

Preliminaries

MT1/1

T : a complete consistent theory, in language L
with infinite models (countable)

that is, $T = \text{Th}(M)$, M : L -structure.
infinite

L denotes also the set of formulas of language L

$M = (|M|; \dots)$, but M also denotes $|M|$.
 $\emptyset \neq \uparrow$ universe of M (for brevity)

usually we omit $| \cdot |$ in $|M|$.

$M \supseteq A$: a set of parameters.

$L_n(A) = \{ \varphi(x_1, \dots, x_n, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in L, \bar{a} \subseteq A \}$

$L(A) = \bigcup_n L_n(A)$, also $L(A)$: language L
extended by names for elements of A .

$L_n(A)$: Lindenbaum algebra.

[formally: on $L_n(A)$: $\varphi \sim \psi \Leftrightarrow T(A) \vdash \varphi \leftrightarrow \psi$
 $\Leftrightarrow M \models \varphi \leftrightarrow \psi$

here: $T(A) = \text{Th}(M, a)_{a \in A}$

a complete theory
in language $L(A)$.



$L_m(A)/\sim$: a Boolean algebra
(Lindenbaum algebra)

$$[\varphi]_{\sim} \wedge [\psi]_{\sim} = [\varphi \wedge \psi]_{\sim} \text{ etc.}$$

shorthand: $L_m(A)$ denotes also $L_m(A)/\sim$.

$$S_m(A) = \{ \text{complete } n\text{-types over } A, \text{ in } \mathcal{M}_n \\ \text{in variables } x_1, \dots, x_n \}$$

consistent n -type over $A \iff$ proper filter in $L_m(A)$

An n -type $p(\bar{x})$ over A is complete if

$$L_m(A) \begin{cases} \cdot p(\bar{x}) : \text{consistent type} \\ \cdot \forall \varphi(\bar{x}) \in L_m(A) (\varphi(\bar{x}) \in p \text{ or } (\neg\varphi(\bar{x})) \in p) \end{cases}$$

$$S(A) := S_1(A)$$

(default)

$S_m(A)$: topological space :

for $\varphi(\bar{x}) \in L_m(A)$

$$[\varphi] = \{ p \in S_m(A) : \varphi \in p \}$$

basic open set [clopen]

closed and open

$S_m(A)$: compact Hausdorff space, 0-dimensional
(i.e. basis of clopen sets)

complete n -types / $A \leftrightarrow$ ultrafilters in $L_n(A)$ MT1/3

So $S_n(A) = S(L_n(A))$, the Stone space
of ultrafilters in $L_n(A)$

• the ~~type~~ topology
on $S_n(A)$ = the Stone space topology.

For $p(\bar{x}) \in S_n(A)$

$$p(M) = \{ \bar{a} \in M^n : \underbrace{\bar{a} \text{ satisfies } p}_{\text{realizes } p} \}$$

$\bar{a} \models p$, i.e. $M \models \varphi(\bar{a})$ for
every $\varphi(\bar{x}) \in p(\bar{x})$

• The same notation for
arbitrary type (also incomplete)

• A formula $\varphi(\bar{x}) \in L(M)$: a special case of a
type $\{ \varphi(\bar{x}) \}$.

$$\varphi(M) = \dots$$

• When $p \in S_n(A)$, $\bar{a} \subseteq M$ and $\bar{a} \models p$, then

$$p = \text{tp}^M(\bar{a}/A) = \{ \varphi(\bar{x}) \in L_n(A) : M \models \varphi(\bar{a}) \}.$$

Example Assume $p(\bar{x})$: a consistent type over M .

Then $\exists N \supseteq M$ p is realized in N

i.e. $p(N) \neq \emptyset$.



From now on "a type" means "a consistent type". MT1/4

Def A type $p(\bar{x})$ over A is isolated, if:

$$\exists \varphi(\bar{x}) \in L_n(A) \left\{ \begin{array}{l} \textcircled{1} \varphi(\bar{x}) \text{ is consistent (wrt } T), \text{ i.e.} \\ \varphi(M) \neq \emptyset \Leftrightarrow T(A) \vdash \exists \bar{x} \varphi(\bar{x}) \end{array} \right.$$

symbolically: $\varphi(\bar{x}) \vdash p(\bar{x}) \rightarrow$

$$\left\{ \begin{array}{l} \textcircled{2} \varphi(\bar{x}) \\ \forall \psi(\bar{x}) \in p(\bar{x}) \quad \varphi(M) \subseteq \psi(M) \\ \updownarrow \\ T(A) \vdash \varphi(\bar{x}) \rightarrow \psi(\bar{x}) \end{array} \right.$$

• When $p(\bar{x})$: a complete type over A , then:

$p(\bar{x})$ is isolated $\Leftrightarrow p$ is isolated in $S_n(A)$
in the topological sense
(i.e. $\{p\}$ is open)

Tarski - Vaught test

Assume $A \subseteq M$. Then $A = |N|$ for some $N \prec M$ iff

$$\forall \varphi(x) \in L_1(A) \quad [\varphi(M) \neq \emptyset \Rightarrow \varphi(M) \cap A \neq \emptyset]$$

Construction of an elementary submodel of M containing A :

• $A_n \subseteq M$, $n < \omega$, increasing chain of sets

recursive construction:

$$A_0 = A$$

$$A_n \subseteq A_{n+1} \subseteq M \text{ such that } \forall \psi(x) \in L_1(A_n)$$

$$[\psi(M) \neq \emptyset \Rightarrow \psi(M) \cap A_{n+1} \neq \emptyset]$$

$$A_\infty = \bigcup_{n < \omega} A_n \text{ satisfies TV-test.}$$

Omitting types theorem

MT1/5

Assume $p_n(\bar{x}_n)$, $n < \omega$: a family of non-isolated types in theory T , over \emptyset . Then:

$(\exists M \models T)$ M omits every p_n [i.e. $p_n(M) = \emptyset$]

Assume $M, N \models T$
 $\underset{A}{\cup}$

Def. $f: A \rightarrow N$ is elementary ($f: A \xrightarrow{\equiv} N$) if:

$$\forall \bar{a} \in A \forall \varphi(\bar{x}) \in L \quad (M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a})))$$

$$(\Leftrightarrow) \text{tp}^M(\bar{a}) = \text{tp}^N(f(\bar{a}))$$

Elementary diagram of $A \subseteq M$:

$$D_e(A) = T(A) = \text{Th}(M, a)_{a \in A}$$

Remark $f: A \rightarrow N$ is elementary $\Leftrightarrow (N, f(a))_{a \in A} \models T(A)$

Atomic diagram of $A \subseteq M$:

$$D_{\text{at}}(A) = \{ \varphi \in D_{\text{el}}(A) : \varphi \text{ is a quantifier free sentence} \}$$
$$= \{ \varphi(\bar{a}) \in L(A) : M \models \varphi(\bar{a}) \text{ and } \varphi(\bar{a}) : \text{q.f. sentence} \}$$

Remark $f: M \rightarrow N$ is a monomorphism (i.e.:

$$f: M \xrightarrow{\cong} f(M) \subseteq N$$

↑ substructure

$$\Leftrightarrow (N, f(a))_{a \in M} \models D_{\text{at}}(M).$$



Here always $f: M \rightarrow N$ denotes a monomorphism. MT 1/6

$M \subseteq N$: M is a submodel (substructure) of N

$M < N$: M is an elementary submodel of N , i.e.:

$$M \subseteq N \text{ and } \text{id}_M: M \xrightarrow{\equiv} N$$

Remark Assume $M < N$, $A \subseteq M$.

(1) Assume $p(\bar{x}) \subseteq L_n(A)$. Then

$p(\bar{x})$ is a consistent type in $M \Leftrightarrow p(\bar{x})$ is a consistent type in N

(2) Assume $A \subseteq B \subseteq M$

• If $p(\bar{x})$: a type over B , then $p \upharpoonright_A \stackrel{\text{def}}{=} p(\bar{x}) \cap L(A)$
a type over A

Let $r: S_n(B) \rightarrow S_n(A)$, $r(p) \stackrel{\text{def}}{=} p \upharpoonright_A$.

Then r : continuous and "onto".

(3) If $p(\bar{x})$: a type over A , then $\exists q(\bar{x}) \in S_n(A)$ $p(\bar{x}) \subseteq q(\bar{x})$

Saturation, universality, (strong) homogeneity.

Let $\kappa \in \mathbb{C}N$, $\kappa \neq \aleph_0$.

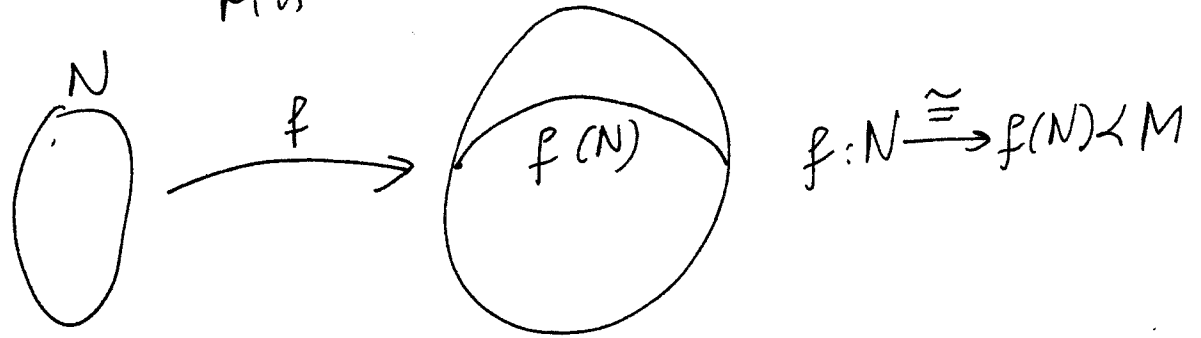
Def. (1) M κ -saturated if $\forall A \subseteq M \forall p \in S_n(A) p(M) \neq \emptyset$
(nasyromy) $|A| < \kappa$

M is saturated if M is $\|M\|$ -saturated

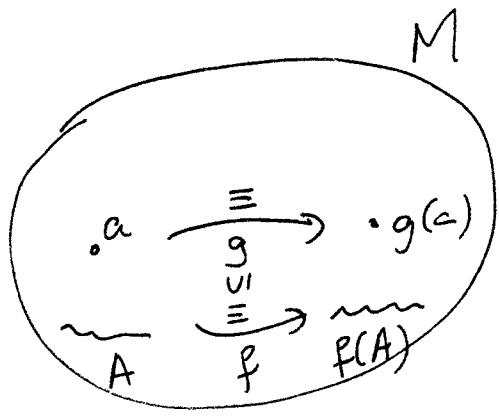
(2) M is κ -universal if $\forall N \equiv M (\|N\| \leq \kappa \Rightarrow \exists f: N \xrightarrow{\equiv} M)$
elementarily equivalent
i.e. $\text{Th}(N) = \text{Th}(M)$

M : universal \Leftrightarrow $\|M\|$ -universal

MT1/7



(3) M : κ -homogeneous if $\forall A \subseteq M \forall a \in M \forall f: A \xrightarrow{\cong} M$
 $|A| < \kappa \quad \exists g: A \cup \{a\} \xrightarrow{\cong} M$
 homogeneous = $\|M\|$ -homogeneous.



4. M strongly κ -homogeneous if $\forall A \subseteq M \forall f: A \xrightarrow{\cong} M$
 $|A| < \kappa \quad \exists g: M \xrightarrow{\cong} M$

strongly homogeneous = strongly $\|M\|$ -homogeneous.

5. M is κ -compact if $(\forall 1$ -type $p(x)$ over $M)$
 $(|p| < \kappa \Rightarrow p(M) \neq \emptyset)$

Elementary chains of structures

Def $\langle M_\alpha : \alpha < \mu \rangle, \mu \in \text{Ord}$, : an elementary chain of structures if $(\forall \alpha < \beta < \mu) M_\alpha \prec M_\beta$.

Union of chain (when $\mu \in \text{Lim}$)

$$M_\mu = \bigcup_{\alpha < \mu} M_\alpha ?$$

• $|M_\mu| := \bigcup_{\alpha < \mu} |M_\alpha|$

• $c \in L$ constant symbol

$c^{M_\mu} = c^{M_\alpha}$ for $\alpha < \mu$

• P : relation symbol

$P^{M_\mu}(a_1, \dots, a_n) \Leftrightarrow M_\alpha \models P(a_1, \dots, a_n)$ for $\alpha < \mu$
 $\bigwedge_{\alpha < \mu} |M_\alpha|$ sufficiently large
 [so that $\bar{a} \subseteq M_\alpha$]

• $f^{M_\mu}(\bar{a}) = b \Leftrightarrow M_\alpha \models f(\bar{a}) = b$ for $\alpha < \mu$
 sufficiently large

Fact (Tarski) $M_\alpha \prec M_\mu$ for all $\alpha < \mu$.

Proof (1) $M_\alpha \subseteq M_\mu$ (substructure): exercise

(2) $\forall \varphi(\bar{x}) \in L \forall \alpha < \mu \forall \bar{a} \subseteq M_\alpha (M_\alpha \models \varphi(\bar{a}) \Leftrightarrow M_\mu \models \varphi(\bar{a}))$

(a) φ atomic: $M_\alpha \subseteq M_\mu \checkmark$

(b) $\varphi = \psi_1 \wedge \psi_2, \varphi = \neg \psi$: easy

(c) $\varphi(\bar{x}) = \exists y \psi(\bar{x}, y)$

$M_\alpha \models \varphi(\bar{a}) \Rightarrow M_\alpha \models \psi(\bar{a}, b)$ for some $b \in M_\alpha$

\Downarrow ind. assumption for ψ

$M_\mu \models \psi(\bar{a}, b)$

\Downarrow
 $M_\mu \models \varphi(\bar{a})$

$$M_\mu \models \varphi(\bar{a}) \Rightarrow M_\mu \models \psi(\bar{a}, b) \text{ for some } b \in M_\mu$$

$$\exists y \psi(\bar{a}, y)$$

\Downarrow ind. assumption

$$\downarrow \\ b \in M_\beta \text{ for some } \alpha \leq \beta < \mu$$

$$M_\beta \models \psi(\bar{a}, b)$$

\Downarrow

$$M_\beta \models \varphi(\bar{a})$$

$$\Downarrow M_\alpha < M_\beta$$

$$M_\alpha \models \varphi(\bar{a})$$

Elementary directed systems of structures:

Let (I, \leq) : a directed set, i.e.:

(1) \leq : partial order on I

(2) $(\forall a, b \in I)(\exists c \in I)(a \leq c \wedge b \leq c)$

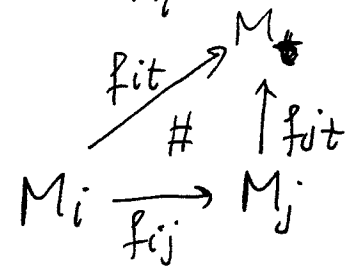
Example J : a set $\mapsto ([J]^{<\omega}, \leq)$: directed set.

Directed system of structures:

$$\mathcal{M} = (M_i, f_{ij})_{i \leq j \in I}$$

connecting functions $f_{ij}: M_i \rightarrow M_j$, $f_{ii} = id_{M_i}$. such that

$$(\forall i \leq j \leq t \in I) f_{it} = f_{jt} \circ f_{ij} \text{ (compatibility)}$$



System \mathcal{M} is elementary if all f_{ij} are elementary.

Example Elementary chain $(M_\alpha)_{\alpha < \mu}$

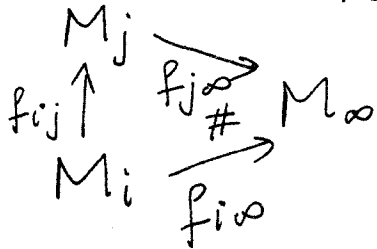
MT1/10

$\mathcal{M} = (M_\alpha, f_{\alpha\beta})_{\alpha \leq \beta < \mu}$ $f_{\alpha\beta} = id_{M_\alpha} : M_\alpha \xrightarrow{\cong} M_\beta$
elementary directed system of structures

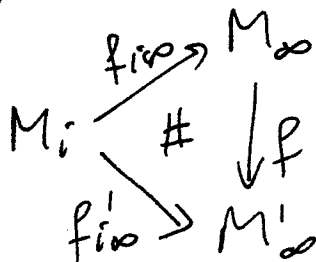
Direct limit of a directed system $\mathcal{M} : M_\infty = \varinjlim \mathcal{M}$

$(M_\infty, f_{i\infty})_{i \in I}$, where $f_{i\infty} : M_i \rightarrow M_\infty$ such that

(1) $\forall i \leq j \in I$ $f_{i\infty} = f_{j\infty} \circ f_{ij}$ [compatible with connecting functions]



(2) $(\forall (M'_\infty, f'_{i\infty})_{i \in I})$ satisfying (1) $\exists ! f : M_\infty \rightarrow M'_\infty$
(universality) $(\forall i \in I) f'_{i\infty} = f \circ f_{i\infty}$



Fact M_∞ exists (and is unique up to \cong).

If \mathcal{M} is elementary, then $f_{i,\infty} : M_i \xrightarrow{\cong} M_\infty$.

Proof 1. Construction of M_∞ :

$S := \dot{\bigcup}_{i \in I} |M_i|$: formally disjoint union.

\sim on S : an equivalence relation

$$M_i \quad M_j \quad \text{MTI/II}$$

$$\downarrow \quad \downarrow \quad \text{def}$$

$$x \sim y \Leftrightarrow f_{it}(x) = f_{jt}(y) \text{ for some } (= \text{every})$$

$$t \geq i, j$$

exercise: \sim is transitive.

$$|M_\infty| := S/\sim$$

- $\sim \upharpoonright |M_i|$: the equality (because f_{ij} : 1-1 (monomorphism))
- $f_{i\infty}(x) = x/\sim$, $f_{i\infty}: |M_i| \xrightarrow{1-1} |M_\infty|$.

L-structure on $|M_\infty|$:

- $c^{M_\infty} = c^{M_i}/\sim$
- $P^{M_\infty}(a_{i_1}/\sim, \dots, a_{i_m}/\sim) \Leftrightarrow M_t \models P(f_{i_1 t}(a_{i_1}), \dots, f_{i_m t}(a_{i_m}))$
 $a_{ij} \in M_{ij}$ for $t \geq i_1, \dots, i_m$
- f^{M_∞} : similarly

the rest is an exercise.

How to extend elementary mappings?

MT2/1

~~Def.~~ \mathbf{BA}_{lg} : Category of Boolean algebras

\mathbf{Comp}_0 : \mathcal{T} of compact Hausdorff 0-dimensional spaces

$$F: \mathbf{BA}_{lg} \rightarrow \mathbf{Comp}_0$$

$$G: \mathbf{Comp}_0 \rightarrow \mathbf{BA}_{lg}$$

$$F(A) = S(A)$$

$$G(X) = C(\text{open}(X))$$

F, G : contravariant functors "inverse" to each other

~~(F, G) is a duality of categories.~~ (look it up)
Categories \mathbf{BA}_{lg} and \mathbf{Comp}_0 are dually equivalent.

A, B : Boolean algebras

$$f: A \rightarrow B \text{ homomorphism} \Rightarrow F(f): S(B) \rightarrow S(A)$$

$$F(f)(p) = f^{-1}[p]$$

continuous.

$$\text{Assume } f: A \xrightarrow{\cong} B$$

$$\begin{matrix} \cap & & \cap \\ T \neq M & , & N \neq T \end{matrix}$$

$$\text{Then } \hat{f}: L_m(A) \rightarrow L_m(B)$$

$$\hat{f}(\varphi(\bar{x}, \bar{a})) = \varphi(\bar{x}, f(\bar{a}))$$

homomorphism

of Boolean algebras.

even: monomorphism.

We skip $\hat{\quad}$ in \hat{f} , so:

$$f: L_m(A) \rightarrow L_m(B) \text{ monomorphism}$$

$$f^*: S_m(B) \rightarrow S_m(A) \text{ epimorphism in } \mathbf{Comp}_0$$

i.e. continuous onto

Lemma (on extensions of elementary mappings) MT2/2

Assume $M, N \models T, A \subseteq M, B \subseteq N, f: A \xrightarrow{\equiv} B$ "onto".

Assume $\overset{\psi}{\underset{a}{\#}}, \overset{\psi}{\underset{b}{\#}}$, $p = \text{tp}(a/A), q = \text{tp}(b/B)$.

Then $f \cup \{Ka, b\}$ is elementary $\Leftrightarrow f^*(q) = p$.

[here $f^*: S(B) \xrightarrow{\cong} S(A)$
homeomorphism]

Proof exercise.

Def. M is $(< \kappa_0)$ -universal $\Leftrightarrow \forall n \forall p \in S_n(\emptyset) p(M) \neq \emptyset$.

Remark $M: \kappa$ -universal $\Rightarrow M: (< \kappa_0)$ -universal.

Proof Let $p \in S_n(\emptyset)$.

Choose a countable $N \models T$ with $p(N) \neq \emptyset$.

$M: \kappa$ -universal $\Rightarrow \exists f: N \xrightarrow{\equiv} M$
 $\overset{\psi}{\underset{a}{\#}} \neq p \mapsto \overset{\psi}{\underset{f(a)}{\#}} \neq p$.

Thm. (1) $M: \kappa$ -saturated $\Rightarrow M: \kappa$ -homogeneous
and κ -universal.

(2) $M: \kappa$ -~~universal~~ ^{homogeneous} and $(< \kappa_0)$ -universal \Rightarrow
 $M: \kappa$ -saturated.

Proof. (1) κ -homogeneity of M :

Assume $f: A \xrightarrow{\equiv} M, A \subseteq M, |A| < \kappa, a \in M$.

We seek $b \in M$ s.t. $g = f \cup \{ \langle a, b \rangle \}$ elementary

MT2/3

\Updownarrow Lemma

$$f^*(tp(b/B)) = tp(a/A).$$

⊗ Let $p = tp(a/A)$, $q = (f^*)^{-1}(p) \in S_1(B)$

\uparrow
 $S_1(A)$

Let $b \in M$ (exists by κ -saturation)
 \uparrow
good. of M

• κ -universality of M :

Assume $N \equiv M$, $\|N\| \leq \kappa$.

We seek $f: N \xrightarrow{\equiv} M$.

Let $\{a_\alpha : \alpha < \mu\}$: an enumeration of N , $\mu = \|N\|$.

We define $f(a_\alpha)$ by induction on $\alpha < \mu$:

• Suppose $f(a_\beta)$ defined for all $\beta < \alpha$ so that

$$f: \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} M$$

Want to find $f(a_\alpha)$ so that

$$f: \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} M.$$

~~By the Lemma it is enough that~~

Let $p = tp(a_\alpha / \{a_\beta : \beta < \alpha\})$.

By the lemma it is enough to find $f(a_\alpha) \in M$

so that $f^*(tp(f(a_\alpha) / \{f(a_\beta) : \beta < \alpha\})) = p$.

So let $q_f = (f^*)^{-1}(p) \in S_{\kappa}(\underbrace{\{f(a_\beta) : \beta < \alpha\}}_{\text{power} < \kappa})$

(MT2/4)

M κ -saturated $\Rightarrow q_f$ realized in M .

Let $f(a_\alpha) \in M$ s.t. $f(a_\alpha) \neq q_f$.

(2) Assume M is κ -homogeneous & $(< \aleph_0)$ -~~saturated~~ ^{universal}.

Want: M : κ -saturated.

So: Let $A \subseteq M$, $|A| < \kappa$, $p \in S_{\kappa}(A)$. Show: $p(M) \neq \emptyset$.

Induction on $|A|$.

Case (a): $|A| < \aleph_0$.

N

$\exists N \supseteq M$ $p(N) \neq \emptyset$. So let $b \in p$.

Let $A^* = A \cup \{b\}$
 $\quad \quad \quad \cup \{a_1, \dots, a_k\}$

Let $q_f = t_p^N(a_1, \dots, a_k, b) \in S_{\kappa+1}(\emptyset)$

q_f is realized in M ($(< \aleph_0)$ -universality),

by $\langle \underbrace{a'_1, \dots, a'_k}_{A'}, b' \rangle$

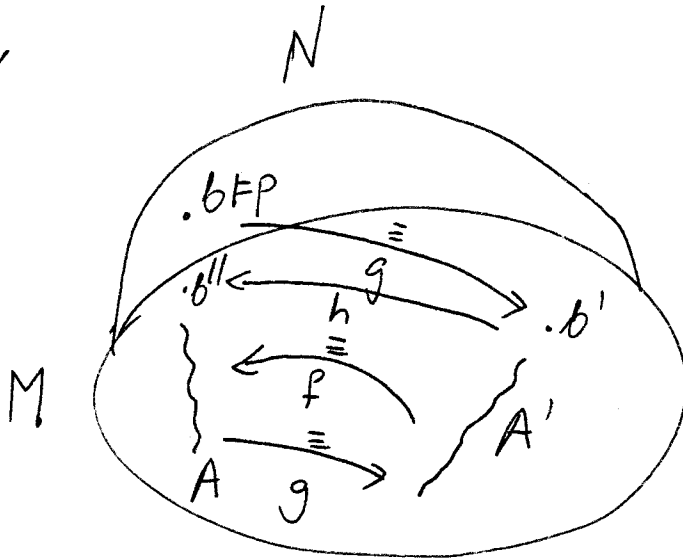
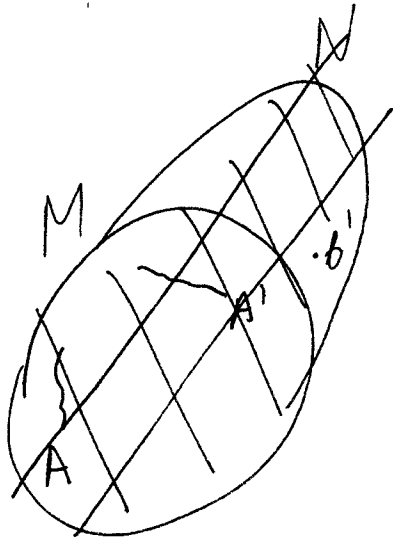
Let $g: A \cup \{b\} \rightarrow A' \cup \{b'\}$, $g(a_i) = a'_i$, $g(b) = b'$.

g : elementary.

$$\Rightarrow g \uparrow_A : A \xrightarrow{\cong} A'$$

$$\Downarrow f := (g \uparrow_A)^{-1} : A' \xrightarrow{\cong} A$$

M: κ -homogeneous $\Rightarrow \exists h : A' \cup \{b''\} \xrightarrow{\cong} A \cup \{b''\}$
for some $b'' \in M$.



$$\begin{array}{ccc} A \cup b & \xrightarrow{\cong} & A \cup b'' \\ g \downarrow \cong & & \cong \uparrow h \\ A' \cup b' & & \end{array}$$

Let $s = h \circ g$

$$s \uparrow_A = \underbrace{(h \uparrow_{A'})}_{\cong} \circ (g \uparrow_A) = \text{id}_A$$

$$s^*(\cancel{tp(b''/A)}) = \cancel{tp(b''/A)}$$

$$s \uparrow_A = \text{id}_A \Rightarrow s^* : S(A) \xrightarrow{\cong} S(A)$$

\cong
 $\text{id}_{S(A)}$

$$\text{hence: } p = tp(b/A) \underset{\uparrow \text{Lemma}}{=} s^*(tp(b''/A)) \underset{\uparrow}{=} tp(b''/A)$$

and $b'' \neq p$ $s^* = \text{id}_{S(A)}$

Case (b) $|A| = \mu$, $x_0 \leq \mu < \kappa$.

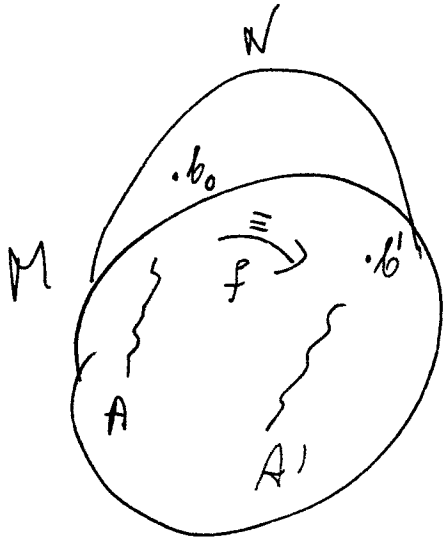
$$A = \{a_\alpha : \alpha < \mu\}, p \in S_1(A).$$

$$p \upharpoonright \emptyset \in S_1(\emptyset) \implies \exists b' \in M \quad b' \neq p \upharpoonright \emptyset,$$

$M: \langle x_0 \rangle$ -universal

$$\exists N \supset M \quad \exists b_0 \in N$$

$\begin{matrix} \pi \\ \downarrow \\ p \end{matrix}$



Will find $A' = \{a'_\alpha : \alpha < \mu\} \subseteq M$

s.t.

$$f: A b_0 \longrightarrow A' b'$$

given by $f(a_\alpha) = a'_\alpha$
 $f(b_0) = b'$

is elementary!

We find $a'_\alpha, \alpha < \mu$ by induction on $\alpha < \mu$.

So suppose $\alpha < \mu$ and a'_β already defined for all $\beta < \alpha$

so that $\boxed{p \upharpoonright \{a_\beta : \beta < \alpha\} b_0} = \{a_\beta : \beta < \alpha\} b_0 \equiv \{a'_\beta : \beta < \alpha\} b'$

$\equiv: f_0$

We look for a'_α .

Let $q = \text{tp}(a_\alpha / \{a_\beta : \beta < \alpha\} \cup \{b_0\})$

then $(f_0^*)^{-1}(q) \in S(\underbrace{\{a'_\beta : \beta < \alpha\} \cup \{b'\}}_{\text{power} < \mu \leq |A|})$

power $< \mu \leq |A|$

By the lemma it is enough that $a'_\alpha \neq (f_0^*)^{-1}(q)$.

But $M: \kappa$ -

M .

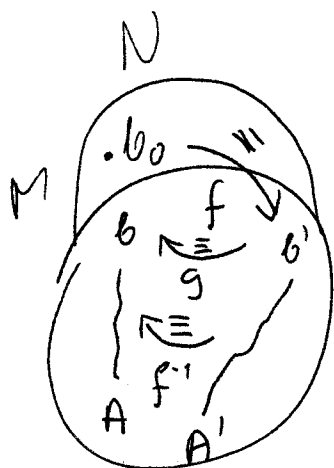
~~Let~~ By the inductive assumption on A :

MT2/7

$(f_0^*)^{-1}(q)$ is realized in M , so we are done with constructing A' .

Now: $f^{-1}: A' \xrightarrow{\cong} A$ in $M \leftarrow \kappa$ -homogeneous, so

$\exists \tilde{g}: A'b' \xrightarrow{\cong} Ab$ for some $b \in M$.



~~Let~~ Let $s = g \circ f$

$s: Ab_0 \xrightarrow{\cong} Ab$

$s \uparrow_A = (g \uparrow_{A'}) \circ (f \uparrow_A) = id_A$,

\parallel
 $(f \uparrow_A)^{-1}$ so $tp(b_0^{\#}/A) = tp(b/A)$
 $p =$

and p is realized in M .

Corollary

M is κ -saturated $\Leftrightarrow M$ is κ -homogeneous and κ -universal and κ -universal.

Proof \Rightarrow by Thm (1).

\Leftarrow κ -homogeneous + κ -universal \Rightarrow

κ -homogeneous + $(\langle \cdot \rangle_0)$ -universal \Rightarrow κ -saturated
 Thm (2)

Properties of saturated models.

Thm. Assume $M, N \models T$ saturated models of the same power. Then $M \cong N$.

Proof $M = \{m_\alpha : \alpha < \kappa\}$, $N = \{n_\alpha : \alpha < \kappa\}$,
 $\kappa = \|M\| = \|N\|$. We find $f: M \xrightarrow{\cong} N$

back-and-forth method:

$$f = \bigcup_{\alpha < \kappa} f_\alpha \quad f_\alpha: M \xrightarrow{\cong} N \quad \text{s.t.}$$

partial, elementary

$$(1) \quad \begin{array}{l} m_\alpha \in \text{Dom } f_{\alpha+1} \\ n_\alpha \in \text{Rng } f_{\alpha+1} \end{array} \quad , \quad |f_\alpha| \leq 2 \cdot |\alpha|$$

$$(2) \quad f_0 = \emptyset$$

$$(3) \quad \text{For } \delta \in \text{Lim}, \quad f_\delta = \bigcup_{\alpha < \delta} f_\alpha.$$

$$(4) \quad f_{\alpha+1} = f_\alpha \cup \left\{ \left\langle \underset{\substack{\uparrow \\ N}}{m_\alpha}, m \right\rangle, \left\langle m, \underset{\substack{\uparrow \\ M}}{n_\alpha} \right\rangle \right\}$$

Inductive step:

Suppose we have f_α . Want: $f_{\alpha+1}$.

Let $A_\alpha = \text{Dom } f_\alpha \subseteq M$, $B_\alpha = \text{Rng } f_\alpha \subseteq N$.

$$f_\alpha: A_\alpha \xrightarrow{\cong} B_\alpha$$

$$\downarrow$$

$$f_\alpha^{\leftarrow}: S(B_\alpha) \xrightarrow{\cong} S(A_\alpha).$$

"forth": Find $n \in N$ st. $f_\alpha \cup \{ \langle m_\alpha, n \rangle \}$ elementary MT2/9

$$\begin{array}{c} \Downarrow \\ (f_\alpha^*)^{-1}(tp(m_\alpha/A_\alpha)) = tp(n/B_\alpha). \end{array}$$

Let $p = tp(m_\alpha/A_\alpha)$.

So $(f_\alpha^*)^{-1}(p) \in S(B_\alpha)$ is realized in N by some n .

"~~back~~": similarly.
back

Thm Assume $M, N \models T$ are homogeneous, of the same power and $\forall n < \omega \forall p \in S_n(\emptyset) (p(M) \neq \emptyset \Leftrightarrow p(N) \neq \emptyset)$.
Then $M \cong N$.

Lemma Under the assumptions of the Thm,

$$\forall A \subseteq M \exists f: A \xrightarrow{\cong} N.$$

Proof. Induction on $|A|$.

Case (a) $|A| < \aleph_0$. $A = \{a_1, \dots, a_n\}$.

Let $p = tp(\langle a_1, \dots, a_n \rangle) \in S_n(\emptyset)$. realized in M
 \Downarrow
realized in N

by some $\langle b_1, \dots, b_n \rangle \in N$.
 $f(a_i) = b_i$ is good.

Case (b) $|A| = \mu \geq \aleph_0$, $A = \{a_\alpha : \alpha < \mu\}$

We find $f(a_\alpha)$ by induction on $\alpha < \mu$.

Inductive step.

Suppose $\alpha < \mu$ and for every $\beta < \alpha$ we have $f(a_\beta)$

$$\text{s.t. } f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} N.$$

We shall find $f(a_\alpha) \in N$ s.t. $f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} N$.

Let $a_{<\alpha} := \{a_\beta : \beta < \alpha\}$. Likewise $a_{\leq\alpha}$.

$$|a_{\leq\alpha}| < \mu = |A|$$

By inductive assumption: $\exists g : a_{\leq\alpha} \xrightarrow{\equiv} N$.

$$\text{Then } f \circ g^{-1} : \underbrace{g(a_{<\alpha})}_N \xrightarrow{\equiv} \underbrace{f(a_{<\alpha})}_N$$

By homogeneity of N : $\exists f(a_\alpha) \in N$ s.t.

$$f \circ g^{-1} : \underbrace{g(a_{<\alpha}) g(a_\alpha)}_{g(a_{\leq\alpha})} \xrightarrow{\equiv} f(a_{<\alpha}) f(a_\alpha) = f(a_{\leq\alpha})$$

$$\text{Then } f = (f \circ g^{-1}) \circ g : a_{\leq\alpha} \xrightarrow{\equiv} N.$$

Proof of the theorem

$$\kappa := \|M\| = \|N\|$$

$f : M \xrightarrow{\cong} N$ constructed by back-and-forth method

$$f = \bigcup_{\alpha < \kappa} f_\alpha, \quad f_\alpha : M \xrightarrow{\cong} N \text{ (partial elementary), } \alpha < \kappa$$

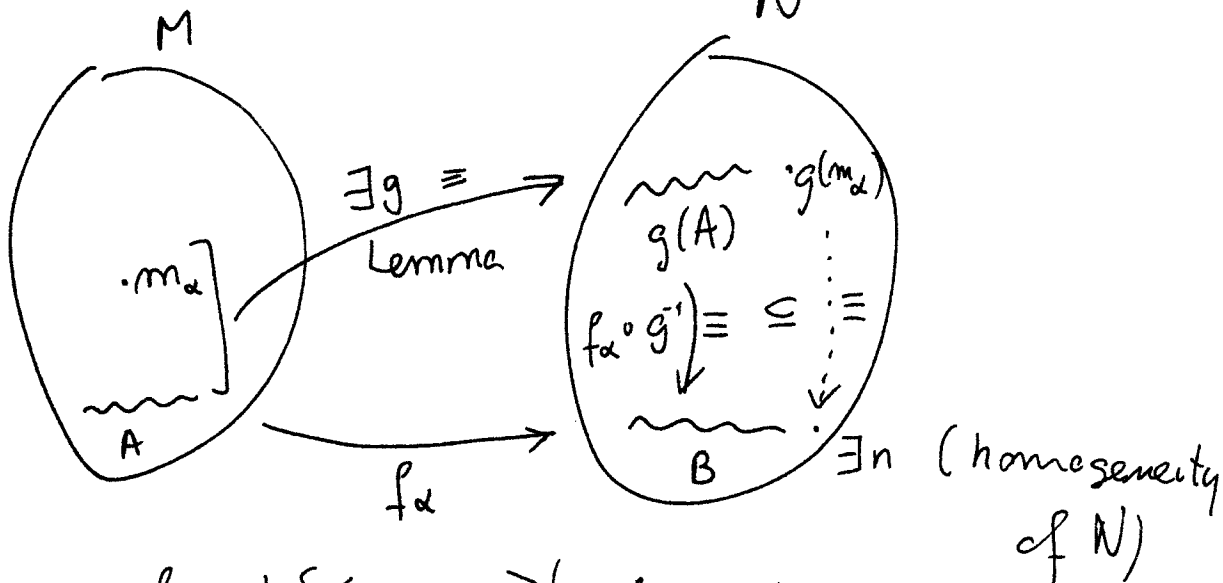
$|f_\alpha| \leq 2 \cdot |\alpha|$ + the same conditions as in the previous thm.

inductive step $f_\alpha \mapsto f_{\alpha+1}$

$A = \text{Dom } f_\alpha$

$B = \text{Rng } f_\alpha$

"forth"



~~$f_\alpha \cup \{ \langle m_\alpha, n \rangle \}$ elementary~~

$h = (f_\alpha \circ g^{-1}) \upharpoonright_{g(A)} \cup \{ \langle g(m_\alpha), n \rangle \}$ elementary

$h \circ g : A \cup \{ m_\alpha \} \xrightarrow{\equiv} B \cup \{ n \} \subseteq N$

\cup
 f_α

"back": similarly.

Constructions of models:

MT3/1

- Saturated \implies • (strongly) homogeneous $\bar{\equiv}$

Thm. $\underbrace{\kappa = 2^{<\kappa}, \kappa \in \text{Reg}}_{\kappa^{<\kappa} = \kappa}, \kappa > \aleph_0 \implies \exists \text{MFT}$
saturated, of power κ .

Proof

(*) $|S_i(A)| \leq 2^{|A| + \aleph_0}$, because: $|L_i(A)| = |A| + \aleph_0$

Here: $|A| < \kappa \implies |S_1(A)| \leq \kappa$.

Lemma NFT, $\|N\| \leq \kappa \implies X_N := \bigcup \{S_i(A) : A \subseteq N \text{ \& } |A| < \kappa\}$
the set has power $\leq \kappa$.

Pf • $|\{A \subseteq N : |A| < \kappa\}| \leq \kappa^{<\kappa} = \kappa$.

• $|S_i(A)| \leq \kappa$ for such A .

Proof of the thm.

$M_\alpha, \alpha < \kappa$: elementary chain of models of T of power κ .

• M_0 : whatever

• $M_\delta = \bigcup_{\alpha < \delta} M_\alpha$, when $\delta < \kappa$ limit.

• $M_{\alpha+1} \supset M_\alpha$ such that $\forall p \in X_{M_\alpha} p(M_{\alpha+1}) \neq \emptyset$:

$$T' = \text{Th}(M_\alpha, m)_{m \in M_\alpha} \cup \bigcup_{\beta < \kappa} \{ \varphi(c_\beta) : \varphi(x) \in p_\beta \},$$

where $X_{M_\alpha} = \{ p_\beta : \beta < \kappa \}$

\uparrow
new constant symbols,

and T' in language $L(M_\alpha) \cup \{ c_\beta : \beta < \kappa \}$.

T^1 : consistent, has model of power $\kappa: M_{\alpha+1}$
such that $M_\alpha < M_{\alpha+1}$.

$M = \bigcup_{\alpha < \kappa} M_\alpha$: of power κ , saturated:

Let $A \subseteq M$, $|A| < \kappa$ and $p \in S_1^M(A)$
 $\kappa \in \text{Reg} \Rightarrow A \subseteq M_\alpha$ for some $\alpha < \kappa$. CW
↓

proof: $A = \{a_\beta : \beta < \mu\}$ for some $\mu < \kappa$.

$\forall \beta < \mu \exists \alpha_\beta < \kappa a_\beta \in M_{\alpha_\beta}$

$\{\alpha_\beta : \beta < \mu\} \subseteq \kappa$, $\mu < \text{cf}(\kappa) = \kappa$

$\Rightarrow \exists \alpha < \kappa \forall \beta < \mu \alpha_\beta < \alpha$
↑
 $A \subseteq M_\alpha$.

~~Let~~ $M_\alpha < M \Rightarrow p \in S_1^{M_\alpha}(A) = S_1^M(A)$

p realized in $M_{\alpha+1}$ by some $a \in M_{\alpha+1}$

$a \models p$ in $M_{\alpha+1} \Rightarrow a \models p$ in M .

$M_{\alpha+1} < M$

Monster model:

Let $\bar{\kappa}$: a large cardinal number.

"Ideal model" $M \models T$: saturated of power $\bar{\kappa}$

because: $\forall M \models T (|M| < \bar{\kappa} \Rightarrow \exists M' < M \ M \cong M')$.

Advantages of saturated model M :

MT 3/3

(i) universality

(ii) strong homogeneity

~~More~~ Weakly (a bit):

(1) $\bar{\kappa}$ -universality

(2) strong $\bar{\kappa}$ -homogeneity

$\text{Aut}(M)$: the group of automorphisms of M

$\text{Aut}(M/A) = \{ f \in \text{Aut}(M) : f|_A = \text{id}_A \}$: automorphisms of M over A
 $A \subseteq M$

Lemma Assume M is strongly κ -homogeneous, κ -saturated, $A \subseteq M$, $|A| < \kappa$. Then:

(1) For $a, b \in M$ ($\text{tp}(a/A) \stackrel{!}{=} \text{tp}(b/A) \Leftrightarrow a, b$ are in the same orbit of $\text{Aut}(M/A)$ on M).

(2) [orbits $\text{Aut}(M/A)$ on M^n] $\overset{1:1}{\underset{\text{onto}}{\longleftrightarrow}} S_n(A)$

Proof (1) \Leftarrow : $f \in \text{Aut}(M/A)$, $f(a) = b$

$$\Downarrow \text{tp}(a/A) = \text{tp}(b/A)$$

\Rightarrow : $\text{tp}(a/A) = \text{tp}(b/A) \Rightarrow f: Aa \xrightarrow{\equiv} Ab$

strong κ -homogeneity $f|_A = \text{id}_A, f(a) = b$

$|A| < \kappa \Rightarrow f \in g \in \text{Aut}(M), g \in \text{Aut}(M/A)$
 $g(a) = b$: a, b in the same orbit of $\text{Aut}(M/A)$

$$(2) M^n \supseteq \mathcal{O} \xrightarrow[\varphi]{(1)} p_{\mathcal{O}} \in S_n(A)$$

\uparrow
 orbit of
 $\text{Aut}(M/A)$

\parallel
 common
 type $tp(a/A)$
 for $a \in \mathcal{O}$.

$$\mathcal{P} : \{ \text{orbits of } \text{Aut}(M/A) \text{ on } M^n \}$$

$\downarrow \varphi$

$$S_n(A)$$

$$\mathcal{O}_1 \neq \mathcal{O}_2 \xrightarrow{(1)} p_{\mathcal{O}_1} \neq p_{\mathcal{O}_2} \quad \boxed{\text{so } \varphi: 1-1}$$

[if $p_{\mathcal{O}_1} = p_{\mathcal{O}_2}$ then let $a \in \mathcal{O}_1, b \in \mathcal{O}_2 \Rightarrow \exists g \in \text{Aut}(M/A)$

$M: \kappa$ -saturated $\Rightarrow \varphi$: "onto".

$$g(a) = b \quad \checkmark.]$$

Def Let $\bar{\kappa}$: a (large) cardinal number,

$M \models T$ monster model, if $M: \bar{\kappa}$ -saturated,
 (w.r. to $\bar{\kappa}$) strongly $\bar{\kappa}$ -homogeneous

Thm. Assume $\aleph_0 \leq \kappa \in \mathcal{C}N$. Then

$\exists M: \kappa$ -saturated $\not\equiv$ strongly $\bar{\kappa}$ -saturated.

Proof $M = \bigcup_{\alpha < \kappa^+} M_\alpha$: union of elementary chain
 s.t.:

(1) $M_0 \models T$ any

(2) $M_\delta = \bigcup_{\alpha < \delta} M_\alpha$ if $\delta \in \text{Lim}$,

(3) $M_{\alpha+1} \supset M_\alpha$ s.t.:

(a) $\forall p \in S_1(M_\alpha)$ p realized in $M_{\alpha+1}$

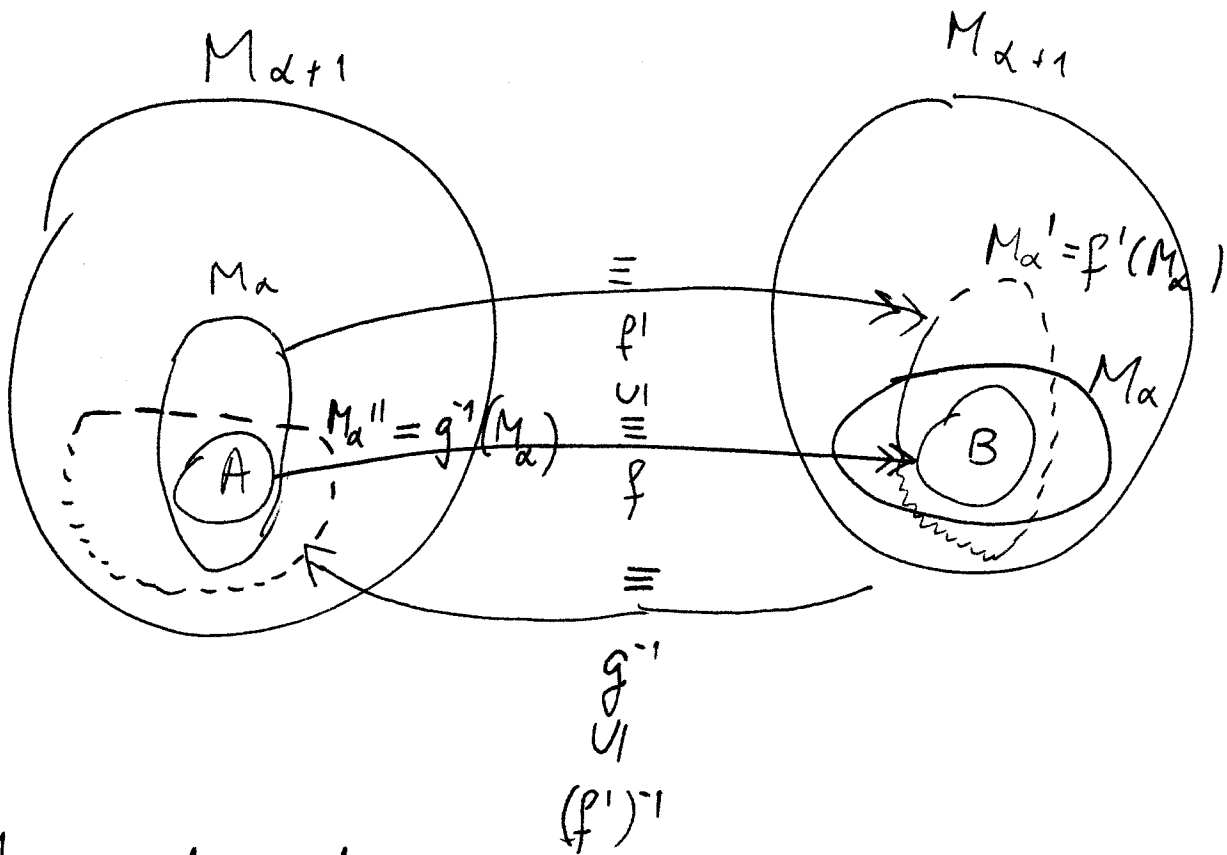
$$(b) \left(\forall f: A \xrightarrow{\equiv} B \right) \left(\exists g \geq f \right) \left(g: A' \xrightarrow{\equiv} B' \text{ in } M_{\alpha+1} \right)$$

$\begin{matrix} \cap & & \cap & & \cup & & \cup \\ M_\alpha & & M_\alpha & & M_\alpha & & M_\alpha \end{matrix}$

MT3/5.

It is enough ~~to~~ that $M_{\alpha+1}$ is $\|M_\alpha\|^+$ -saturated,
 To satisfy (a), (b). $M_{\alpha+1} \succ M_\alpha$

Proof of (b) for such $M_{\alpha+1}$:



I. M : κ -saturated: clear

II. M strongly κ -homogeneous:

Assume $A \subset M$, $|A| < \kappa$. Then $A \subseteq M_\alpha$, $B \subseteq M_\alpha$
 $f: A \xrightarrow{\equiv} M$ for some $\alpha < \kappa$.
 $B = f[A]$

• we construct

a sequence f_β , $\alpha \leq \beta < \kappa^+$:-

• increasing $f_\beta : M \xrightarrow{\equiv} M$

• ~~f_α~~ $f \subseteq f_\alpha$ partial elementary

(*) $M_\beta \subseteq \text{dom } f_\beta \cap \text{rng } f_\beta$.

• ~~f_α~~ f_α constructed according to 3)(b)

$$f_\alpha : M_{\alpha+1} \xrightarrow{\equiv} M_{\alpha+1}$$

• $f_\beta : M_{\beta+1} \xrightarrow{\equiv} M_{\beta+1}$, as in 3.(b)

when β : successor.

• $f_\delta = \bigcup_{\beta < \delta} f_\beta$ when δ limit, still $f_\delta : M_\delta \xrightarrow{\equiv} M_\delta$.

$$f_\infty = \bigcup_{\alpha \leq \beta < \kappa^+} f_\beta, \quad f_\beta \in \text{Aut}(M), \quad f \subseteq f_\beta.$$

Assumptions let $\bar{\kappa}$: a cardinal number large enough
so that:

(1) We consider only small models of T

||
of power $< \bar{\kappa}$, or even $\ll \bar{\kappa}$

(2) We work within a monster model $\mathcal{M} \models T$ (w.r. to $\bar{\kappa}$)

(3) We consider only small models $M \prec \mathcal{M}$

||
of power $< \bar{\kappa}$, or even $\ll \bar{\kappa}$,

Consequences:

(1) For $M, N < \mathcal{M}$, $M \subseteq N \Leftrightarrow M < N$

(2) Convention: For $\bar{a} \in \mathcal{M}$
 $\vDash \varphi(\bar{a})$ means $\mathcal{M} \vDash \varphi(\bar{a})$

(3) For $A \subseteq M < \mathcal{M}$:

$$S_m^M(A) = S_m^{\mathcal{M}}(A) =: S_m(A)$$

Notation Assume $p(\bar{x}), q(\bar{x})$ types (small, over \mathcal{M})

• $p(\bar{x}) \vdash q(\bar{x}) \Leftrightarrow p(\mathcal{M}) \subseteq q(\mathcal{M})$
 "p implies q"

• $p(\bar{x}) \equiv q(\bar{x}) \Leftrightarrow p \vdash q \ \& \ q \vdash p$
 ↑
 equivalent

Special case: $p(\bar{x}) = \{ \varphi(\bar{x}) \}$.

$\varphi(\bar{x}) \vdash q(\bar{x})$: "φ isolates q".

Remark: Syntactically:

$$p(\bar{x}) \vdash q(\bar{x}) \Leftrightarrow \forall \varphi(\bar{x}) \in q \ \exists p_0(\bar{x}) \subseteq p(\bar{x}) \text{ finite} \\
 \begin{matrix} \uparrow & \uparrow \\ \text{types over } A & \end{matrix} \quad T(A) \vdash \bigwedge p_0(x) \rightarrow \varphi(x)$$

Remark (exercise)

$$p(\bar{x}) \vdash q(\bar{x}) \Leftrightarrow \forall M \models T \text{ IA-saturated } p(M) \subseteq q(M).$$

Def. (reminder)

Let $p(\bar{x})$: a type over A .

p is isolated over $A \iff \exists \varphi(\bar{x}) \in L(A)$ $\varphi \vdash p$.
consistent (with T)

Thm (omitting types, Ehrenfeucht)

Assume $p_n(\bar{x}_n), n < \omega$: ~~non~~ a family of non-isolated types over \emptyset . Then $\exists M \models T \forall n \underbrace{p_n(M) = \emptyset}$,
 M omits p_n .

Lemma

Assume A is stable, $p_n(\bar{x}_n), n < \omega$: a family of non-isolated types over A , $\varphi(\bar{x}) \in L_1(A)$, $\underbrace{\varphi(M) \neq \emptyset}$.

Then $\exists c \in \varphi(M) \forall n$ p_n non-isolated over $A \cup \{c\}$.
i.e. φ : consistent

Proof [Lemma \Rightarrow Thm]

By the lemma: $\exists \underbrace{\{a_n : n < \omega\}}_A \subseteq M$ s.t.

(1) A satisfies the TV-test A

(2) $p_n, n < \omega$: non-isolated over A .

Construction of $a_n, n < \omega$: recursion on n .

Let $\{ \varphi_n(x, \bar{y}) : n < \omega \}$: all formulas of L of this form.

Suppose $n < \omega$ and $\{a_i : i < n\} = a_{<n}$ already
 so that all $p_k, k < \omega$ ^{constructed}
 still non-isolated over $a_{<n}$.

MT3/9

We consider a consistent formula $\varphi_n(x) \in L_1(a_{<n})$

By the Lemma we find $c \in \varphi_n(\mathcal{M})$ so that

all $p_k, k < \omega$, still non-isolated over $a_{<n}$.

• the formulas $\varphi_n(x), n < \omega$ may be chosen so that
 after ω steps:

$$\forall \varphi(x) \in L(A) \exists n \varphi = \varphi_n.$$

consistent

Then $A = \{a_n : n < \omega\}$ satisfies TV-test

$A = M \prec \mathcal{M}$, every p_k still non-isolated
 over A .

$$p_k(M) = \emptyset \text{ [if not,}$$

some $\bar{m} \models p_k$. then $\bar{m} \in \bar{a}_0$

$$(\bar{x}_k = \bar{m}) \vdash p_k(\bar{x}_k) \downarrow)$$

Proof of the Lemma.

Let $p(\bar{x})$: one of the types $p_n(\bar{x}_n)$.

Let $h(\bar{x}, y, \bar{a}) \in L(A)$

$$h(\bar{x}, c, \bar{a}) \vdash p(\bar{x}) \Leftrightarrow h(\mathcal{M}, c, \bar{a}) \subseteq p(\mathcal{M}).$$

$$\Leftrightarrow \forall \psi(\bar{x}) \in p(\bar{x}) \quad h(\mathcal{M}, c, \bar{a}) \subseteq \psi(\mathcal{M})$$

$$\Leftrightarrow \forall \psi \in p \quad \mathcal{M} \models \forall \bar{x} (h(\bar{x}, c, \bar{a}) \rightarrow \psi(\bar{x}))$$

$$\Leftrightarrow \forall \psi \in p \quad \psi_h(y) \in t_p(c/A)(y)$$

$$\text{where } \psi_h(y) = \forall \bar{x} (h(\bar{x}, y, \bar{a}) \rightarrow \psi(\bar{x}))$$

hence:

$$t_p(c/A) = t_p(c'/A) \Rightarrow [h(\bar{x}, c, \bar{a}) \vdash p \Leftrightarrow h(\bar{x}, c', \bar{a}) \vdash p]$$

$$h(\bar{x}, c, \bar{a}) \text{ consistent} \Leftrightarrow (\exists \bar{x} h(\bar{x}, y, \bar{a})) \in t_p(c/A)(y).$$

$$\text{Let } X_{h,p} = \{q \in S_1(A) : \text{For } c \models q, h(\bar{x}, c, \bar{a}) \vdash p(\bar{x}) \text{ and } h(\bar{x}, c, \bar{a}) \text{ consistent.}\}$$

"bad types"

$$\text{Let } q \in S_1(A) \text{ then} \quad \text{For } c \models q, h(\bar{x}, c, \bar{a}) \text{ consistent}$$

$$q \in X_{h,p} \Leftrightarrow q(y) \in S_1(A) \cap [\exists \bar{x} h(\bar{x}, y, \bar{a})] \cap \bigcap_{\psi \in p} [\psi_h(y)]$$

$$\text{For } c \models q, h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$$

(*) $X_{h,p}$: nowhere dense in $S_1(A)$.

Proof of (*): (a.a.)

Suppose $\theta(y) \in L_1(A)$ and $\emptyset \neq S_1(A) \cap [\theta] \subseteq X_{h,p}$.

$$\text{Let } \alpha(\bar{x}) = \exists y (h(\bar{x}, y, \bar{a}) \wedge \theta(y))$$

$$\uparrow \\ L(A)$$

• $\alpha(\bar{x})$: consistent :

MT3 / M

Let $c \in \Theta(\mathcal{M})$

$$\Downarrow [\theta] \subseteq X_{n,p}$$

$$\mathcal{M} \models \exists \bar{x} h(\bar{x}, c, \bar{a}).$$

Let $\bar{d} \in \mathcal{M}$ s.t. $\mathcal{M} \models h(\bar{d}, c, \bar{a}).$

$$\bar{d} \text{ satisfies in } \mathcal{M} : \underbrace{\exists y h(\bar{x}, y, \bar{a})}_{\alpha(\bar{x})}$$

• $\alpha(\bar{x}) \vdash p(\bar{x})$, i.e. $\alpha(\mathcal{M}) \subseteq p(\mathcal{M}).$

Let $\bar{d} \in \alpha(\mathcal{M})$. So there is $c \in \mathcal{M}$ s.t.

$$\models h(\bar{d}, c, \bar{a}) \wedge \theta(c)$$

$$\Downarrow [\theta] \subseteq X_{n,d}$$

$$\forall \psi \in p \models \psi_n(c)$$

$$\forall \psi \in p \quad \downarrow \quad \models h(\bar{x}, c, \bar{a}) \vdash \psi(\bar{x}) \Rightarrow \underbrace{\models h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})}_{\neq \bar{d}} \Rightarrow \frac{\perp}{\bar{d}} \quad \textcircled{y}$$

as: $p(\bar{x})$ non-isolated

$$\text{Let } X = \bigcup_{h,p_n} X_{h,p_n} \subseteq S_1(A).$$

meager. Let $q_i \in S_1(A) \cap [\varphi] \setminus X$

$c \models q_i$ good.

\textcircled{pf} (a.e) Suppose $p = p_n$ isolated over $A \cup \Sigma c \mathcal{L}$.

$$\exists h(\bar{x}, c, \bar{a}) \quad \underbrace{h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})}_{\text{consistent}} \Rightarrow \underbrace{q_i \in X_{h,p_n}}_{\substack{\neq \\ \vdash \bar{c}/A}} \quad \downarrow$$

14.03.2022

Def T is quantifier eliminable if $\forall \varphi \in L \exists \psi \in L$

$$T \vdash \varphi \leftrightarrow \psi$$

ψ open
 \equiv q.f.

Def. For $p(\bar{x}) \in S_n(\emptyset)$ let $p_o(\bar{x}) = \exists \varphi(\bar{x}) \in p(\bar{x})$.

Remark T is q.e. $\Leftrightarrow \forall n \forall p \in S_n(\emptyset) p_o \vdash p$ φ open.

Proof " \Rightarrow " Obvious. " \Leftarrow " Let $\varphi(\bar{x}) \in L$.

$$\bullet \forall p \in [\varphi] \cap S_n(\emptyset) \exists \psi \in p \quad p \in [\psi] \subseteq [\varphi]$$

Why? $p_o \vdash p$
 $p_o \vdash \varphi$

by compactness

$$\exists \text{ finite } p'_o \subseteq p_o \text{ s.t. } p'_o \vdash \varphi, \text{ i.e.}$$

$$p'_o(\mathcal{M}) \subseteq \varphi(\mathcal{M})$$

\Downarrow

$$\varphi(\mathcal{M}) = \left(\bigwedge_{\psi' \in p'_o} \psi' \right) (\mathcal{M}) \subseteq \varphi(\mathcal{M}) \rightsquigarrow \mathcal{M} \models \psi(\bar{x}) \rightarrow \varphi(\bar{x})$$

$$\varphi \vdash \varphi \Rightarrow [\varphi] \cap S_n(\emptyset) \subseteq [\varphi] \cap S_n(\emptyset)$$



Application $L = \{+, \cdot, 0, 1\}$: the language of rings.

ACF_p : the theory of algebraically closed fields of char p , in L .

Axioms:

1) field axioms

2) char $= p \neq 0$: $\underbrace{1 + \dots + 1}_p = 0$

2') $p = 0$: $\underbrace{1 + \dots + 1}_n = 0$ for $n \geq 1$

3) Every polynomial of deg n has a root:
 $0 < n$
 $\forall y_{n-1}, y_{n-2}, \dots, y_0 \exists x \quad x^n + y_{n-1}x^{n-1} + \dots + y_0 = 0.$

Fact ACF_p is complete.

Proof Let $M, N \models ACF_p$. Enough to show that $M \equiv N$. Let $\varepsilon > \|M\|, \|N\|$ and

let $M' \succ M, N' \succ N$.

power ε .

M', N' : uncountable acl fields of the same power and char

$$\begin{array}{c}
 \Downarrow \text{algebra} \\
 M' \cong N' \Rightarrow M' \equiv N' \\
 \quad \quad \quad \Downarrow \\
 \quad \quad \quad M \equiv N
 \end{array}$$

Fact ACF_p is q.e. (Chevalley, Tarski)

Proof (in \mathcal{M}) We will show that $\forall p \in S_n(\emptyset)$

$p_0 + p \Leftrightarrow p_0(\mathcal{M}) \subseteq p(\mathcal{M})$. Let $\bar{a} \models p_0$,

$\bar{b} \models p$, $\bar{a}, \bar{b} \in \mathcal{M}$. It's enough to prove

that $\exists f \in \text{Aut}(\mathcal{M}) f(\bar{a}) = \bar{b}$.

$$\begin{array}{l}
 \bar{a} = (a_1, \dots, a_n) \\
 \bar{b} = (b_1, \dots, b_n)
 \end{array}$$

Let $\langle \bar{a} \rangle, \langle \bar{b} \rangle$: the subfields with \mathbb{A}

of \mathcal{M} generated by \bar{a}, \bar{b} . $\bar{a}, \bar{b} \models p_0 \Rightarrow \langle \bar{a} \rangle \cong \langle \bar{b} \rangle$

$$\langle \bar{a} \rangle \cong \langle \bar{b} \rangle$$

\Downarrow unique \Downarrow

Some algebraic magic.

$$\mathcal{M} \cong \langle \bar{a} \rangle_0 \cong \langle \bar{b} \rangle_0 \subseteq \mathcal{M}$$

(fraction field)

$$\mathcal{M} \cong \underbrace{\langle \bar{a} \rangle_0^{\text{alg}}}_{F_a} \cong \underbrace{\langle \bar{b} \rangle_0^{\text{alg}}}_{F_b} \subseteq \mathcal{M}$$

(not unique)

$$\text{trdeg}(\mathcal{M}/\mathbb{F}_a) = \|\mathcal{M}\| = \text{trdeg}(\mathcal{M}/\mathbb{F}_b)$$

$$\begin{array}{ccc} \mathcal{M} \cong \mathbb{F}_a(X_\alpha, \alpha < \lambda)^{\text{alg}} & \Downarrow & \\ f \cong \downarrow & \curvearrowright & \cong \\ \mathcal{M} \cong \mathbb{F}_b(X_\beta, \beta < \lambda)^{\text{alg}} & & \end{array}$$

$$f \in \text{Aut}(\mathcal{M}), f(\bar{a}) = \bar{b}.$$

■

Types in $T = \text{ACF}_p$

Let $\mathcal{M} \models T$: a monster model.

subfield K , ^{UI} We will describe $S_n(K)$.
(small)

Let $\bar{a} \subseteq \mathcal{M}$, $|\bar{a}| = n$.

$$K[\bar{x}] \triangleright I(\bar{a}/K) = \{ f \in K[\bar{x}] : f(\bar{a}) = 0 \}$$

Remark 1) $\text{tp}(\bar{a}/K) = \text{tp}(\bar{a}'/K) \Leftrightarrow I(\bar{a}/K) = I(\bar{a}'/K)$

2) $\forall I \triangleleft K[\bar{x}]$ _{prime} $\exists \bar{a} \subseteq \mathcal{M} \ I(\bar{a}/K) = I$.

Proof 1) " \Rightarrow " $\text{tp}(\bar{a}/K) = \text{tp}(\bar{a}'/K) \Rightarrow \exists f \in \text{Aut}(\mathcal{M}/K)$
 $I(\bar{a}/K) = I(\bar{a}'/K) \stackrel{f}{=} f(\bar{a}) = \bar{a}'$

[alternatively: $f \in I(\bar{a}/K) \Leftrightarrow "f(\bar{x}) = 0" \in \text{tp}(\bar{a}/K)$]

" \Leftarrow " Assume $I(\bar{a}/K) = I(\bar{a}'/K) = I$.

$$K[\bar{a}] \cong_K K[\bar{x}] / I \cong K[\bar{a}']$$

$$\Downarrow \\ \exists f \in \text{Aut}(K[\bar{x}]/K) \quad f(\bar{a}) = \bar{a}'$$

$$\Downarrow \\ \text{tp}(\bar{a}/K) = \text{tp}(\bar{a}'/K).$$

2) $K \subseteq K[\bar{x}] / I = K[\bar{a}] \cong \bar{a} = \bar{x} / I$ and $I(\bar{a}/K) = I$.

•
•
•

□

$$S_1(K) = \{ \text{tp}(a/K) : a \in \mathcal{M} \} : a \text{ top-space.}$$

$$\Downarrow \\ p(x) = \text{tp}(a/K), \quad I_p = I(a/K) \triangleleft K[x]$$

a) $I_p \neq \{0\}$, i.e. a is algebraic / K , so

$0 \neq f \in I_p$ " $f(x) = 0$ " $\in p(x)$. In fact,

irreducible over K " $f(x) = 0$ " $\vdash p(x)$ (isolates)

Let $a' \in \mathcal{M}$ s.t. $f(a') = 0 \Rightarrow I(a'/K) \ni f \Rightarrow$
 $p = \text{tp}(a/K) = \text{tp}(a'/K) \Leftarrow I(a/K) = I(a'/K) \Leftarrow \begin{matrix} f \text{ generates} \\ I(a'/K) \end{matrix}$

$p(x)$ here is called algebraic.

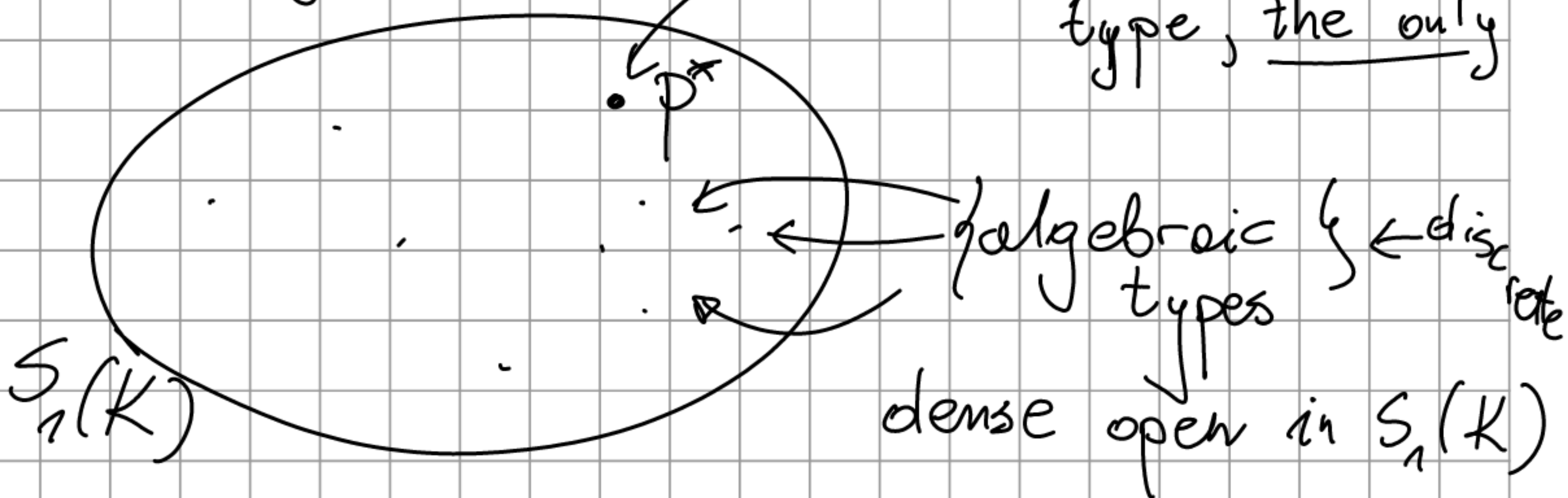
More generally (T : arbitrary)

Def $\varphi(\bar{x}) \in L(A)$ is called algebraic, if
 $0 < |\varphi(\mathcal{M})| < \aleph_0$, similarly for q : a type.

b) $I_p = \{0\}$: $p = \text{tp}(a/K)$ s.t. $a \in \mathcal{M}$ transcendental over K

↑
the transcendental type over K .

transcendental type, the only



If K cble then $S_n(K)$ cble

Corollary ACF_p is \aleph_0 -stable [recall: T is κ -stable $\Leftrightarrow \forall A \subseteq \mathcal{M}, |A| \leq \kappa$
 $|S_n(A)| \leq \kappa$]

Proof Let $A \subseteq \mathcal{M}$.
 A is abelian

$$A \subseteq K \subseteq \mathcal{M} \quad |S_n(A)| \leq |S_n(K)| = \aleph_0$$

\uparrow
 A is abelian subfield

Remark T is totally transcendental $\Leftrightarrow T: \aleph_0$ -stable

Proof " \Rightarrow " from def " \Leftarrow ": (A.a.) Let $\kappa > \aleph_0$.

Suppose $|A| \leq \kappa < |S_n(A)|$ for some $A \subseteq \mathcal{M}$.

Shall find $A_0 \subseteq A$ with $|S_n(A_0)| \geq 2^{\aleph_0}$.
 A_0 is abelian

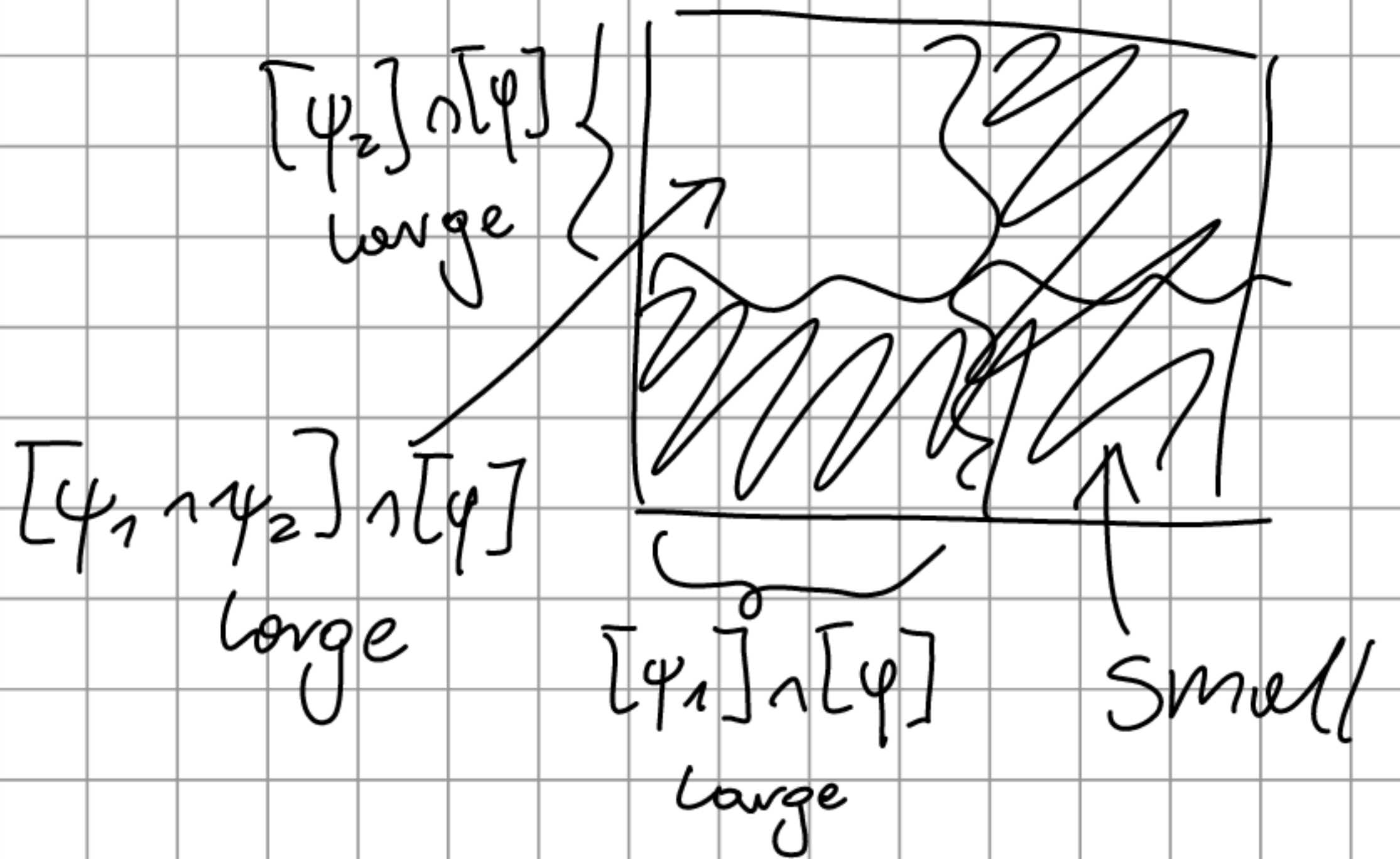
(def) $\varphi(x) \in L(A)$ is large iff $|S_n(A) \cap [\varphi]| > \kappa$
 otherwise: $\varphi(x)$ small.

(a) " $x=x$ " is large

(b) if $\varphi(x)$ is large, then $\exists \psi_1, \psi_2 \in L(A)$ large
 s.t. $\varphi(\mathcal{M}) = \psi_1(\mathcal{M}) \cup \psi_2(\mathcal{M})$

Pf. (b) if not then $\exists \psi \in L_n(A)$: $\psi \wedge \varphi$ is large
 is a complete type is $S_n(A) \cap [\psi]$

1° \mathcal{P}^* : consistent: $\psi_1, \psi_2 \in \mathcal{P}^* \Rightarrow \psi_1 \wedge \psi_2 \in \mathcal{P}^*$



2° \mathcal{P}^* : complete OK

$\mathcal{P}^* \in S_n(A) \cap [\phi]$: the only large type.

$$S_n(A) \cap [\phi] = \underbrace{\mathcal{P}^*}_{\text{large}} \cup \underbrace{\bigcup_{\substack{\psi \in L_n(A) \\ \psi \vdash \phi}}}_{\text{small}} (S_n(A) \cap [\phi])$$

$\leq \aleph$ $\leq \aleph$

$> \aleph$

$\leq \aleph$

↘

c) a tree of large formulas in $L_1(A) \varphi_\eta(x)$,
 $\eta \in 2^{<\omega}$ st. $\varphi_\eta(\mathcal{M}) = \varphi_{\eta_0}(\mathcal{M}) \cup \varphi_{\eta_1}(\mathcal{M})$
↑
by (b)

Let $A_0 \subseteq A$: the set of all params of $\varphi_\eta, \eta \in 2^{<\omega}$

Then $|A_0| \leq \aleph_0$.

For $\eta = 2^{<\omega}$: $\mathcal{P}_\eta^0 = \{ \varphi_{\eta|n}(x) : x < \omega \}$: a consistent
 \mathcal{L} -type over A_0

When $\nu \neq \eta$
then $\mathcal{P}_\eta \neq \mathcal{P}_\nu$

$\mathcal{P}_\eta \in S_\eta(A_0)$

$$\Downarrow \quad |S_\eta(A_0)| \geq 2^{\aleph_0} > \aleph_0 \quad \Downarrow$$

21.03.2022

CONSTRUCTION OF SPECIAL MODELS: N, M ⊨ T

Def. M is atomic if $\forall \bar{a} \in M$ $\text{tp}(\bar{a}/\emptyset) = \text{tp}(\bar{a})$ is isolated.

(2) M is prime if $\forall N \models T \exists f: M \xrightarrow{\cong} N$

Example $T = \text{ACF}_p$, F_p : prime field of char p

a) F_p : atomic (exercise)

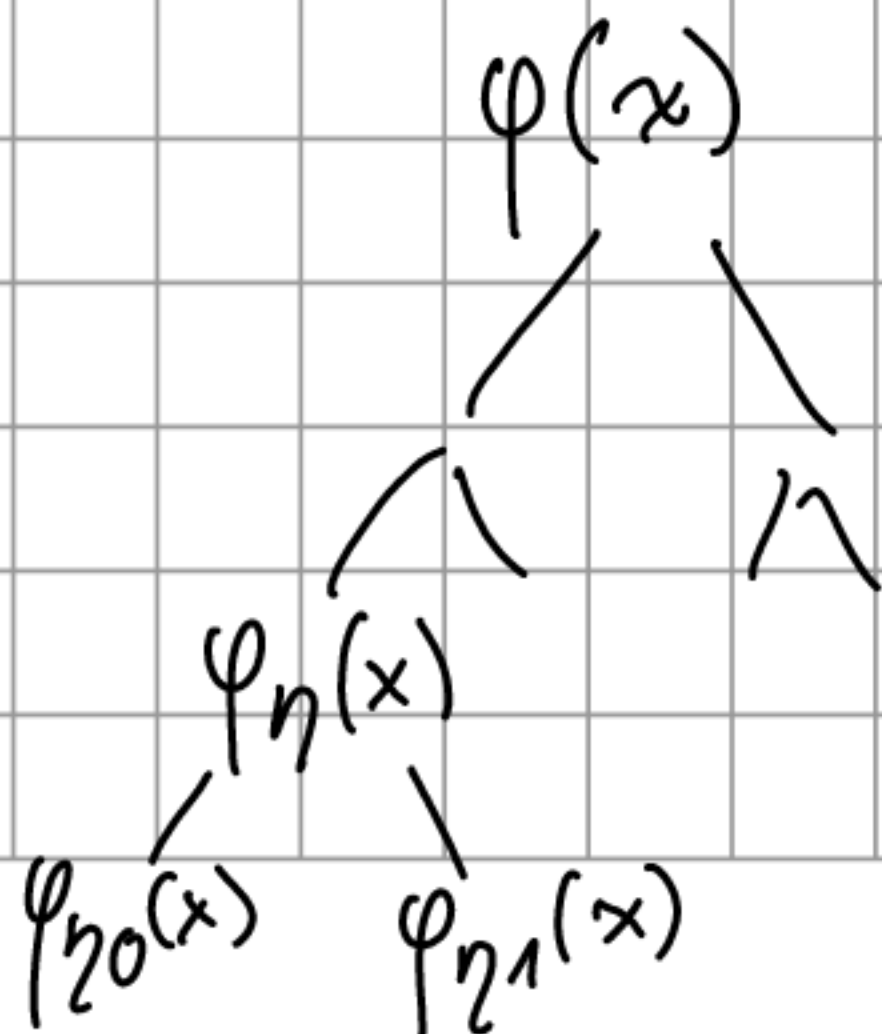
b) F_p : prime (exercise)

Thm. $T: \aleph_0$ -stable $\Rightarrow T$ has a prime model.

Lemma 1 $T: \aleph_0$ -stable $\Rightarrow \forall A \subseteq \mathcal{M}$ $\{ \text{isolated types} \} \subseteq S_n(A)$ dense

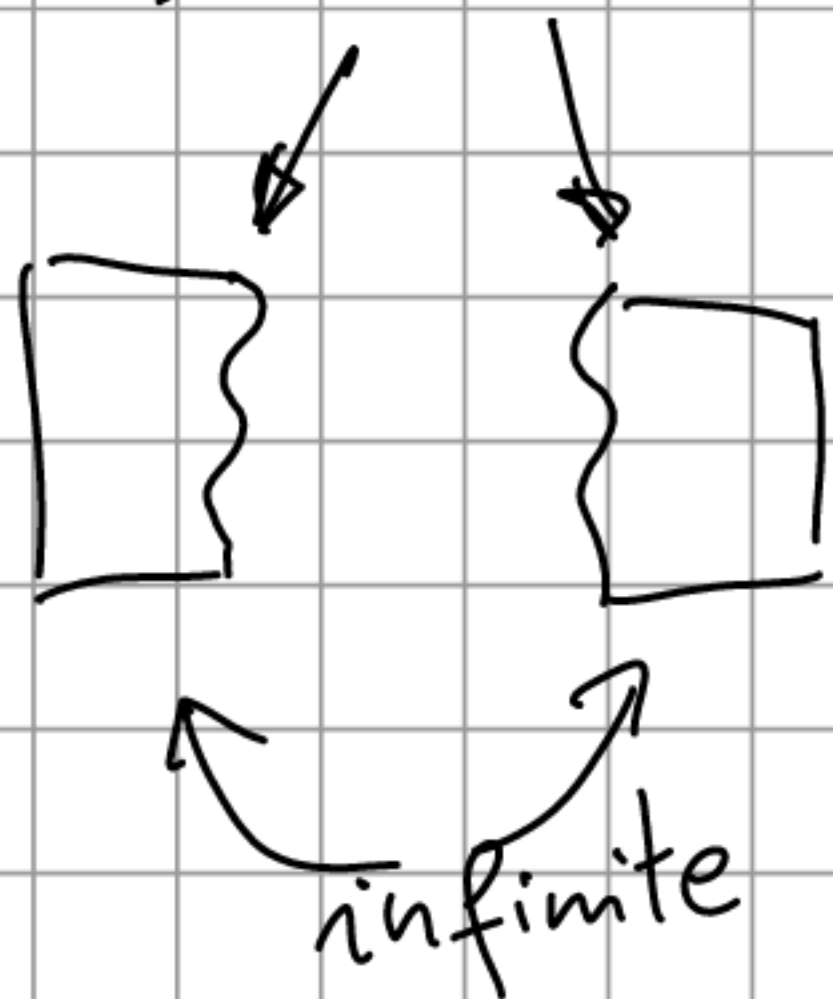
Pf. Suppose $\varphi(x) \in L_n(A)$ s.t. in $S_n(A) \cap [\varphi]$ (consistent with T)

there is no isolated types \Rightarrow a tree of formulas $\varphi_\eta(x) \in L_n(A)$, $\eta \in 2^{<\omega}$



\rightsquigarrow contradicts \aleph_0 -stability of T

clopen $\{ \} \subseteq S_1(A) \cap [\varphi]$



□

Lemma 2 $(a, b \in \mathcal{M})$ $tp(a)$ isolated and $tp(b/a)$ isolated

$\Leftrightarrow tp(ab)$ isolated

Pf " \Rightarrow ": $\varphi(x) \vdash tp(a), \psi(a, y) \vdash tp(b/a)(y)$

$p_a(y) \subseteq S_y(a)$

Then: $\varphi(x) \wedge \psi(x, y) \vdash tp(ab)$

Let $a', b' \in \mathcal{M}$ satisfy $\varphi(x) \wedge \psi(x, y)$

$\models \varphi(a') \Rightarrow tp(a') = tp(a) \Rightarrow \psi(a', y) \vdash p_{a'}(y)$

$\supseteq S_y(a')$

$\models \psi(a', b') \Rightarrow \models p_{a'}(b')$

\Downarrow

$ab \equiv a'b'$

and $tp(ab) = tp(a'b')$

" \Leftarrow ": $\Theta(x, y) \vdash \text{tp}(a, b)$.

(a) " $\exists y \Theta(x, y)$ " $\vdash \text{tp}(a)$.

because Let $a' \in \mathcal{M}$ satisfy " $\exists y \Theta(x, y)$ ".

So there is b' s.t. $\models \Theta(a', b')$

$\Rightarrow \text{tp}(ab) = \text{tp}(a', b') \Rightarrow \text{tp}(a) = \text{tp}(a')$.

(b) $\Theta(a, y) \vdash \text{tp}(b/a)(y)$

because: similar to (a) ▀

Proof of the thm. Construction of a prime model of T :

$A = \{a_n : n < \omega\} \subseteq \mathcal{M}$ so that:

1) A satisfies TV-test

2) $\forall n$ $\text{tp}(a_n/a_{<n})$ is isolated

At step n choose a_n : $[a_{<n} = \{a_k : k < n\}]$

Let $\varphi(x) \in L_n(a_{<n})$ consistent.

Let $a_n \in \varphi(\mathcal{M})$ s.t. $\text{tp}(a_n/a_{<n})$ is isolated (lemma 1)

Suitable choice of φ 's ensures (1).

$M \models T$ is prime. Let $N \models T$. We find
 $f(a_n) \in N$ for $n < \omega$ s.t. \exists ^{arbitrary} $f: M \equiv \rightarrow N$.

At step n $f[a_{<n}] \subseteq N$ with $f: a_{<n} \equiv \rightarrow N$

Let $p(x) = \text{tp}(a_n/a_{<n})$ (isolated)

$$\Downarrow \\ f^*(p)(x) \in S(f(a_{<n}))$$

isolated

too, hence realised by $f(a_n)$

$$\Downarrow \\ f: a_{\leq n} \equiv \rightarrow N$$



Remark (1) A prime model $M \models T$ is atomic

(2) If $M \models T$ is atomic, then M prime

Corollary \hat{F}_p is atomic.

Proof of remark

(1) Let $p(\bar{x}) \in S_n(\emptyset)$ non-isolated. Will show

$p(M) = \emptyset$. Let $N \models T$ be omitting p .

$\exists f: M \xrightarrow{\equiv} N \Rightarrow p(M) = \emptyset$.

(2) Let $M = \prod_T \{a_n : n < \omega\}$ atomic.

Then $\forall n$ $tp(a_n)$ is isolated

\Downarrow lemma 2
 $\forall n$ $tp(a_n/a_{<n})$ is isolated

\Downarrow pf of thm

M prime. □

Corollary A prime model of T is unique (up to \cong)

Proof Let $M, N \models T$ both prime $\stackrel{\text{remark}}{\Rightarrow} M, N$

are stable and atomic, so we have embeddings

in both directions, using back-and-forth

we get the iso.

Def $M \models T$ is minimal if $\neg \exists N \cong M$

Example $\hat{\mathbb{F}}_p$ is minimal.

Fact T has a prime model $\Leftrightarrow \forall n$ $\{ \text{isolated types} \} \subseteq S_n(\emptyset)$
dense

Proof " \Rightarrow ": Let $M \models T \Rightarrow M$: atomic
prime

what we need

$N \models T$, then $\forall n \exists p \in S_n(\emptyset)$:
any $p(N) \neq \emptyset$
exercise | is dense in $S_n(\emptyset)$

" \Leftarrow ": Claim Assume $\bar{a} \subseteq \mathcal{M}$ and $\text{tp}(\bar{a})$ is isolated.
finite

Then $\{ \text{isolated types} \} \subseteq S_n(\bar{a})$.
dense

Proof of claim Let $n = |\bar{a}|$, $\varphi(\bar{x}) \vdash \text{tp}(\bar{a})$.

Let $\psi(\bar{x}, y) \in L_{n+1}(\emptyset)$ s.t. $\psi(\bar{a}, y)$ is consistent.

We seek $q(y) \in S_1(A) \cap [\psi(\bar{a}, y)]$ isolated.

Let $\chi(\bar{x}, y) = \varphi(\bar{x}) \wedge \psi(\bar{x}, y)$.

By assumptions of $\Leftarrow \exists p(\bar{x}, y) \in S_{\bar{x}, y}(\emptyset) \cap [\chi(\bar{x}, y)]$.
isolated

Let $\bar{a}', b' \models p(\bar{x}, y)$. Then $\bar{a}' \models p(\bar{x}, y) \upharpoonright_{\bar{x}} = \text{tp}(\bar{a})$
 \wedge
 $\varphi(\bar{x})$

Let $f \in \text{Aut}(\mathcal{M}) : f(\bar{a}') = \bar{a}$
 $b = f(b')$

Then $\bar{a}'b' \stackrel{\equiv}{\underset{f}{\rightarrow}} \bar{a}b \Rightarrow \bar{a}b \models p(\bar{x}, y)$

so $\text{tp}(\bar{a}b)$ is isolated $\stackrel{\text{lemma 2}}{\Rightarrow}$ $\text{tp}(b/\bar{a})$ isolated
 \Downarrow
 $\psi(\bar{a}, y)$

So $q(y) = \text{tp}(b/\bar{a})$

Given $n \mapsto$ we construct a model $M = \{a_n : n < \omega\}$

s.t. $\forall n \text{ tp}(a_n/a_{<n})$ is isolated

\Downarrow lemma 2

M atomic dble $\Rightarrow M$ prime.

Corollary If $\forall n |S_n(\emptyset)| \leq \aleph_0$, then T has a
 prime model.

Corollary A prime model (of a dble T) is
 homogeneous (exercise).

The number of countable models of $T: I(T, \aleph_0), n(T)$.

Remark $1 \leq n(T) \leq 2^{\aleph_0}$

$$M \models T \Rightarrow M \cong \underbrace{(N, \dots)}_{\leq 2^{\aleph_0} \text{ L-structures like that}}$$

Recall $n(T) = 1 \Leftrightarrow \forall n \ |S_n(\emptyset)| < \aleph_0$

($T: \aleph_0$ -categorical)

Vaught conjecture (1961)

$$n(T) > \aleph_0 \Rightarrow n(T) = 2^{\aleph_0}$$

Thm (M. Morley, 1971) $\aleph_0 < n(T) < 2^{\aleph_0} \Rightarrow n(T) = \aleph_1$

Thm (Vaught, 1961) $n(T) \neq 2$

Proof (A.a) suppose $n(T) = 2$.

$$n(T) < 2^{\aleph_0} \Rightarrow T \text{ small (i.e. } \forall n \ |S_n(\emptyset)| \leq \aleph_0)$$

⋮