ \delta\text{-formulas}}}{\psi_1, \psi_2, \dots} \ \bigwedge_i R(\varphi \wedge \psi_i) \geq \alpha$

(c) $R_\delta(\varphi \vee \psi) = \max\{R_\delta(\varphi), R_\delta(\psi)\}$

(d) $p(x)$: a type $/A \implies \exists \underset{p \subseteq}{q(x)}$: a complete type $/A$ $R_\delta(p) = R_\delta(q)$

<u>Example</u>

$$R_\delta = R_{\underbrace{\{\delta, \neg\delta\}}_{\Delta}} \underset{\substack{\uparrow \\ \text{Shelah trick}}}{=} R_{\delta'} \text{ (for some } \delta')$$

$\rightsquigarrow$ We assume that "δ is closed under negation": i.e. $\forall \bar{a} \ \exists \bar{b} \models \delta(x,\bar{a}) \leftrightarrow \neg\delta(x,\bar{b})$

(1) $R_\delta(\varphi(x,\bar{a})) \geq \underset{\omega \ni}{k}+1 \iff \forall n<\omega \ \exists \underset{\substack{\delta\text{-formulas} \\ \text{pairwise contr.}}}{\varphi_1, \dots, \varphi_n} \ \bigwedge_i R_\delta(\varphi(x,\bar{a}) \wedge \varphi_i) \geq k$

$\Leftrightarrow \forall n\ \exists \underset{\text{distinct}}{p_1, \dots, p_n} \in S_\delta(M)\ \hat{\bigwedge}\ R_\delta(p_i \cup \{\varphi\}) \geq k$

$$\left[\begin{array}{l} p_i \neq p_j \quad \text{for } i \neq j \\ \exists c_{ij}\ \delta(x, c_{ij}) \quad \neg\delta(x, c_{ij}) \end{array}\right]$$

$\Leftrightarrow \forall n\ \exists \langle c_{ij} : 1 \leq i \neq j \leq n \rangle \bigwedge_i \big[\overbrace{\bigwedge_{j \neq i} \underbrace{\delta(x, c_{ij}) \wedge \neg\delta(x, c_{ji})}_{\psi_i(x, \bar{c}) \text{: a } \delta\text{-formula}}}^{\text{only a def. of } \psi_i} \in p_i \big]$ and $\overbrace{R_\delta(\varphi(x, \bar{a}) \wedge \psi_i(x, \bar{c})) \geq k}^{\text{enough to have this in the condition}}$

[Notice: $\psi_i(x, \bar{c})$, $i = 1, \dots, n$ are explicitely pairwise $\not\downarrow$]

<u>Lemma</u> For every $\varphi(x, \bar{y})$, $k < \omega$ there is a type $\Phi_{\varphi,k}(\bar{y})$ s.t.

$$\forall \bar{a} \subseteq M\ R_\delta(\varphi(x, \bar{a})) \geq k \Leftrightarrow \models \Phi_{\varphi,k}(\bar{a})$$

<u>Pf</u> Induction on k.

<u>$k=0$</u> $R_\delta(\varphi(x, \bar{a})) \geq 0 \Leftrightarrow \models \exists x\, \varphi(x, \bar{a})$ so $\Phi_{\varphi,0}(\bar{y}) = \{\exists x\, \varphi(x, \bar{y})\}$

<u>$k \to k+1$</u> $R_\delta(\varphi(x, \bar{a})) \geq k+1 \Leftrightarrow_{\models} \forall n\ \underbrace{\exists \bar{c} \bigwedge_{1 \leq i \leq n} \Phi_{\varphi \wedge \psi_i, k}(\bar{a}, \bar{c})}_{\models \Psi_n(\bar{a}) \text{ for a type } \Psi_n(\bar{y})}$

$$\Phi_{\varphi,k+1}(\bar{y}) = \bigcup_n \Psi_n(\bar{y})$$

□

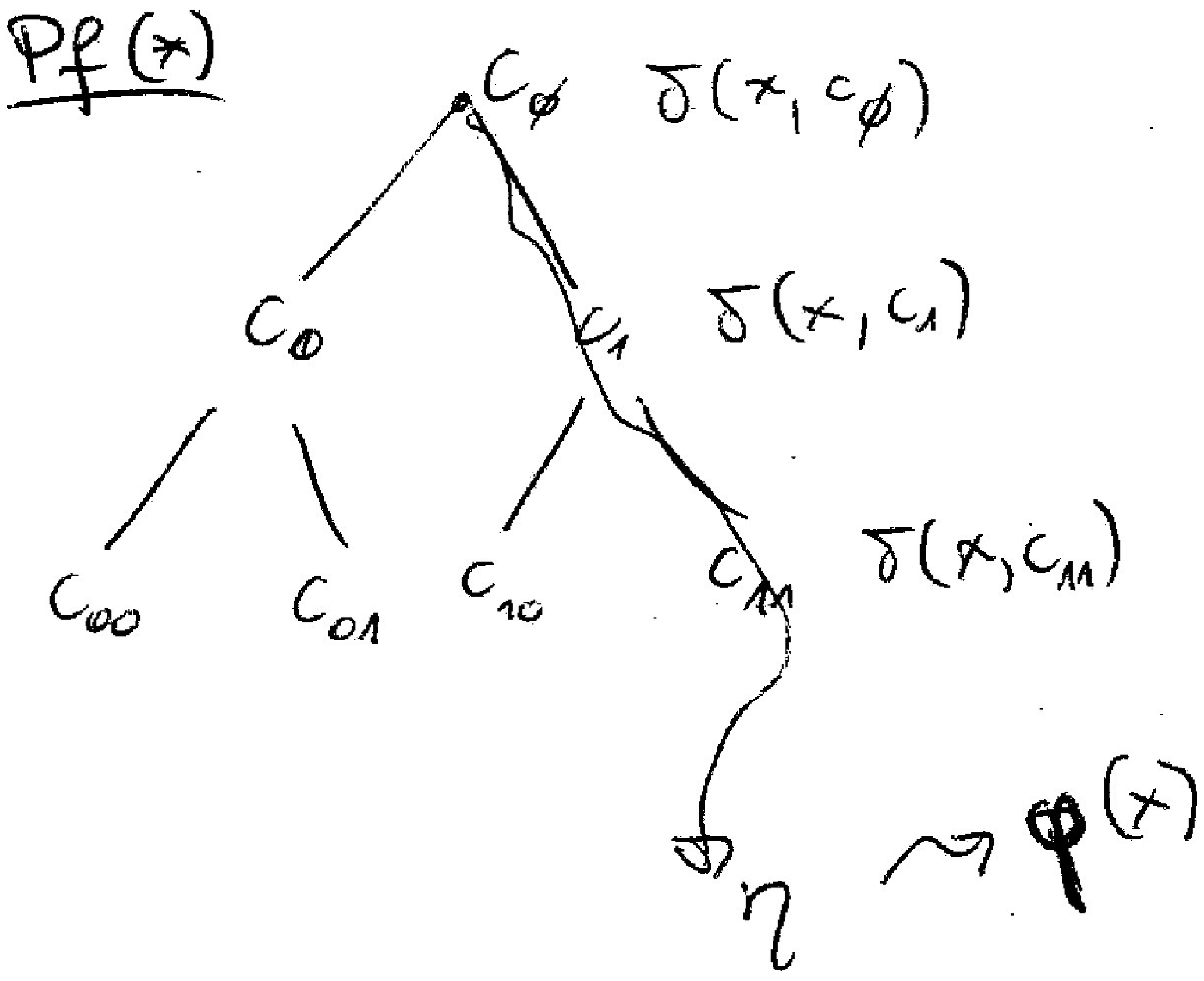
<u>Corollary</u> (1) $R_\delta(\varphi) \geq \omega \Leftrightarrow R_\delta(\varphi) = \infty$

(2) δ is stable $\Leftrightarrow R_\delta(x = x) < \omega$

here $\begin{cases} \chi^0 = \chi \\ \chi^1 = \neg\chi \end{cases}$

<u>Proof</u> (1) $\Leftrightarrow$ $(*)$: $\exists (c_\eta, \eta \in 2^{<\omega})\ \forall \eta \in 2^{\omega}\ p_\eta(x) = \{\varphi\} \cup \{\delta(x, c_{\eta|n})^{\eta(n)} : n < \omega\}$ is consistent

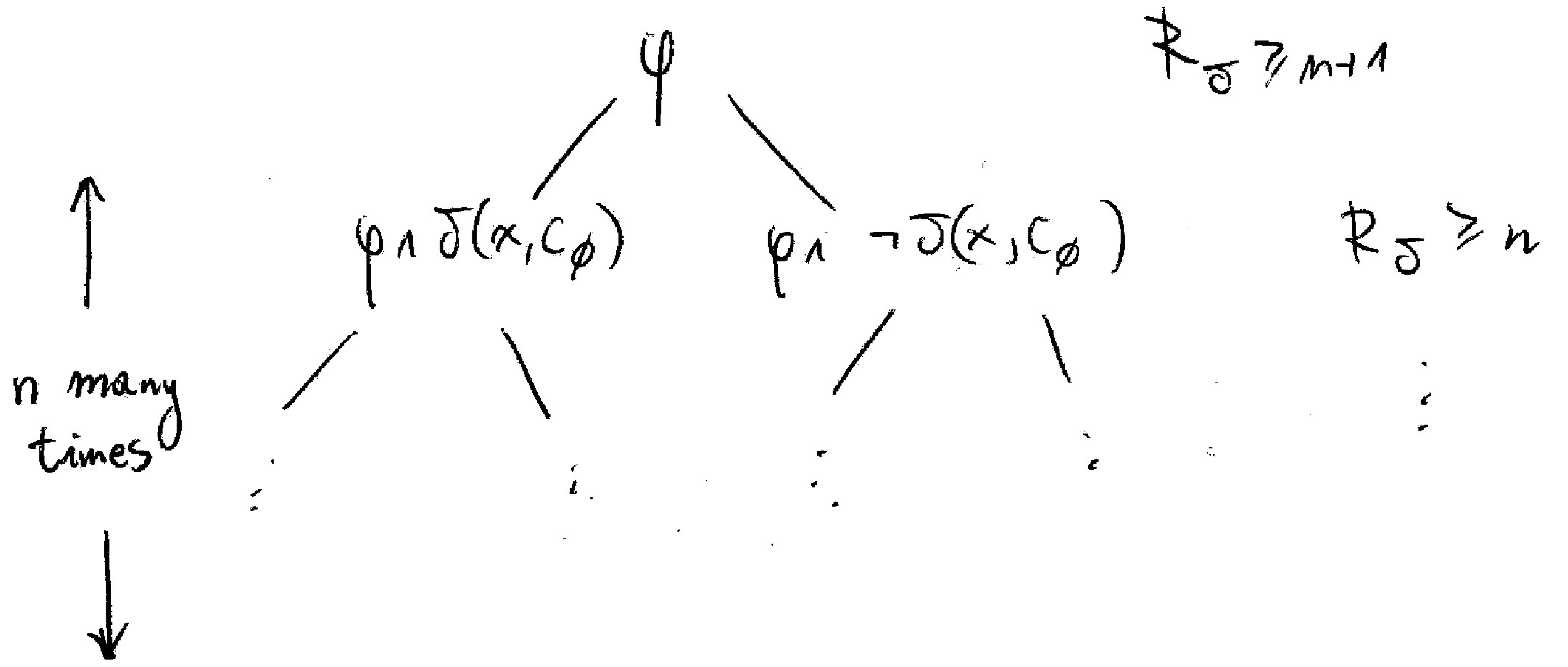
<u>Pf $(*)$</u>



$\rightsquigarrow \varphi(x) \cup \{\neg\delta(x, c_\emptyset), \neg\delta(x, c_1), \neg\delta(x, c_{11}) \dots\}$ consistent

good

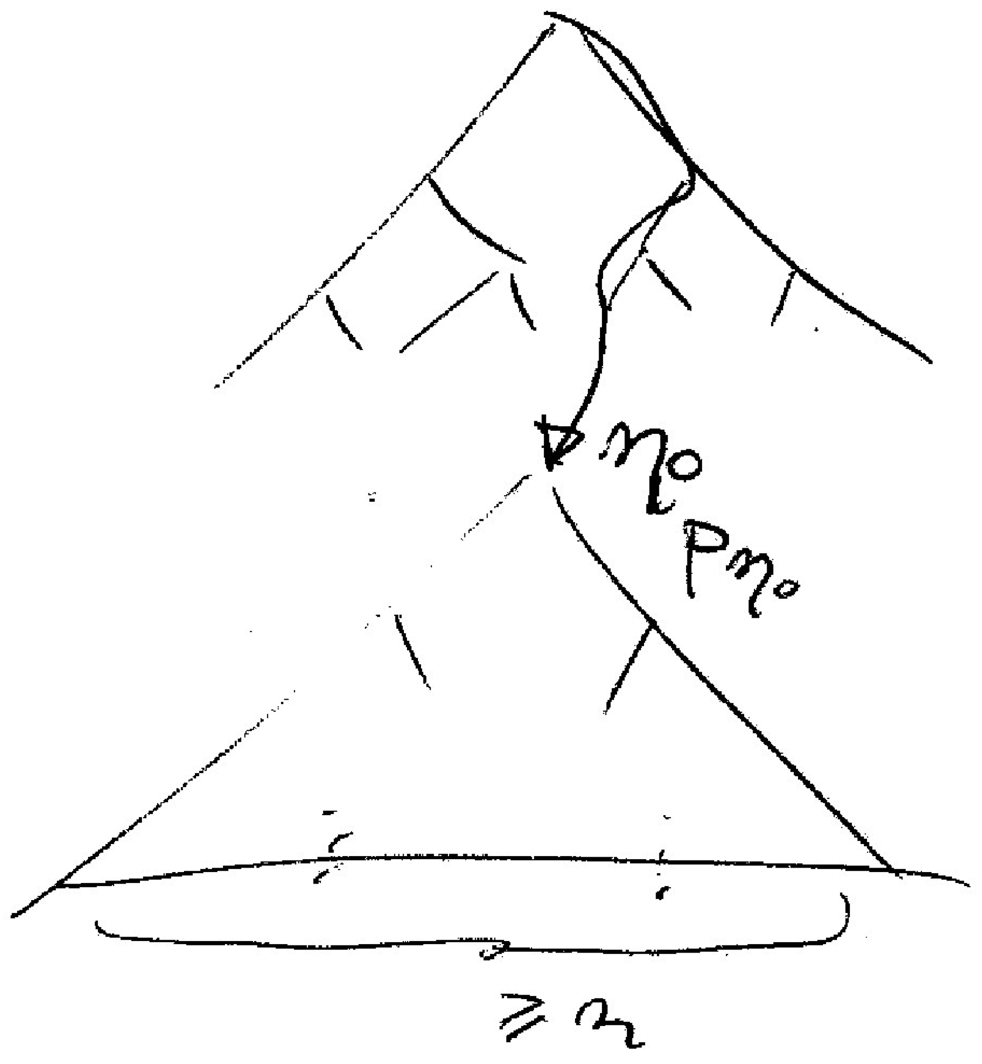
Enough to show $\forall n\ \exists c_\eta, \eta \in 2^{<n}\ \forall \eta \in 2^n\ p_\eta(x)$ consistent. But this follows from $R_\delta(\varphi) \geq n+1$



(*)□

Proof of $\Rightarrow$ in (1): (A.c.) Suppose $R_\delta(\varphi) < \infty$. For $\eta \in 2^{<\omega}$ $R_\delta(p_\eta(x)) < \infty$.

Let $\eta_0 \in 2^{<\omega}$ s.t. $\alpha := R_\delta(p_{\eta_0})$ is minimal.



So $\forall n \ \exists \varphi_1, \ldots, \varphi_n$ $\bigwedge_i R_\delta(p_{\eta_0} \wedge \varphi_i) \geq \alpha$ ↯

δ-formulas pairwise ↯

Pf (2) ~~δ not stable ⇔ R_δ~~ ($\Rightarrow$)(A.c.) $R_\delta(x=x) \geq \omega \overset{(1)}{\Rightarrow} R_\delta(x=x) = \infty$

the proof of (1) get a tree of instances of $\delta \Rightarrow |S_\delta(\{c_\eta : \eta \in 2^{<\omega}\})| = 2^{\aleph_0}$

$\Rightarrow \delta$ unstable

($\Leftarrow$) $R_\delta(x=x) < \omega$, A: a set, $|A| \leq \varkappa$, shall prove $|S_\delta(A)| \leq \varkappa$

(a) $p \in S_\delta(A) \rightsquigarrow \exists \varphi \in p$ $R_\delta(p) = R_\delta(\varphi) \leq k < \omega$ (for some)

There are $\leq \varkappa$-many such $\varphi \in L(A)$

(b) For $\varphi \in L_\delta(A)$ $|\{p \in S_\delta(A) : \varphi \in p \text{ and } R_\delta(p) = R_\delta(\varphi)\}| < \aleph_0$

□

<u>Corollary</u> $R_\delta(p) \geqslant k+1 \iff \forall n \, \exists \varphi_1, \dots, \varphi_n \; R_\delta(p \cup \{\varphi_i\}) \geqslant k$

any type $\nearrow$ (for p); φ_i: δ-formulas pairwise $\not\parallel$

BINARY RANKS : $R_{\delta,2}$

$$R_{\delta,2}(\varphi) \geqslant \alpha+1 \iff \exists c \left[R_{\delta,2}(\varphi \wedge \delta(x,c)) \geqslant \alpha \text{ and } R_{\delta,2}(\varphi \wedge \neg\delta(x,c)) \right]$$

<u>Lemma</u> (1) $R_{\delta,2}(\varphi(x,\bar a)) \geqslant k \iff \models \Theta_{\varphi,k}(\bar a)$ for a formula $\Theta_{\varphi,k}(\bar y)$

(2) $R_{\delta,2}(\varphi(x,\bar a)) = k \iff \models \Theta_{\varphi,k}(\bar a) \wedge \neg \Theta_{\varphi,k+1}(\bar a)$

(3) $R_{\delta,2}(\varphi) \geqslant \omega \iff R_{\delta,2}(\varphi) = \infty$

(4) $R_{\delta,2}(x=x) < \infty \iff \delta$ is stable

<u>Comments</u> (a) Usually $R_{\delta,2}(\varphi \vee \psi) \neq \max\{R_{\delta,2}(\varphi), R_{\delta,2}(\psi)\}$.

(b) So usually it is not true that $\forall p \, \forall A \, \exists_{p \not\parallel} q \in S(A) \; R_{\delta,2}(p) = R_{\delta,2}(q)$

Definability again

<u>Lemma (Shelah)</u> Let $\delta(x,y)$: stable, $p(x) \in S_\delta(A)$. Then

$$\exists \chi \in L(A) \, \forall \bar a \subseteq A \models \chi(\bar a) \iff \delta(x,\bar a) \in p(x)$$

a δ-definition (for χ)

<u>Proof</u> $R_{\delta,2}(p) = k < \omega$. Choose $\psi \in p$ with $R_{\delta,2}(\psi) = k$. Then for $\bar a \subseteq A$

$$\delta(x,\bar a) \in p \iff R_{\delta,2}(\psi \wedge \delta(x,\bar a)) \geqslant k \iff \underbrace{\Theta_{\psi\wedge\delta,k}(\bar a)}_{\chi(\bar a)}$$

$\blacksquare$

<u>Corollary</u> $[\delta$: stable, $a \in \mathcal{M}$, $A \subseteq]$ $\delta(a,A) = \{b \in A : \models \delta(a,b)\} = \{b \in A : \models \chi(b)\}$

for some $\chi(y) \in L(A)$

<u>Proof</u> $p = tp_\delta(a/A) \rightsquigarrow \chi$ from the lemma

Many-sorted language: L

- $\{S_i : i \in I\}$ a set of sorts
- each sort S_i has a specified set of variables $x_n^{S_i}$, $n<\omega$
- each predicate symbol $P(x_1, \ldots, x_n)$

  prescribed sorts

- each function symbol $f(x_1, \ldots, x_n)$

  value, sorts

- likewise constants

A structure for L: [many-sorted structure] $|M| = M = \dot{\bigcup}_{i\in I} \underbrace{M_{S_i}}_{\neq \emptyset}$; the S_i-sort of M. Predicates, functions and constants are interpreted respecting sorts

Formulas of L:

- $t(\bar{x}) = t(\bar{x})$ — terms of the same sorts
- $P(\vec{t}(x))$
- complex formulas as before (but quantify over some fixed sort)

KRL as before

Example: "Plain" $= (P, L, \in)$ — points, lines, incidence relation

M for L, $T = Th(M)$, $ER(T) = \{E(\bar{x},\bar{y}) \in L : M \models$ "E is an equivalence relation"$\}$

$\overset{\in}{E} \rightsquigarrow S_E$: new sort symbol.

$L \rightsquigarrow L^{eq}$: many sorted language.

L^{eq}:

- the standard sort $S_=$
- S_E, $E \in ER(T)$ (those are imaginary sort symbols)
- symbols of L refer to $S_=$ ~~(the standard sort)~~
- new function symbols of L^{eq}:
  - $\underset{E(\bar{x},\bar{y})}{\overset{\shortparallel}{E}} \in ER(T) \rightsquigarrow F_E(\bar{x})$ — F_E of sort S_E, $\bar{x}$ of $S_=$

$M \models T \rightsquigarrow \underset{\text{for } L^{eq}}{M^{eq}} = \dot{\bigcup}_{E \in ER(T)} S_E^{M^{eq}}$, where $S_E^{M^{eq}} = \{\bar{a}/E : \bar{a} \in M^n\}$

$S_=^{M^{eq}} = M$

the home sort of M^{eq}
standard

$M_E := S_E^{M^{eq}}$, elements of M_E: imaginary elements, imaginaries where $E \neq x=y$

$F_E^{M^{eq}}(\bar{a}) = \bar{a}/E$. The symbols of L are naturally interpreted in M.

~~$T^{eq} := Th(M^{eq})$~~

Fact (1) For $\varphi(\bar{x}) \in L$ and $\bar{a} \in M^n$ $M \models \varphi(\bar{a}) \iff M^{eq} \models \varphi(\bar{a})$

Def $T^{eq} := Cn\Big(T \cup \{$"$F_E : S_= \xrightarrow{\text{onto}} S_E$"$\}_{E \in ER(T)} \cup \{F_E(\bar{x}) = F_E(\bar{y}) \leftrightarrow E(\bar{x},\bar{y})\}_{E \in ER(T)}\Big)$

Fact (1) $M \models T \iff M^{eq} \models T^{eq}$

(2) $\forall M^* \models T^{eq}\ \exists M \models T\ \big(M^* \underset{\text{essentially}}{=} M^{eq}$ and $M^*_= = M\big)$

i.e. $\exists f : M^* \xrightarrow{\cong} M^{eq}$

$f{\restriction}_M = id_M$

(3) T^{eq}: complete

(4) If $E(1), \dots, E(k) \in ER(T)$, $\varphi(x_1, \dots, x_k) \in L^{eq}$, $x_i \in E(i)$

then there's $\psi(y_1, \dots, y_k) \in L$ s.t. $T^{eq} \vdash (\forall \underbrace{y_1, \dots, y_k}_{\text{of sort } S_=})\big(\psi(y_1, \dots, y_k) \leftrightarrow$

$$\varphi(F_{E(1)}(y_1), \dots, F_{E(k)}(y_k))\big)$$

Corollary (1) $\forall M \prec \mathcal{M}$ $\quad M^{eq} \prec \mathcal{M}^{eq}$

(2) $\mathcal{M}^{eq} = dcl^{eq}(\mathcal{M}_=)$, $M^{eq} = dcl^{eq}(M_=)$

(3) $A \subseteq \mathcal{M} = \mathcal{M}_=$ is definable [with parameters] in $\mathcal{M}^{eq}$

$\iff A$ is definable [with parameters] in $\mathcal{M}$

(4) $\mathcal{M}^{eq}$ is $\varkappa$-saturated and strongly $\varkappa$-homogeneous

Proof of fact (1), (2): obvious

(3) Let $M^*, N^* \models T^{eq} \overset{(2)}{\Longrightarrow} M^* = M^{eq}, N^* = N^{eq}$ for some $M, N \models T$

$\overset{T \text{ complete}}{\Longrightarrow} M \equiv N \overset{(4)}{\Longrightarrow} M^{eq} \equiv N^{eq}$

(4) Induction on φ:

- φ: atomic
- the connectives

] easy

- φ: $\exists \underset{\substack{\uparrow \\ \text{of sort } S_{E_0}}}{x_0} \varphi_0(x_0, x_1, \dots, x_k)$. By the ind. hyp. φ_0 is "equivalent" in T^{eq} to $\psi_0(\bar{y}_0, \dots, \bar{y}_k)$ Σ

So: $T^{eq} \vdash \exists x_0\, \varphi_0(x_0, F_{E(1)}(\bar{y}_1), \dots, F_{E(k)}(\bar{y}_k))$

$\qquad ||| \text{ (by axioms of } T^{eq})$

$$\exists \bar{y}_0\, \varphi_0(F_{E(0)}(\bar{y}_0), \dots, F_{E(k)}(\bar{y}_k))$$

$$|||$$

$$\exists \bar{y}_0\, \psi(\bar{y}_0, \dots, \bar{y}_k)$$

$\blacksquare$

Proof of corollary (1) Goal: $M^{eq} \prec \mathcal{M}^{eq}$. Take $\bar{a}^{eq} \subseteq M^{eq} \models \varphi(\bar{a}^{eq})$, "$\bar{a} = \bar{b}/E$"

by fact (4) $M^{eq} \models \varphi(\bar{a}^{eq}) \iff M^{eq} \models \psi(\bar{b}) \iff M \models \psi(\bar{b}) \iff \mathcal{M} \models \psi(\bar{b})$

$\iff \mathcal{M}^{eq} \models \psi(\bar{b}) \iff \mathcal{M}^{eq} \models \varphi(\bar{a})$

<u>Fact</u> Assume $E \in ER(T^{eq})$. Then there is $\psi \in ER(T)$ on $\mathcal{M}^n$ and

$\exists f: \mathcal{M}^n \xrightarrow{\text{onto}} \mathrm{dom}(E)$ s.t. $\forall \bar{a}, \bar{b} \in \mathcal{M}\ (\mathcal{M}^{eq} \models \psi(\bar{a}, \bar{b}) \leftrightarrow f(\bar{a})\, E\, f(\bar{b}))$

f: 0-definable in T^{eq}. So: $\exists g: \mathcal{M}_\psi \xrightarrow[\text{onto}]{1-1} \mathcal{M}^{eq}_E$ in $(\mathcal{M}^{eq})^{eq}$ g: 0-definable

<u>Def</u> Assume $X \subseteq \mathcal{M}^{eq}$ definable (with parameters) and $c \in \mathcal{M}^{eq}$. c is a <u>name (code)</u> of X iff

$$\forall f \in Aut(\mathcal{M}^{eq})\ (f[X] = X \Leftrightarrow f(c) = c)$$

<u>Remark</u> (1) $X \subseteq_{def} \mathcal{M}^{eq} \Rightarrow X$ has a name $c \in \mathcal{M}^{eq}$

(2) If c: name of X then $X = \varphi(\mathcal{M}^{eq}, c)$ for some $\varphi \in L^{eq}$

<u>Proof</u> (1) 1° $X \subseteq_{def} \mathcal{M}$, $X = \psi(\mathcal{M}, \bar{a})$, $\bar{a} \subseteq \mathcal{M}$. $E_\psi \in ER(T)$,

$$E_\psi(\bar{y}_0, \bar{y}_1) = \forall x\ (\psi(x, \bar{y}_0) \leftrightarrow \psi(x, \bar{y}_1))$$

So for $\bar{a}_0, \bar{a}_1 \in \mathcal{M}$ $\quad \models E_\psi(\bar{a}_0, \bar{a}_1) \Leftrightarrow \psi(\mathcal{M}, \bar{a}_0) = \psi(\mathcal{M}, \bar{a}_1)$

$$c = \bar{a}/E_\psi = F_{E_\psi}(\bar{a}) \in \mathcal{M}_{E_\psi}$$

good, name for X

2° $X \subseteq \mathcal{M}^{eq} \overset{1°}{\rightsquigarrow}$ a name of X in $(\mathcal{M}^{eq})^{eq}$ $c^\in$ $\Rightarrow$ get $c' \in \mathcal{M}^{eq}$ a name of X

$(\mathcal{M}^{eq})^{eq} = \mathcal{M}^{eq}$

(2) $X \subseteq_{def} \mathcal{M}^{eq}$, c: a name of X in $\mathcal{M}^{eq}$. By def. $\forall f \in Aut(\mathcal{M}^{eq})$ $(f[X] = X \Leftrightarrow f(c) = c)$

hence X is invariant under $Aut(\mathcal{M}^{eq}/c)$

$\Downarrow$

X definable over c

<u>Example</u> $X = \psi(\mathcal{M}, \bar{a})$, $c = \bar{a}/E_\psi$ a name of X

$$x \in X \Leftrightarrow \underbrace{\exists \bar{y}\ (\psi(x, \bar{y}) \wedge F_{E_\psi}(\bar{y}) = c)}_{\sigma(x, c) \text{ defines } X}$$

From now on we work in $\mathcal{M}^{eq}, T^{eq}$
$\overset{\shortparallel}{\mathcal{M}^*}$

<u>Definition</u> Assume $\underset{\text{definable}}{X \subseteq \mathcal{M}}$. X is <u>definable almost over</u> $A \subseteq \mathcal{M}$

$\iff$ the set $\{f[X] : f \in Aut(\mathcal{M}/A)\}$ is finite

<u>Lemma</u> X is definable almost over $A \iff X$ is definable over $acl^{eq}(A)$

<u>Proof</u> $(\Leftarrow)$ $X = \varphi(\mathcal{M}, \underset{\underset{acl^{eq}(A)}{\cap}}{\bar{a}})$ $\quad f[X] = \varphi(\mathcal{M}, \underbrace{f(\bar{a})}_{\text{finitely many possibillities}})$, $f \in Aut(\mathcal{M}/A)$

$(\Rightarrow)$ $X = \varphi(\mathcal{M}, \bar{b})$, $|\{f[X] : f \in Aut(\mathcal{M}/A)\}| \underset{k}{<} \aleph_0$. A type over A in $\bar{y}_1, \ldots, \bar{y}_{k+1}$

$\{\varphi(\mathcal{M}, \bar{y}_i) \neq \varphi(\mathcal{M}, \bar{y}_j)\}_{1 \leq i \neq j \leq k+1}$ ① $\cup \{``\bar{y}_i \models tp(\bar{b}/A)"\}_{1 \leq i \leq k+1}$.

This type has to be inconsistent. So for some $\Theta(\bar{x}) \in \underbrace{tp(\bar{b}/A)}_{p(\bar{x})}$

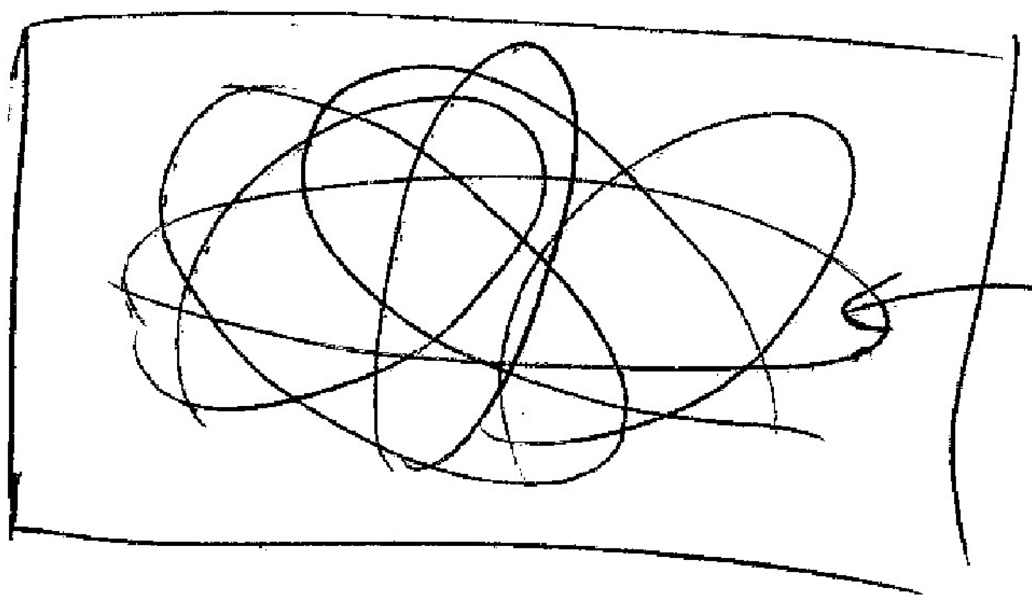
① $\cup \{\bigwedge_{1 \leq i \leq k+1} \Theta(\bar{y}_i)\}$ is inconsistent with T.

So: $|\underbrace{\{\varphi(\mathcal{M}, \bar{c}) : \bar{c} \in \Theta(\mathcal{M})\}}_{(*)}| \leq k$

$\underset{\underset{L(A)}{\cap}}{E}(x, x') : \forall \bar{c} \in \Theta(\mathcal{M}) \, (\varphi(x, \bar{c}) \leftrightarrow \varphi(x', \bar{c}))$.

E defines in $\mathcal{M}^{eq}$ an equiv. rel. with finitely many classes.

$E \in \underbrace{FE(A)}_{\text{formulas over } A \text{ defining in } \mathcal{M} \text{ an equiv. rel. with finitely many classes}}$



E-classes are atoms of the algebra of sets generated by $(*)$

X = union of some E-classes $\underset{\text{definable in } \mathcal{M}}{\mathcal{E}_1, \ldots, \mathcal{E}_n}$ $\Rightarrow$ have names $c_1, \ldots, c_n \in \mathcal{M}^{eq}$

$\Rightarrow c_1, \ldots, c_n \in acl^{eq}(A)$

So X is definable over $acl^{eq}(A)$

FORKING OF TYPES

<u>Example</u> $p \in S(A)$, $A \subset B$, $RM(p) < \infty$

$\cap$

$q \in S(B)$

1° $RM(p) = RM(q)$: "q is a free extension* of p"

2° $RM(q) < RM(p)$: "q is a $\underbrace{\text{degenerated}}_{\text{forking}}$ extension of p"

<u>Def</u> (1) $\{\varphi(x, a_i) : i < \omega\}$ is <u>k-contradictory</u> ($k < \omega$) if every k-element subset is inconsistent (with T)

(2) $\varphi(x, a)$ <u>divides over A</u> if

$$\exists k < \omega \ \exists \{\varphi(x, a_i) : i < \omega\} : k\text{-contradictory} \wedge \bigwedge_{i<\omega} tp(a_i/A) = tp(a/A)$$

(3) A type $p(x)$ (over $\mathcal{M}$) <u>divides over A</u> if $p \vdash \varphi(x, \bar{a})$ for some $\varphi(x, \bar{a})$ that divides over A

<u>Remark</u> In (2) can request $a_0 = a$ and $\langle a_i, i < \omega \rangle$ is order indiscernible

<u>Def</u> (1) $\varphi(x, \bar{a})$ <u>forks over A</u> if $\varphi(x, \bar{a}) \vdash \underbrace{\bigvee_{i \leq n} \varphi_i(x, \bar{b}_i)}_{\text{divides over } A}$ for some $\varphi_i(x, \bar{b}_i)$, $i \leq n$

(2) A type $p(x)$ <u>forks$/A$</u> if $p(x) \vdash \underbrace{\varphi(x, \bar{a})}_{\text{forks}/A}$ for some $\varphi(x, \bar{a})$