Preliminaries

T : a complete consistent theory, in language L (countable)
with infinite models.

that is, $T = Th(M)$, M : L-structure.
infinite

L denotes also the set of formulas of language L

$M = (|M|; \dots)$, but M also denotes $|M|$.
(for brevity)

$\emptyset \neq$ universe of M

usually we omit $|\cdot|$ in $|M|$.

$M \supseteq A$: a set of parameters.

$L_n(A) = \{\varphi(x_1, \dots, x_n, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in L, \ \bar{a} \subseteq A\}$

$L(A) = \bigcup_n L_n(A)$, also $L(A)$: language L extended by names for elements of A.

$L_n(A)$: Lindenbaum algebra.

formally: on $L_n(A)$: $\varphi \sim \psi \iff T(A) \vdash \varphi \leftrightarrow \psi$
$\iff M \models \varphi \leftrightarrow \psi$

here: $T(A) = Th(M, a)_{a \in A}$
a complete theory in language $L(A)$.

$L_n(A)/\sim$: a Boolean algebra
(Lindenbaum algebra)

$[\varphi]_\sim \wedge [\psi]_\sim = [\varphi \wedge \psi]_\sim$ etc.

shortly: $L_n(A)$ denotes also $L_n(A)/\sim$.

$S_n(A) = \{$complete n-types over A, in $M\}$.
in variables $x_1, \dots, x_n$

consistent n-type over A $\rightsquigarrow$ proper filter in $L_n(A)$

An n-type $p(\bar{x})$ over A is <u>complete</u> if
$\cap$
$L_n(A)$
$\begin{cases} \cdot\ p(\bar{x}) : \text{consistent type} \\ \cdot\ \forall \varphi(\bar{x}) \in L_n(A)\ (\varphi(\bar{x}) \in p \text{ or } (\neg\varphi(\bar{x})) \in p) \end{cases}$

$S(A) := S_1(A)$
(default)

$S_n(A)$: topological space:
for $\varphi(\bar{x}) \in L_n(A)$

$$[\varphi] = \{p \in S_n(A) : \varphi \in p\}$$

basic open set [clopen]
$\shortparallel$
closed and open

$S_n(A)$: compact Hausdorff space, 0-dimensional
(i.e. basis of clopen sets)

complete n-types $/A$ $\longleftrightarrow$ ultrafilters in $L_n(A)$

So $S_n(A) = S(L_n(A))$, the Stone space of ultrafilters in $L_n(A)$

- the ~~topo~~ topology on $S_n(A)$ $=$ the Stone space topology.

For $p(\bar{x}) \in S_n(A)$

$$p(M) = \{ \bar{a} \in M^n : \overset{\text{realizes } p}{\underbrace{\bar{a} \text{ satisfies } p}_{\bar{a} \models p}} \}$$

$\bar{a} \models p$, i.e. $M \models \varphi(\bar{a})$ for every $\varphi(\bar{x}) \in p(\bar{x})$

- The same notation for arbitrary type (also incomplete)
- A formula $\varphi(\bar{x}) \in L(M)$: a special case of a type $\{\varphi(\bar{x})\}$.

$\varphi(M) = \ldots$

- When $p \in S_n(A)$, $\bar{a} \subseteq M$ and $\bar{a} \models p$, then

$$p = tp^M(\bar{a}/A) = tp(\bar{a}/A) = \{\varphi(\bar{x}) \in L_n(A) : M \models \varphi(\bar{a})\}.$$

<u>Example</u> Assume $p(\bar{x})$: a consistent type over M.

Then $\exists N \succcurlyeq M$ $\underbrace{p \text{ is realized in } N}_{\text{i.e. } p(N) \neq \emptyset}$

i.e. $p(N) \neq \emptyset$.

From now on "a type" means "a consistent type".

<u>Def</u> A type $p(\overline{x})$ over A is <u>isolated</u>, if:

$\exists\, \varphi(\overline{x}) \in L_n(A)$
① $\varphi(\overline{x})$ is consistent (with T), i.e. $\varphi(M) \neq \emptyset$ ($\Leftrightarrow T(A) \vdash \exists \overline{x}\, \varphi(\overline{x})$)

(2) ~~$\varphi(M)$~~ $\forall \psi(\overline{x}) \in p(\overline{x})$ $\varphi(M) \subseteq \psi(M)$ $\Updownarrow$ $T(A) \vdash \varphi(\overline{x}) \to \psi(\overline{x})$

symbolically: $\varphi(\overline{x}) \vdash p(\overline{x})$

- When $p(\overline{x})$: a complete type over A, then:

  $p(\overline{x})$ is isolated $\Leftrightarrow$ p is isolated in $S_n(A)$ in the topological sense (i.e. $\{p\}$ is open)

<u>Tarski – Vaught test</u>

Assume $A \subseteq M$. Then $A = |N|$ for some $N \prec M$ <u>iff</u>

$$\forall \varphi(x) \in L_1(A)\ [\varphi(M) \neq \emptyset \Rightarrow \varphi(M) \cap A \neq \emptyset]$$

Construction of an elementary submodel of M containing A:

- $A_n \subseteq M$, $n < \omega$, increasing chain of sets

recursive construction:

$A_0 = A$

$A_n \subseteq A_{n+1} \subseteq M$ such that $\forall \psi(x) \in L_1(A_n)$
$$[\psi(M) \neq \emptyset \Rightarrow \psi(M) \cap A_{n+1} \neq \emptyset]$$

$A_\infty = \bigcup_{n<\omega} A_n$ satisfies TV-test.

Omitting types theorem

Assume $p_n(\bar{x}_n)$, $n<\omega$: a family of non-isolated types in theory T, over $\emptyset$. Then:

$(\exists M \models T)$ M omits every p_n [i.e. $p_n(M)=\emptyset$]

Assume $\underset{\underset{A}{\cup\!\!\mid}}{M}, N \models T$

Def. $f: A \to N$ is *elementary* ($f: A \xrightarrow{\equiv} N$) if:

$$\forall \bar{a} \subseteq A\ \forall \varphi(\bar{x}) \in L\ \big(M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a}))\big)$$

$(\Leftrightarrow\ tp^M(\bar{a}) = tp^N(f(\bar{a}))\ )$

Elementary diagram of $A \subseteq M$:

$D_e(A) = T(A) = Th(M, a)_{a\in A}$

Remark $f: A \to N$ is elementary $\Leftrightarrow (N, f(a))_{a\in A} \models T(A)$

Atomic diagram of $A \subseteq M$:

$D_{at}(A) = \{\varphi \in D_{el}(A) : \varphi \text{ is a quantifier free sentence}\}$

$= \{\varphi(\bar{a}) \in L(A) : M \models \varphi(\bar{a}) \text{ and } \varphi(\bar{a}) : \text{q.f.-sentence}\}$

Remark $f: M \to N$ is a monomorphism (i.e: $f: M \xrightarrow{\cong} f(M) \subseteq N$) ↑substructure

$\Leftrightarrow (N, f(a))_{a\in M} \models D_{at}(M)$.

Here always $f: M \to N$ denotes a <u>monomorphism</u>.

$M \subseteq N$: M is a submodel (substructure) of N

$M \prec N$: M is an elementary submodel of N, i.e.:

$$M \subseteq N \text{ and } id_M : M \xrightarrow{\equiv} N$$

<u>Remark</u> Assume $M \prec N$, $A \subseteq M$.

(1) Assume $p(\bar{x}) \subseteq L_n(A)$. Then

$p(\bar{x})$ is a consistent type in $M \Leftrightarrow p(\bar{x})$ is a consistent type in N

(2) Assume $A \subseteq B \subseteq M$

- If $p(\bar{x})$: a type over B, then $p\restriction_A \overset{def}{=} p(\bar{x}) \cap L(A)$ a type over A
- Let $r: S_n(B) \to S_n(A)$, $r(p) \overset{def}{=} p\restriction_A$.
  Then r: continuous and "onto".

(3) If $p(\bar{x})$: a type over A, then $\exists q(\bar{x}) \in S_n(A)$ $p(\bar{x}) \subseteq q(\bar{x})$

Saturation, universality, (strong) homogeneity.

Let $\varkappa \in CN$, $\varkappa \geq \aleph_0$.

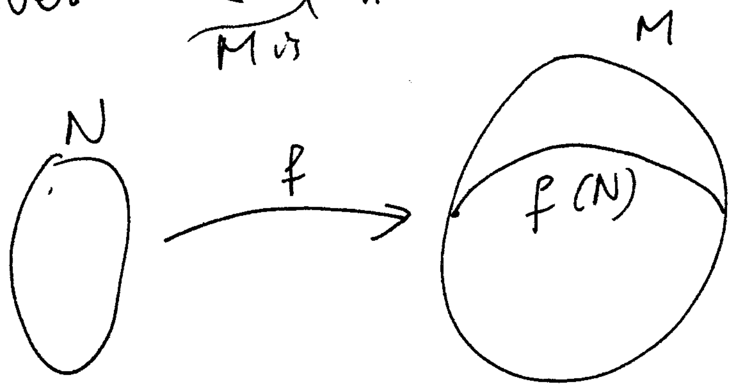
<u>Def.</u> (1) M $\varkappa$-<u>saturated</u> (nasycony) if $\forall A \subseteq M$, $|A| < \varkappa$ $\forall p \in S_1(A)$ $p(M) \neq \emptyset$

M is <u>saturated</u> if M is $\|M\|$-saturated

(2) M is $\varkappa$-universal if $\forall N \equiv M$ $(\|N\| \leq \varkappa \Rightarrow \exists f: N \xrightarrow{\equiv} M)$

elementarily equivalent
i.e. $Th(N) = Th(M)$

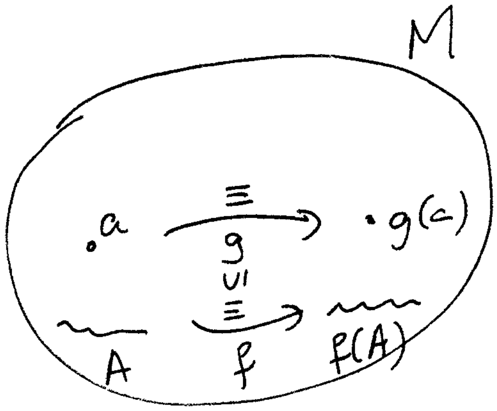
M: universal $\underset{M \text{ is}}{\iff}$ $\|M\|$-universal



$f : N \xrightarrow{\cong\,\equiv} f(N) \prec M$

(3) M : κ-homogeneous if $\forall A \subseteq M$, $|A| < \kappa$, $\forall a \in M$ $\forall f : A \xrightarrow{\equiv} M$ $\exists\, g \supseteq f : A \cup \{a\} \xrightarrow{\equiv} M$

homogeneous = $\|M\|$-homogeneous.



4. M strongly κ-homogeneous if $\forall A \subseteq M$, $|A| < \kappa$, $\forall f : A \xrightarrow{\equiv} M$ $\exists\, g \supseteq f : M \xrightarrow{\cong\,\equiv} M$

strongly homogeneous =: strongly $\|M\|$-homogeneous.

5. M is κ-compact if ($\forall$ 1-type $p(x)$ over M)
$(|p| < \kappa \Rightarrow p(M) \neq \emptyset)$

Elementary chains of structures

Def $\langle M_\alpha : \alpha < \mu \rangle$, $\mu \in \mathrm{Ord}$, : an elementary chain of structures if $(\forall \alpha < \beta < \mu)\, M_\alpha \prec M_\beta$.

Union of chain (when $\mu \in \mathrm{Lim}$)

$$M_\mu = \bigcup_{\alpha < \mu} M_\alpha \ :$$

- $|M_\mu| := \bigcup\limits_{\alpha<\mu}^{\nearrow} |M_\alpha|$
- $c \in L$ constant symbol $\quad c^{M_\mu} = c^{M_\alpha}$ for $\alpha < \mu$
- P : relation symbol

$$\underset{\substack{\cap \\ |M_\mu|}}{P^{M_\mu}(a_1, \dots, a_n)} \iff M_\alpha \models P(a_1, \dots, a_n) \text{ for } \alpha < \mu$$

sufficiently large [so that $\bar{a} \subseteq M_\alpha$]

- $f^{M_\mu}(\bar{a}) = b \iff M_\alpha \models f(\bar{a}) = b$ for $\alpha < \mu$ sufficiently large

Fact (Tarski) $M_\alpha \prec M_\mu$ for all $\alpha < \mu$.

<u>Proof</u> (1) $M_\alpha \subseteq M_\mu$ (substructure): exercise

(2) $\forall \varphi(\bar{x}) \in L \; \forall \alpha < \mu \; \forall \bar{a} \subseteq M_\alpha \; (M_\alpha \models \varphi(\bar{a}) \iff M_\mu \models \varphi(\bar{a}))$

(a) φ atomic : $M_\alpha \subseteq M_\mu$ ✓

(b) $\varphi = \psi_1 \wedge \psi_2$, $\varphi = \neg\psi$: easy

(c) $\varphi(\bar{x}) = \exists y \, \psi(\bar{x}, y)$

$M_\alpha \models \varphi(\bar{a}) \Rightarrow M_\alpha \models \psi(\bar{a}, b)$ for some $b \in M_\alpha$

$\Downarrow$ ind. assumption for ψ

$M_\mu \models \psi(\bar{a}, b)$

$\Downarrow$

$M_\mu \models \varphi(\bar{a})$

$$
\begin{array}{lll}
M_\mu \models \varphi(\bar a) \Longrightarrow & M_\mu \models \psi(\bar a, b) \text{ for some } b \in M_\mu & \\
\quad \| & \Downarrow \text{ ind. assumption} & \quad \Downarrow \\
\exists y\, \psi(\bar a, y) & & b \in M_\beta \text{ for some } \\
& M_\beta \models \psi(\bar a, b) & \quad \alpha \le \beta < \mu \\
& \Downarrow & \\
& M_\beta \models \varphi(\bar a) & \\
& \Downarrow\ M_\alpha \prec M_\beta & \\
& M_\alpha \models \varphi(\bar a) &
\end{array}
$$

<u>Elementary directed systems of structures:</u>

Let $(I, \le)$: a <u>directed set</u>, i.e.:

(1) $\le$: partial order on I

(2) $(\forall a, b \in I)(\exists c \in I)(a \le c \wedge b \le c)$

<u>Example</u> J : a set $\rightsquigarrow$ $([J]^{<\omega}, \subseteq)$: directed set.

Directed system of structures:

- $\mathcal{M} = (M_i, f_{ij})_{i \le j \in I}$

connecting functions $f_{ij} : M_i \longrightarrow M_j$, $f_{ii} = id_{M_i}$ such that

$$(\forall\, i \le j \le t \in I)\ f_{it} = f_{jt} \circ f_{ij}$$

(compatibility)

$$
\begin{tikzcd}
 & M_t \\
M_i \arrow[ur, "f_{it}"] \arrow[r, "f_{ij}"'] & M_j \arrow[u, "f_{jt}"']
\end{tikzcd}
$$

- System $\mathcal{M}$ is <u>elementary</u> if all f_{ij} are elementary.

<u>Example</u> Elementary chain $(M_\alpha)_{\alpha<\mu}$

$\mathcal{M} = (M_\alpha, f_{\alpha\beta})_{\alpha\leq\beta<\mu}$ $f_{\alpha\beta} = id_{M_\alpha} : M_\alpha \xrightarrow{\equiv} M_\beta$
elementary directed system of structures

Direct limit of a directed system $\mathcal{M}$: $M_\infty = \varinjlim \mathcal{M}$.

$(M_\infty, f_{i\infty})_{i\in I}$, where $f_{i\infty} : M_i \longrightarrow M_\infty$ such that

(1) $\forall i \leq j \in I$ $f_{i\infty} = f_{j\infty} \circ f_{ij}$ [compatible with connecting functions]

$$\begin{tikzcd} M_j \arrow[dr, "f_{j\infty}"] & \\ & M_\infty \\ M_i \arrow[uu, "f_{ij}"] \arrow[ur, "f_{i\infty}"'] \arrow[ur, phantom, "\#"] & \end{tikzcd}$$

(2) $(\forall (M'_\infty, f'_{i\infty})_{i\in I}$ satisfying (1)) $\exists! f : M_\infty \to M'_\infty$
(universality) $(\forall i\in I)$ $f'_{i\infty} = f \circ f_{i\infty}$

$$\begin{tikzcd} & M_\infty \arrow[dd, "f"] \\ M_i \arrow[ur, "f_{i\infty}"] \arrow[dr, "f'_{i\infty}"'] \arrow[r, phantom, "\#"] & \\ & M'_\infty \end{tikzcd}$$

<u>Fact</u> M_∞ exists (and is unique up to $\cong$).
If $\mathcal{M}$ is elementary, then $f_{i,\infty} : M_i \xrightarrow{\equiv} M_\infty$.

<u>Proof</u> 1. Construction of M_∞:

$S := \dot{\bigcup}_{i\in I} |M_i|$: formally disjoint union.

$\sim$ on S : an equivalence relation

$$\underset{\substack{\Downarrow \\ x}}{M_i} \sim \underset{\substack{\Downarrow \\ y}}{M_j} \overset{\text{def}}{\iff} f_{it}(x) = f_{jt}(y) \text{ for some } (\equiv \text{every}) \; t \geq i, j$$

exercise: $\sim$ is transitive.

$$|M_\infty| := S/\sim$$

- $\sim \restriction_{|M_i|}$: the equality (because f_{ij} : 1-1 (monomorphism))
- $f_{i\infty}(\underset{\substack{\Uparrow \\ M_i}}{x}) = x/\sim$, $\quad f_{i\infty} : |M_i| \xrightarrow{1-1} |M_\infty|$.

L-structure on $|M_\infty|$:

- $c^{M_\infty} = c^{M_i}/\sim$
- $P^{M_\infty}(a_{i_1}/\sim, \dots, a_{i_n}/\sim) \iff M_t \models P(f_{i_1 t}(a_{i_1}), \dots, f_{i_n t}(a_{i_n}))$

  $a_{i_j} \in M_{i_j}$ $\qquad$ for $t \geq i_1, \dots, i_n$

- f^{M_∞} : similarly

the rest is an exercise.

How to extend elementary mappings?

~~Bat.~~ BAlg : Category of Boolean algebras

Comp$_0$: —//— of compact Hausdorff 0-dimensional spaces

$F : \mathrm{BAlg} \longrightarrow \mathrm{Comp}_0$ $\quad$ $G : \mathrm{Comp}_0 \longrightarrow \mathrm{BAlg}$

$F(A) = S(A)$ $\quad$ $G(X) = \mathrm{Clopen}(X)$.

F, G : contravariant functors "inverse" to each other

~~(F, G) is a duality of categories... (look it up)~~

Categories BAlg and Comp$_0$ are dually equivalent.

A, B : Boolean algebras

$f : A \longrightarrow B$ homomorphism $\Rightarrow$ $\underset{f^*}{F(f)} : S(B) \longrightarrow S(A)$

$F(f)(p) = f^{-1}[p]$

continuous.

Assume $f : A \xrightarrow{\equiv} B$, $A \subseteq M$, $B \subseteq N$, $T \models M$, $N \models T$. Then $\hat{f} : L_n(A) \longrightarrow L_n(B)$

$\hat{f}(\varphi(\bar{x}, \bar{a})) = \varphi(\bar{x}, f(\bar{a}))$

homomorphism of Boolean algebras.

even: monomorphism.

We skip ^ in $\hat{f}$, so:

$f : L_n(A) \longrightarrow L_n(B)$ monomorphism

$\Downarrow$

$f^* : S_n(B) \longrightarrow S_n(A)$ epimorphism in Comp$_0$

i.e: continuous onto

Lemma (on extensions of elementary mappings)

Assume $M, N \models T$, $A \subseteq M$, $B \subseteq N$, $f : A \xrightarrow{\equiv} B$ "onto".

Assume $\underset{a}{\overset{\Psi}{M}}, \underset{b}{\overset{\Psi}{N}}$, $p = tp(a/A)$, $q = tp(b/B)$.

Then $f \cup \{\langle a, b\rangle\}$ is elementary $\iff f^*(q) = p$.

[here $f^* : S(B) \xrightarrow{\approx} S(A)$].
homeomorphism

Proof exercise.

Def. M is $(<\aleph_0)$-universal $\iff \forall n\, \forall p \in S_n(\emptyset)\; p(M) \neq \emptyset$.

Remark $M : \varkappa$-universal $\Rightarrow M : (<\aleph_0)$-universal.

Proof Let $p \in S_n(\emptyset)$.

Choose a countable $N \models T$ with $p(N) \neq \emptyset$.

$M : \varkappa$-universal $\Rightarrow \exists f : N \xrightarrow{\equiv} M$

$\overset{\cup\!\shortmid}{\bar{a}} \models p \overset{f}{\rightsquigarrow} \overset{\cup\!\shortmid}{f}(a) \models p$.

Thm. (1) $M : \varkappa$-saturated $\Rightarrow M : \varkappa$-homogeneous and $\varkappa$-universal.

(2) $M : \varkappa$-~~universal~~ homogeneous and $(<\aleph_0)$-universal $\Rightarrow$ $M : \varkappa$-saturated.

Proof. (1) $\varkappa$-homogeneity of M:

Assume $f : A \xrightarrow{\equiv} M$, $A \subseteq M$, $|A| < \varkappa$, $a \in M$.

We seek $b \in M$ s.t. $g = f \cup \{\langle a, b\rangle\}$ elementary

$$\Updownarrow \text{ Lemma}$$

$$f^*(tp(b/B)) = tp(a/A).$$

Let $p = tp(a/A)$, $q = (f^*)^{-1}(p) \in S_1(B)$

$p \in S_1(A)$

Let $b \in M$ (exists by κ-saturation of M) realise q

b good.

• ~~κ-universality~~ of M:

Assume $N \equiv M$, $\|N\| \leq \kappa$.

We seek $f : N \xrightarrow{\equiv} M$.

Let $\{a_\alpha : \alpha < \mu\}$: an enumeration of N, $\kappa \geq \mu = \|N\|$.

We define $f(a_\alpha)$ by induction on $\alpha < \mu$:

• Suppose $f(a_\beta)$ defined for all $\beta < \alpha$ so that

$$f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} M$$

Want to find $f(a_\alpha)$ so that

$$f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} M.$$

~~By the Lemma it is enough that~~

Let $p = tp(a_\alpha / \{a_\beta : \beta < \alpha\})$.

By the lemma it is enough to find $f(a_\alpha) \in M$ so that $f^*(tp(f(a_\alpha)/\{f(a_\beta) : \beta < \alpha\})) = p$.

So let $q = (f^*)^{-1}(p) \in S_1(\underbrace{\{f(a_\beta) : \beta < \alpha\}}_{\text{power} < \kappa})$

M κ-saturated $\Rightarrow$ q realized in M.

Let $f(a_\alpha) \in M$ s.t. $f(a_\alpha) \models q$.

(2) Assume M is κ-homogeneous & $(<\aleph_0)$-~~saturated~~ universal.

Want: M: κ-saturated.

So: Let $A \subseteq M$, $|A| < \kappa$, $p \in S_1(A)$. Show: $p(M) \neq \emptyset$.

Induction on $|A|$.

Case (a): $|A| < \aleph_0$.

$\exists N \succ M$ $p(N) \neq \emptyset$. So let $b \models p$.

Let $A^* = \underset{\{a_1,\dots,a_k\}}{\overset{\shortparallel}{A}} \cup \{b\}$

Let $q = tp^N(a_1,\dots,a_k,b) \in S_{k+1}(\emptyset)$

q is realized in M ((by $<\aleph_0$)-universality),

by $\langle \underbrace{a_1',\dots,a_k'}_{A'}, b' \rangle$

Let $g: A \cup \{b\} \longrightarrow A' \cup \{b'\}$, $g(a_i) = a_i'$, $g(b) = b'$.

g: elementary.

$\Rightarrow g\restriction_A : A \xrightarrow{\equiv} A'$

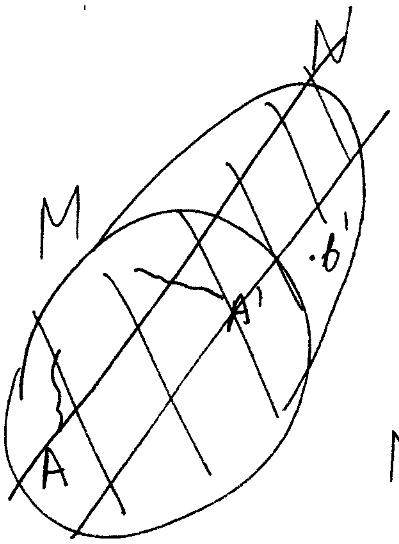
$\Downarrow$

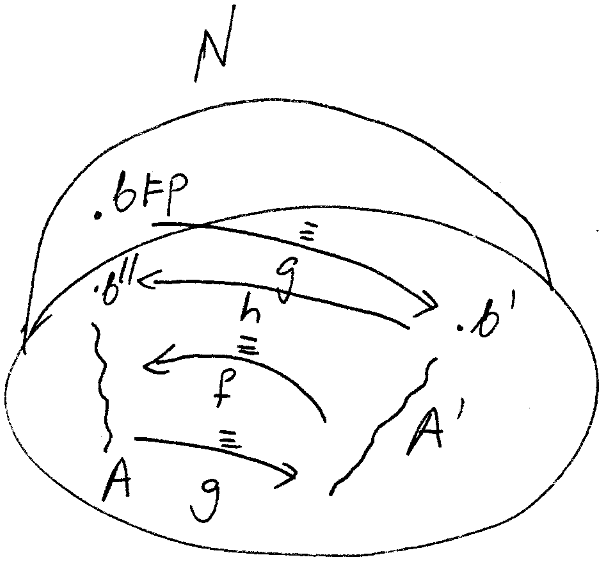
$f := (g\restriction_A)^{-1} : A' \xrightarrow{\equiv} A$

$\cap\!|$

M: $\varkappa$-homogeneous $\Rightarrow \exists\, h : A' \cup \{b'\} \xrightarrow{\equiv} A \cup \{b''\}$

for some $b'' \in M$.





$$\begin{tikzcd} Ab \arrow[rr, "\equiv", "h\circ g"'] \arrow[dr, "g"', "\equiv"] & & Ab'' \\ & A'b' \arrow[ur, "\equiv", "h"'] & \end{tikzcd}$$

Let $s = h \circ g$

$s\restriction A = \underbrace{(h\restriction_{A'})}_{\overset{\shortparallel}{(g\restriction_A)^{-1}}} \circ (g\restriction_A) = id_A.$

$\Downarrow$

$\cancel{s^*(tp(b''/A)) = tp(b}$

$s\restriction_A = id_A \Rightarrow \underset{\overset{\shortparallel}{id_{S(A)}}}{s^*} : S(A) \xrightarrow{\simeq} S(A)$

hence: $p = tp(b/A) \underset{\substack{\uparrow \\ \text{Lemma} \\ \text{and } b'' \models p}}{=} s^*(tp(b''/A)) \underset{\substack{\uparrow \\ s^* = id_{S(A)}}}{=} tp(b''/A)$

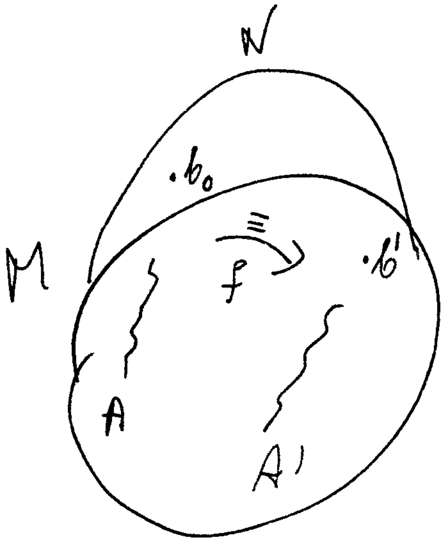
Case (b) $|A|=\mu$, $\aleph_0 \le \mu < \kappa$.

$A=\{a_\alpha : \alpha<\mu\}$, $p\in S_1(A)$.

$p\restriction\emptyset \in S_1(\emptyset) \Longrightarrow \exists b'\in M \ \ b'\models p\restriction\emptyset$.

M: $(<\aleph_0)$-universal

$\exists N \succ M \ \ \exists b_0\in N$, $b_0 \models p$



Will find $A'=\{a'_\alpha:\alpha<\mu\}\subseteq M$

s.t. $f: Ab_0 \longrightarrow A'b'$

given by $f(a_\alpha)=a_\alpha'$

$f(b_0)=b'$

is <u>elementary</u>!

We find a'_α, $\alpha<\mu$ by induction on $\alpha<\mu$.

So suppose $\alpha<\mu$ and a'_β already defined for all $\beta<\alpha$

so that $\underbrace{f\restriction_{\{a_\beta:\beta<\alpha\}b_0}}_{=: f_0} : \{a_\beta:\beta<\alpha\}b_0 \xrightarrow{\equiv} \{a'_\beta:\beta<\alpha\}b'$

We look for a'_α.

Let $q = tp(a_\alpha/\{a_\beta:\beta<\alpha\}\cup\{b_0\})$

Then $(f_0^*)^{-1}(q) \in S(\underbrace{\{a'_\beta:\beta<\alpha\}\cup\{b'\}}_{\text{power } < \cancel{\kappa}\, \mu \le |A|})$

By the lemma it is enough that $\underset{\in M}{a'_\alpha} \models (f_0^*)^{-1}(q)$.

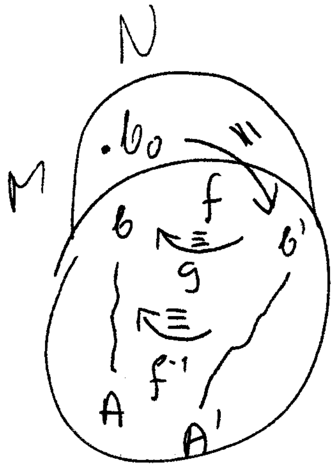
~~But M: κ-~~

~~Let A~~ By the inductive assumption on A:

$(f_0^*)^{-1}(q)$ is realized in M, so we are done with constructing A'.

Now: $f^{-1} : A' \xrightarrow{\equiv} A$ in M ← κ-homogeneous, so

$\exists\ \hat{g}' : A'b' \xrightarrow{\equiv} Ab$ for some $b \in M$.



~~Let~~ Let $s = g \circ f$

$s : Ab_0 \xrightarrow{\equiv} Ab$

$$s\restriction_A = \underset{\substack{\| \\ (f\restriction_A)^{-1}}}{(g\restriction_{A'})} \circ (f\restriction_A) = id_A ,$$

so $p = tp(b_0/A) = tp(b/A)$

and p is realized in M.

<u>Corollary</u>

M is κ-saturated $\iff$ M is κ-homogeneous and ~~κ-universal~~. and κ-universal.

<u>Proof</u> $\Rightarrow$ by Thm (1).

$\Leftarrow$ κ-homogeneous + κ-universal $\Rightarrow$

κ-homogeneous + $(<\aleph_0)$-universal $\underset{\text{Thm }(2)}{\Rightarrow}$ κ-saturated

Properties of saturated models.

Thm. Assume $M, N \models T$ saturated models of the same power. Then $M \cong N$.

Proof $M = \{m_\alpha : \alpha < \varkappa\}$, $N = \{n_\alpha : \alpha < \varkappa\}$,

$\varkappa = \|M\| = \|N\|$. We find $f : M \xrightarrow{\cong} N$

back-and-forth method:

$$f = \bigcup_{\alpha<\varkappa}^{\nearrow} f_\alpha \qquad f_\alpha : M \xrightarrow[o]{\equiv} N \quad \text{s.t.:}$$

partial, elementary

(1) $m_\alpha \in \mathrm{Dom}\, f_{\alpha+1}$

$n_\alpha \in \mathrm{Rng}\, f_{\alpha+1}$, $|f_\alpha| \le 2\cdot|\alpha|$

(2) $f_0 = \emptyset$

(3) For $\delta \in \mathrm{Lim}$, $f_\delta = \bigcup_{\alpha<\delta}^{\nearrow} f_\alpha$.

(4) $f_{\alpha+1} = f_\alpha \cup \{\langle m_\alpha, \underset{N}{\underset{\Uparrow}{n}}\rangle, \langle \underset{M}{\underset{\Uparrow}{m}}, n_\alpha\rangle\}$

Inductive step:

Suppose we have f_α. Want: $f_{\alpha+1}$.

Let $A_\alpha = \mathrm{Dom}\, f_\alpha \subseteq M$, $B_\alpha = \mathrm{Rng}\, f_\alpha \subseteq N$.

$$f_\alpha : A_\alpha \xrightarrow{\equiv} B_\alpha$$
$$\Downarrow$$
$$f_\alpha^{*} : S(B_\alpha) \xrightarrow{\cong} S(A_\alpha).$$

"forth": Find $n \in N$ st. $f_\alpha \cup \{\langle m_\alpha, n\rangle\}$ elementary

$$\Updownarrow$$

$$(f_\alpha^*)^{-1}(tp(m_\alpha/A_\alpha)) = tp(n/B_\alpha).$$

Let $p = tp(m_\alpha/A_\alpha)$.

So $(f_\alpha^*)^{-1}(p) \in S(B_\alpha)$ is realized in N by some n.

"~~forth~~ back": similarly.

<u>Thm</u> Assume $M, N \models T$ are homogeneous, of the same power and $\forall n<\omega\ \forall p \in S_n(\emptyset)\ (p(M) \neq \emptyset \Leftrightarrow p(N) \neq \emptyset)$.
Then $M \cong N$.

<u>Lemma</u> Under the assumptions of the Thm,

$$\forall A \subseteq M\ \exists f : A \xrightarrow{\equiv} N.$$

<u>Proof</u>. Induction on $|A|$.

Case (a) $|A| < \aleph_0$. $A = \{a_1, \dots, a_n\}$.

Let $p = tp(\langle a_1, \dots, a_n\rangle) \in S_n(\emptyset)$. realized in M

$$\Downarrow$$

realized in N

by some $\langle b_1, \dots, b_n\rangle \subseteq N$.

$f(a_i) = b_i$ is <u>good</u>.

Case (b) $|A| = \mu \geq \aleph_0$, $A = \{a_\alpha : \alpha < \mu\}$

We find $f(a_\alpha)$ by induction on $\alpha < \mu$.

Inductive step.

Suppose $\alpha < \mu$ and for every $\beta < \alpha$ we have $f(a_\beta)$

s.t. $f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} N$.

We shall find $f(a_\alpha) \in N$ s.t. $f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} N$.

Let $a_{<\alpha} := \{a_\beta : \beta < \alpha\}$. Likewise $a_{\leq\alpha}$.

$|a_{\leq\alpha}| < \mu = |A|$

By inductive assumption: $\exists g : a_{\leq\alpha} \xrightarrow{\equiv} N$.

Then $f \circ g^{-1} : \underset{\underset{N}{\cap}}{g(a_{<\alpha})} \xrightarrow{\equiv} \underset{\underset{N}{\cap}}{f(a_{<\alpha})}$

By homogeneity of N: $\exists f(a_\alpha) \in N$ s.t.

$$f \circ g^{-1} : \underset{\underset{g(a_{\leq\alpha})}{\shortparallel}}{\underline{g(a_{<\alpha})\, g(a_\alpha)}} \xrightarrow{\equiv} f(a_{<\alpha}) f(a_\alpha) = f(a_{\leq\alpha})$$

Then $f = (f \circ g^{-1}) \circ g : a_{\leq\alpha} \xrightarrow{\equiv} N$.

Proof of the theorem $\quad \varkappa := \|M\| = \|N\|$

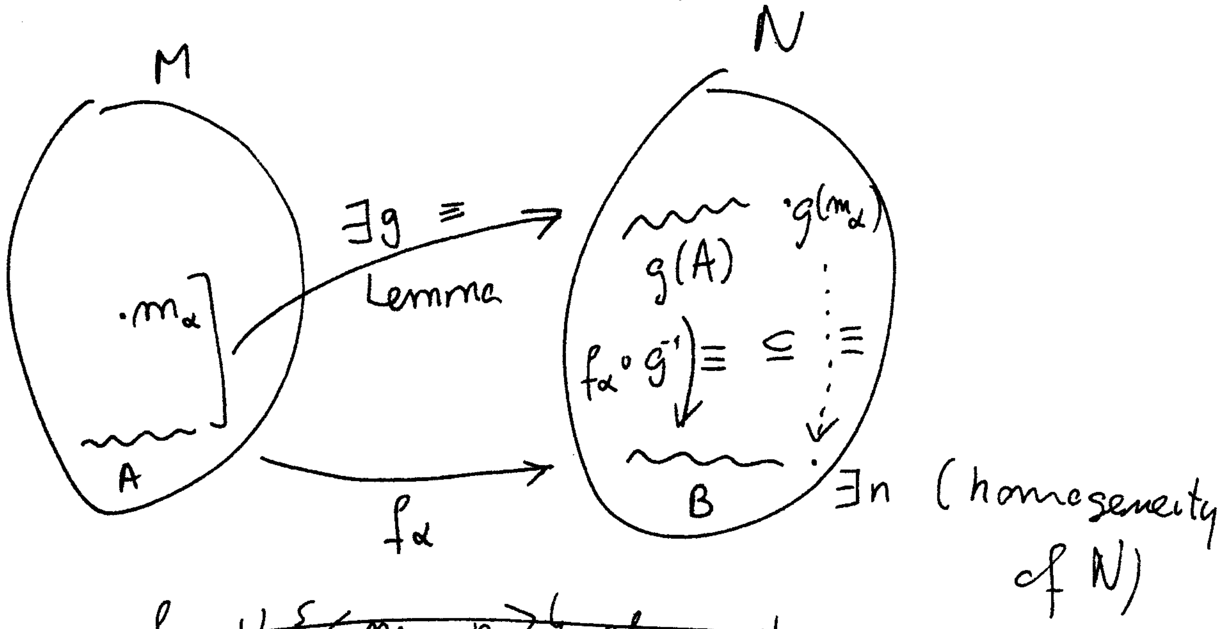
$f : M \xrightarrow{\cong} N$ constructed by back-and-forth method

$f = \bigcup\limits_{\alpha<\varkappa}^{\uparrow} f_\alpha$, $f_\alpha : M \xrightarrow[\circ]{\equiv} N$ (partial elementary), $\alpha < \varkappa$

$|f_\alpha| \leq 2 \cdot |\alpha|$ + the same conditions as in the previous thm.

Inductive step $f_\alpha \longmapsto f_{\alpha+1}$

$A = \mathrm{Dom}\, f_\alpha \qquad B = \mathrm{Rng}\, f_\alpha$ $\qquad \boxed{\text{"forth":}}$



~~$f_\alpha \cup \{\langle m_\alpha, n\rangle\}$ elementary~~

$h = (f_\alpha \circ g^{-1})\restriction_{g(A)} \cup \{\langle g(m_\alpha), n\rangle\}$ elementary

$$\underset{\displaystyle f_\alpha}{\underset{\cup\!\!|}{h \circ g}} : A m_\alpha \xrightarrow{\equiv} B n \subseteq N$$

"back": similarly.

Constructions of models:

- saturated $\Rightarrow$ • (strongly) homogeneous $\not\Leftarrow$

Thm. $\underbrace{\varkappa = 2^{<\varkappa},\ \varkappa \in Reg}_{\varkappa^{<\varkappa} = \varkappa}$, $\varkappa > \aleph_0 \Rightarrow \exists M \models T$ saturated, of power $\varkappa$.

Proof

(*) $|S_1(A)| \leq 2^{|A|+\aleph_0}$, because: $|L_1(A)| = |A| + \aleph_0$

Here: $|A| < \varkappa \Rightarrow |S_1(A)| \leq \varkappa$.

Lemma $N \models T$, $\|N\| \leq \varkappa \Rightarrow X_N := \bigcup \{S_1(A) : A \subseteq N \ \&\ |A| < \varkappa\}$ the set has power $\leq \varkappa$.

Pf • $|\{A \subset N : |A| < \varkappa\}| \leq \varkappa^{<\varkappa} = \varkappa$.

• $|S_1(A)| \leq \varkappa$ for such A.

Proof of the thm.

M_α, $\alpha < \varkappa$: elementary chain of models of T of power $\varkappa$.

- M_0 : whatever
- $M_\delta = \bigcup\limits_{\alpha<\delta}^{\nearrow} M_\alpha$, when $\delta < \varkappa$ limit.
- $M_{\alpha+1} \succ M_\alpha$ such that $\forall p \in X_{M_\alpha}\ p(M_{\alpha+1}) \neq \emptyset$:

$T' = Th(M_\alpha, m)_{m \in M_\alpha} \cup \bigcup\limits_{\beta<\varkappa} \{\varphi(c_\beta) : \varphi(x) \in p_\beta\}$,

where $X_{M_\alpha} = \{p_\beta : \beta < \varkappa\}$ new constant symbols,

and T' in language $L(M_\alpha) \cup \{c_\beta : \beta < \varkappa\}$.

T': consistent, has model of power $\varkappa$: $M_{\alpha+1}$
such that $M_\alpha \prec M_{\alpha+1}$.

$M = \bigcup\limits_{\alpha<\varkappa}^{\nearrow} M_\alpha$: of power $\varkappa$, saturated:

Let $A \subseteq M$, $|A| < \varkappa$ and $p \in S_1^M(A)$

$\varkappa \in Reg \Rightarrow A \subseteq M_\alpha$ for some $\alpha < \varkappa$.

proof: $A = \{a_\beta : \beta < \mu\}$ for some $\underset{\text{CN}}{\mu} < \varkappa$.

- $\forall \beta < \mu \;\; \exists\, \alpha_\beta < \varkappa \;\; a_\beta \in M_{\alpha_\beta}$
- $\{\alpha_\beta : \beta < \mu\} \subseteq \varkappa$, $\mu < cf(\varkappa) = \varkappa$

$\Rightarrow \exists\, \alpha < \varkappa \;\; \forall \beta < \mu \;\; \alpha_\beta < \alpha$

$A \subseteq M_\alpha$.

Let $M_\alpha \prec M \Rightarrow p \in S_1^{M_\alpha}(A) = S_1^M(A)$

p realized in $M_{\alpha+1}$ by some $a \in M_{\alpha+1}$

$a \models p$ in $M_{\alpha+1}$ $\underset{M_{\alpha+1} \prec M}{\Longrightarrow}$ $a \models p$ in M.

Monster model:

Let $\bar\varkappa$: a large cardinal number.

"Ideal model" $\mathcal{M} \models T$: saturated of power $\bar\varkappa$

because: $\forall M \models T \; (\|M\| < \bar\varkappa \Rightarrow \exists M' \prec \mathcal{M} \;\; M \cong M')$.

Advantages of saturated model $\mathcal{M}$:

(i) universality

(ii) strong homogeneity

~~More~~ More Weakly (a bit):

(1) $\bar{\kappa}$ - universality

(2) strong $\bar{\kappa}$ - homogeneity

$Aut(M)$: the group of automorphisms of M

$Aut(M/A) = \{ f \in Aut(M) : f\restriction_A = id_A \}$: automorphisms of M over A.
$A \subseteq M$

Lemma Assume M is strongly κ-homogeneous, κ-saturated, $A \subseteq M$, $|A| < \kappa$. Then:

(1) For $a, b \in M$ ($tp(a/A) = tp(b/A) \Leftrightarrow a, b$ are in the same orbit of $Aut(M/A)$ on M.

(2) $[\text{orbits } Aut(M/A) \text{ on } M^n] \underset{\text{onto}}{\overset{1\text{-}1}{\longleftrightarrow}} S_n(A)$

Proof (1) $\Leftarrow$: $f \in Aut(M/A)$, $f(a) = b$

$\Downarrow$

$tp(a/A) = tp(b/A)$.

$\Rightarrow$: $tp(a/A) = tp(b/A) \Rightarrow f: Aa \xrightarrow{\equiv} Ab$
$f\restriction_A = id_A$, $f(a) = b$

strong κ-homogeneity, $|A| < \kappa$ $\Rightarrow$ $f \subseteq g \in Aut(M)$, $g \in Aut(M/A)$
$g(a) = b$: a, b in the same orbit of $Aut(M/A)$

(2) $M^n \supseteq \mathcal{O} \xrightarrow[\varphi]{(1)} p_{\mathcal{O}} \in S_n(A)$

$\mathcal{O}$: orbit of $\mathrm{Aut}(M/A)$; $p_{\mathcal{O}}$ = common type $tp(a/A)$ for $a \in \mathcal{O}$.

Φ : {orbits of $\mathrm{Aut}(M/A)$ on M^n} $\xrightarrow{\varphi} S_n(A)$

$\mathcal{O}_1 \neq \mathcal{O}_2 \overset{(1)}{\Rightarrow} p_{\mathcal{O}_1} \neq p_{\mathcal{O}_2}$ $\boxed{\text{so } \varphi : 1\text{-}1}$

[if $p_{\mathcal{O}_1} = p_{\mathcal{O}_2}$ then let $a \in \mathcal{O}_1, b \in \mathcal{O}_2 \Rightarrow \exists g \in \mathrm{Aut}(M/A)$ $g(a) = b$ ↯.]

M: κ-saturated $\Rightarrow \varphi$: "onto".

<u>Def</u> Let $\bar{\kappa}$: a (large) cardinal number, $\mathcal{M} \models T$ <u>monster model</u> (w.r. to $\bar{\kappa}$), if $\mathcal{M}$: $\bar{\kappa}$-saturated, strongly $\bar{\kappa}$-homogeneous

Thm. Assume $\aleph_0 \leq \kappa \in CN$. Then $\exists M$: κ-saturated ~~&~~ strongly $\bar{\kappa}$-saturated.

<u>Proof</u> $M = \bigcup\limits_{\alpha < \kappa^+}^{\uparrow} M_\alpha$: union of elementary chain s.t.:

(1) $M_0 \models T$ any

(2) $M_\delta = \bigcup\limits_{\alpha<\delta} M_\alpha$ if $\delta \in Lim$,

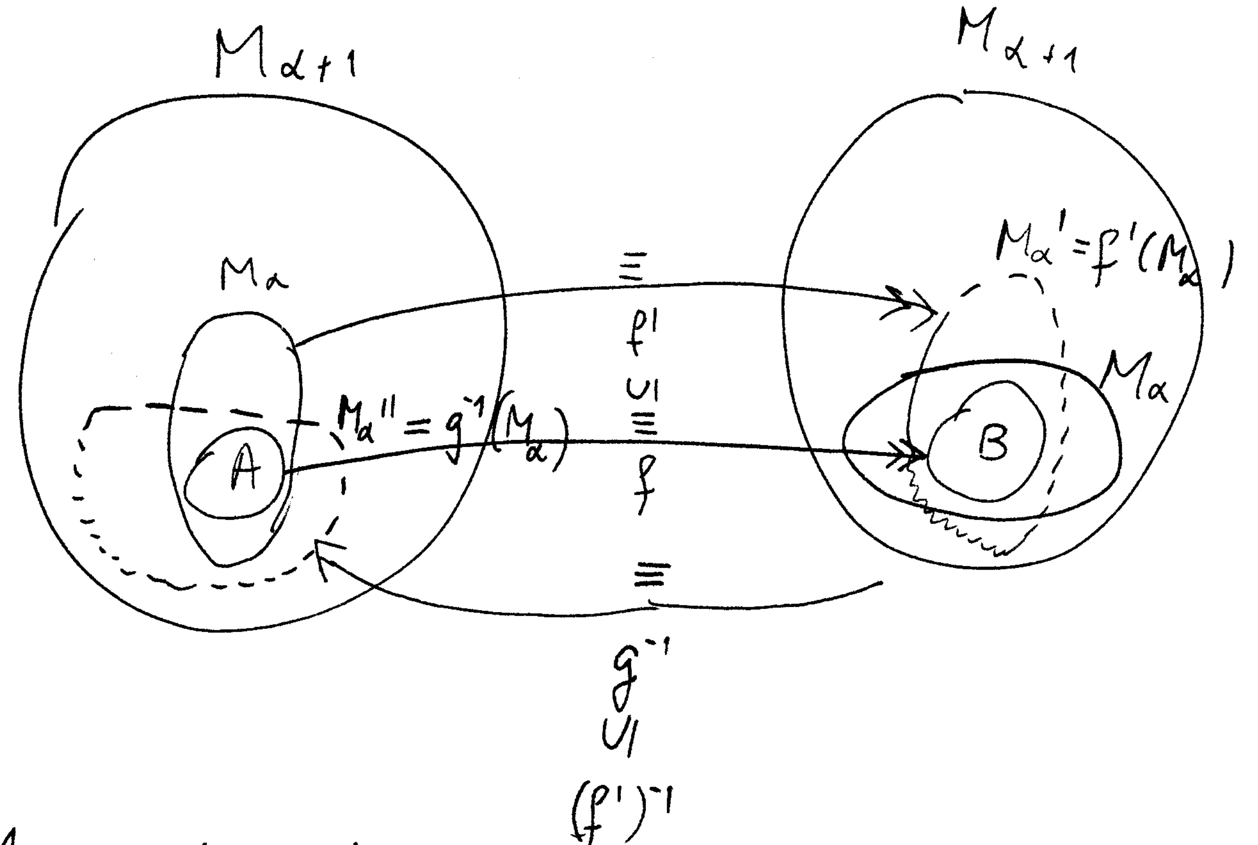
(3) $M_{\alpha+1} \succ M_\alpha$ s.t.:

(a) $\forall p \in S_1(M_\alpha)$ p realized in $M_{\alpha+1}$

(b) $\left(\forall f : \underset{M_\alpha}{\overset{\cap}{A}} \xrightarrow{\equiv} \underset{M_\alpha}{\overset{\cap}{B}}\right)(\exists\, g \supseteq f)\left(g : \underset{M_\alpha}{\overset{\cup}{A'}} \xrightarrow{\equiv} \underset{M_\alpha}{\overset{\cup}{B'}} \text{ in } M_{\alpha+1}\right)$

It is enough that $M_{\alpha+1}$ is $\|M_\alpha\|^+$-saturated,
To satisfy (a), (b). $M_{\alpha+1} \succ M_\alpha$

Proof of (b) for such $M_{\alpha+1}$:



I. M: $\varkappa$-saturated: <u>clear</u>

II M strongly $\varkappa$-homogeneous:

Assume $A \subset M$, $|A| < \varkappa$. Then $A \subseteq M_\alpha$, $B \subseteq M_\alpha$
$f : A \xrightarrow{\equiv} M$ for some $\alpha < \varkappa$.
$B = f[A]$

• we construct

a sequence f_β, $\alpha \le \beta < \varkappa^+$:

• increasing $f_\beta : M \xrightarrow{\overset{\equiv}{\emptyset}} M$ partial elementary

• $f \subseteq f_\alpha$

(*) $M_\beta \subseteq \mathrm{dom}\, f_\beta \cap \mathrm{rng}\, f_\beta$.

• f_α constructed according to 3)(b)

$$f_\alpha : M_{\alpha+1} \xrightarrow{\overset{\equiv}{\emptyset}} M_{\alpha+1}$$

• $f_\beta : M_{\beta+1} \xrightarrow{\overset{\equiv}{\emptyset}} M_{\beta+1}$, as in 3.(b) when β : successor.

• $f_\delta = \bigcup_{\beta<\delta} f_\beta$ when δ limit., still $f_\delta : M_\delta \xrightarrow{\overset{\equiv}{\emptyset}} M_\delta$.

$f_\infty = \bigcup^{\nearrow}_{\alpha \le \beta < \varkappa^+} f_\beta$, $f_\beta \in \mathrm{Aut}(M)$, $f \subseteq f_\beta$.

<u>Assumptions</u> Let $\bar\varkappa$: a cardinal number <u>large enough</u> so that:

(1) We consider only small models of T
$\quad$ small $\Vert$ of power $< \bar\varkappa$, or even $\ll \bar\varkappa$

(2) We work within a monster model $\mathfrak{M} \models T$ (w.r.to $\bar\varkappa$)

(3) We consider only small models $M \prec \mathfrak{M}$
$\quad$ small $\Vert$ of power $< \bar\varkappa$, or even $\ll \bar\varkappa$,

Consequences:

(1) For $M, N \prec \mathcal{M}$, $M \subseteq N \iff M \prec N$

(2) Convention: For $\bar{a} \subseteq \mathcal{M}$

$\models \varphi(\bar{a})$ means $\mathcal{M} \models \varphi(\bar{a})$

(3) For $A \subseteq M \prec \mathcal{M}$:

$S_n^M(A) = S_n^{\mathcal{M}}(A) =: S_n(A)$

<u>Notation</u> Assume $p(\bar{x})$, $q(\bar{x})$ types (small, over $\mathcal{M}$)

- $p(\bar{x}) \vdash q(\bar{x}) \iff p(\mathcal{M}) \subseteq q(\mathcal{M})$
  "p implies q"
- $p(\bar{x}) \underset{\substack{\uparrow \\ \text{equivalent}}}{\equiv} q(\bar{x}) \iff p \vdash q \;\&\; q \vdash p$

Special case: $p(\bar{x}) = \{\varphi(\bar{x})\}$.

$\varphi(\bar{x}) \vdash q(\bar{x})$: "φ isolates q".

<u>Remark</u>: Syntactically:

$p(\bar{x}) \vdash q(\bar{x}) \iff \forall \varphi(\bar{x}) \in q \;\; \exists \underset{\text{finite}}{p_0(\bar{x}) \subseteq p(\bar{x})}$

types over A

$T(A) \vdash \bigwedge p_0(x) \longrightarrow \varphi(x)$

Remark (exercise)

$p(\bar{x}) \vdash q(\bar{x}) \iff \underset{|A|^+\text{-saturated}}{\forall M \models T} \;\; p(M) \subseteq q(M)$.

Def. (reminder)
Let $p(\bar{x})$: a type over A.
p is isolated over $A \iff \exists \underbrace{\varphi(\bar{x}) \in L(A)}_{\text{consistent (with } T)}$ $\varphi \vdash p$.

Thm (omitting types, Ehrenfeucht)
Assume $p_n(\bar{x}_n)$, $n<\omega$: ~~non~~ a family of non-isolated types over $\emptyset$. Then $\exists M \models T$ $\forall n$ $\underbrace{p_n(M)=\emptyset}_{M \text{ omits } p_n}$.

Lemma
Assume A is ctble, $p_n(\bar{x}_n)$, $n<\omega$: a family of non-isolated types over A, $\varphi(x) \in L_1(A)$, $\underbrace{\varphi(\mathcal{M}) \neq \emptyset}_{\text{i.e. } \varphi \text{ : consistent}}$.

Then $\exists c \in \varphi(\mathcal{M})$ $\forall n$ p_n non-isolated over $A \cup \{c\}$.

Proof [Lemma $\Rightarrow$ Thm]
By the Lemma: $\exists \underbrace{\{a_n : n<\omega\}}_{= A} \subseteq \mathcal{M}$ s.t.

(1) A satisfies the TV-test

(2) p_n, $n<\omega$: non-isolated over A.

Construction of a_n, $n<\omega$: recursion on n.
Let $\{\varphi_n(x, \bar{y}) : n<\omega\}$: all formulas of L of this form.

Suppose $n<\omega$ and $\{a_i : i<n\} = a_{<n}$ already constructed

so that all $p_k, k<\omega$ still non-isolated over $a_{<n}$.

We consider a consistent formula $\varphi_n(x) \in L_1(a_{<n})$

By the Lemma we find $\underset{a_n}{\overset{\shortparallel}{c}} \in \varphi_n(\mathcal{M})$ so that

all p_k, $k<\omega$, still non-isolated over $a_{\leq n}$.

- the formulas $\varphi_n(x)$, $n<\omega$ may be chosen so that after ω steps:

  $\underset{\text{consistent}}{\forall \varphi(x) \in L(A)}\ \exists n\ \varphi = \varphi_n$.

Then $A = \{a_n : n<\omega\}$ satisfies TV-test

$A = M \prec \mathcal{M}$, every p_k still non-isolated over A.

$p_k(M) = \emptyset$ [if not, some $\underset{M}{\overset{\cap}{\overline{m}}} \models p_k$. then $(\overline{x}_k = \overline{m}) \vdash p_k(\overline{x}_k)$ ↯]

<u>Proof of the Lemma.</u>

Let $p(\overline{x})$: one of the types $p_n(\overline{x}_n)$.

Let $h(\overline{x}, y, \overline{a}) \in L(A$

$h(\overline{x}, c, \overline{a}) \vdash p(\overline{x}) \iff h(\mathcal{M}, c, \overline{a}) \subseteq p(\mathcal{M})$.

$\iff \forall \psi(\overline{x}) \in p(\overline{x})\ h(\mathcal{M}, c, \overline{a}) \subseteq \psi(\mathcal{M})$

$\iff \forall \psi \in p\ \ \mathcal{M} \models \forall \overline{x}\,(h(\overline{x}, c, \overline{a}) \rightarrow \psi(\overline{x}))$

$\iff \forall \psi \in p \quad \psi_h(y) \in tp(c/A)(y)$

where $\psi_h(y) = \forall \bar{x}\,(h(\bar{x}, y, \bar{a}) \to \psi(\bar{x}))$

<u>hence</u>:

$tp(c/A) = tp(c'/A) \Rightarrow [\,h(\bar{x}, c, \bar{a}) \vdash p \iff h(\bar{x}, c', \bar{a}) \vdash p\,]$

$h(\bar{x}, c, \bar{a})$ consistent $\iff (\exists \bar{x}\, h(\bar{x}, y, \bar{a})) \in tp(c/A)(y)$.

Let $\underset{\substack{\uparrow \\ \text{"bad types"}}}{X_{h,p}} = \{q \in S_1(A) :$ For $c \models q$, $h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$ and $h(\bar{x}, c, \bar{a})$ consistent.$\}$

Let $q \in S_1(A)$ then

$$q \in X_{h,p} \iff q(y) \in S_1(A) \cap \underbrace{[\exists \bar{x}\, h(\bar{x}, y, \bar{a})]}_{\substack{\text{For } c \models q \;\; h(\bar{x}, c, \bar{a}) \text{ consistent}}} \cap \underbrace{\bigcap_{\psi \in p} [\psi_h(y)]}_{\substack{\text{For } c \models q \;\; h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})}}$$

(*) $X_{h,p}$: nowhere dense in $S_1(A)$.

Proof of (*) : (a.a.)

Suppose $\theta(y) \in L_1(A)$ and $\emptyset \neq S_1(A) \cap [\theta] \subseteq X_{h,p}$.

Let $\underset{\substack{\cap \\ L(A)}}{\alpha(\bar{x})} = \exists y (h(\bar{x}, y, \bar{a}) \wedge \theta(y))$

• $\alpha(\bar{x})$: consistent :

Let $c \in \theta(\mathcal{M})$

$\Downarrow$ $[\theta] \subseteq X_{h,p}$)

$\mathcal{M} \models \exists \bar{x}\ h(\bar{x}, c, \bar{a})$.

Let $\bar{d} \subseteq \mathcal{M}$ s.t. $\mathcal{M} \models h(\bar{d}, c, \bar{a})$.

$\bar{d}$ satisfies in $\mathcal{M}$: $\underbrace{\exists y\, h(\bar{x}, y, \bar{a})}_{= \alpha(\bar{x})}$.

• $\alpha(\bar{x}) \vdash p(\bar{x})$, i.e. $\alpha(\mathcal{M}) \subseteq p(\mathcal{M})$.

Let $\bar{d} \in \alpha(\mathcal{M})$. So there is $c \in \mathcal{M}$ s.t.

$\models h(\bar{d}, c, \bar{a}) \wedge \theta(c)$

$\Downarrow$ $[\theta] \subseteq X_{h,d}$

$\forall \psi \in p \ \models \psi_h(c)$

$\Downarrow$

$\forall \psi \in p \ h(\bar{x}, c, \bar{a}) \vdash \psi(\bar{x}) \implies h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$

$\bar{d} \implies d$ ↯.

as : $p(\bar{x})$ non-isolated

Let $X = \bigcup_{h, p_n} X_{h,p_n} \subseteq S_1(A)$.

↑ meager. Let $q \in S_1(A) \cap [\varphi] \setminus X$

$c \models q$ good.

Pf (a.a) Suppose $p = p_n$ isolated over $A \cup \{c\}$.

$\exists\ h(\bar{x}, c, \bar{a})$ $h(\bar{x}, c, \bar{a}) \vdash p(x)$ $\implies q \in X_{h,p_n}$ ↯.

Consistent $\qquad q = tp(c/A)$

14.03.2022

Def T is *quantifier eliminable* if $\forall \varphi \in L \ \exists \underset{\text{open}}{\psi} \in L$ $\equiv$ q.f.

$T \vdash \varphi \leftrightarrow \psi$

Def. For $p(\bar{x}) \in S_n(\emptyset)$ let $\underline{p_0(\bar{x})} = \{\varphi(\bar{x}) \in p(\bar{x}) : \varphi \text{ open}\}$.

Remark T is q.e. $\iff \forall n \, \forall p \in S_n(\emptyset) \quad p_0 \vdash p$

Proof „$\Rightarrow$" Obvious. „$\Leftarrow$" Let $\varphi(\bar{x}) \in L$.

- $\forall p \in [\varphi] \cap S_n(\emptyset) \ \exists \underset{\text{open}}{\psi} \in p \quad p \in [\psi] \subseteq [\varphi]$

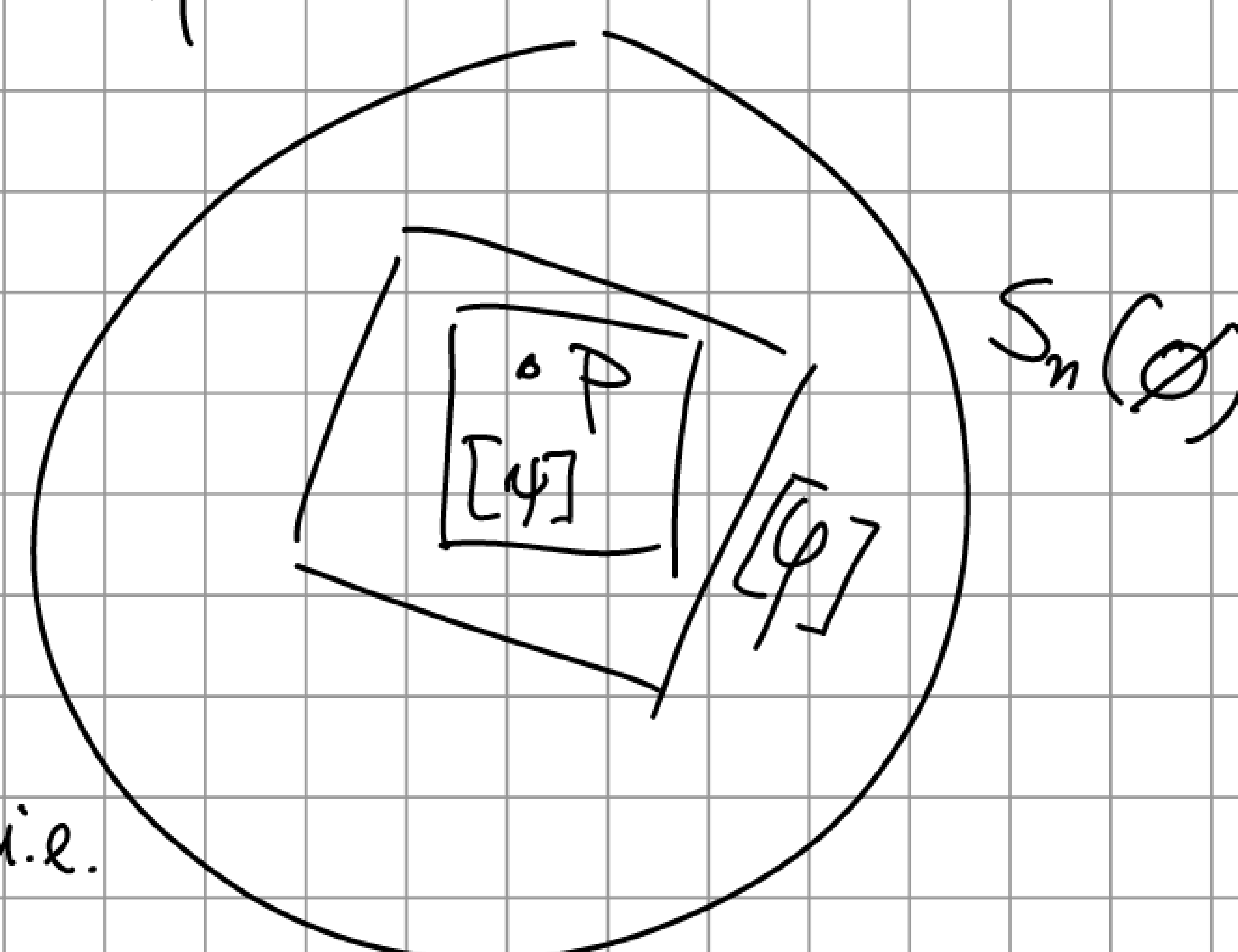
Why? $p_0 \vdash p$
$$\Downarrow$$
$p_0 \vdash \varphi$

by compactness

$\exists$ finite $p_0' \subseteq p_0$ s.t.

$p_0' \vdash \varphi$, i.e.

$p_0'(\mathcal{M}) \subseteq \varphi(\mathcal{M})$



$$\psi(\mathcal{M}) = \Big(\bigwedge_{\psi' \in p_0'} \psi'\Big)(\mathcal{M}) \subseteq \varphi(\mathcal{M}) \rightsquigarrow \mathcal{M} \models \psi(\bar{x}) \to \varphi(\bar{x})$$

$$\psi \vdash \varphi \Rightarrow [\psi] \cap S_n(\emptyset) \subseteq [\varphi] \cap S_n(\emptyset)$$

Application $L = \{+, \cdot, 0, 1\}$: the language of rings.

ACF_p: the theory of algebraically closed fields of char p, in L.

Axioms:

1) field axioms

2) char $= p \neq 0$: $\underbrace{1 + \dots + 1}_{p} = 0$

2') $p = 0$: $\underbrace{1 + \dots + 1}_{n} \neq 0$ for $n \geq 1$

3) Every polynomial of deg n has a root:
$0 < n$

$$\forall y_{n-1}, y_{n-2}, \dots, y_0 \, \exists x \; x^n + y_{n-1} x^{n-1} + \dots + y_0 = 0.$$

Fact ACF_p is complete.

Proof Let $M, N \models ACF_p$. Enough to show that $M \equiv N$. Let $\kappa > \|M\|, \|N\|$ and let $\underbrace{M' \succ M, \; N' \succ N}_{\text{power } \kappa}$.

M', N': uncountable acl fields of the same power and char

$$\Downarrow \text{algebra}$$

$$M' \cong N' \Rightarrow M' \equiv N'$$

$$\begin{array}{ccc} M' & \equiv & N' \\ \text{\rotatebox{90}{$\equiv$}} & \Downarrow & \text{\rotatebox{90}{$\equiv$}} \\ M & \equiv & N \end{array}$$

<u>Fact</u> ACF_p is q.e. (Chevalley, Tarski)

<u>Proof</u> (in $\mathcal{M}$) We will show that $\forall p \in S_n(\emptyset)$ $p_0 \vdash p$ ($\Leftrightarrow p_0(\mathcal{M}) \subseteq p(\mathcal{M})$). Let $\bar{a} \models p_0$, $\bar{b} \models p$, $\bar{a}, \bar{b} \subseteq \mathcal{M}$. It's enough to prove that $\exists f \in Aut(\mathcal{M})\ f(\bar{a}) = \bar{b}$.

$\bar{a} = (a_1, \dots, a_n)$
$\bar{b} = (b_1, \dots, b_n)$

Let $\langle \bar{a} \rangle, \langle \bar{b} \rangle$: the subrings with 1 of $\mathcal{M}$ generated by $\bar{a}, \bar{b}$. $\bar{a}, \bar{b} \models p_0 \Rightarrow \langle \bar{a} \rangle \cong \langle \bar{b} \rangle$

$a_i \overset{f}{\longmapsto} b_i$

Some algebraic magic.

$$\langle \bar{a} \rangle \cong \langle \bar{b} \rangle$$

$$\cap \quad \Downarrow \text{unique} \quad \cap$$

$$\mathcal{M} \supseteq \langle \bar{a} \rangle_0 \cong \langle \bar{b} \rangle_0 \subseteq \mathcal{M}$$

(fraction field) (not unique) $\mathcal{M} \supseteq \underbrace{\langle \bar{a} \rangle_0^{alg}}_{F_a} \cong \underbrace{\langle \bar{b} \rangle_0^{alg}}_{F_b} \subseteq \mathcal{M}$

$$\operatorname{trdeg}(\mathcal{M}/F_a) = \|\mathcal{M}\| = \operatorname{trdeg}(\mathcal{M}/F_b)$$

$$\Downarrow$$

$$\begin{array}{ccc} & \mathcal{M} \cong F_a(X_\alpha,\ \alpha<\lambda)^{alg} \\ f & \cong\downarrow \qquad\qquad \cong \\ & \mathcal{M} \cong F_b(X_\beta,\ \beta<\lambda)^{alg} \end{array}$$

$$f \in \operatorname{Aut}(\mathcal{M}),\ f(\bar{a}) = \bar{b}.$$

■

Types in $T = ACF_p$

Let $\mathcal{M} \models T$: a monster model.

subfield (small) $K \subseteq \mathcal{M}$. We will describe $S_n(K)$.

Let $\bar{a} \subseteq \mathcal{M}$, $|\bar{a}| = n$.

$K[\bar{x}] \rhd I(\bar{a}/K) = \{ f \in K[\bar{x}] : f(\bar{a}) = 0 \}$

Remark 1) $tp(\bar{a}/K) = tp(\bar{a}'/K) \iff I(\bar{a}/K) = I(\bar{a}'/K)$

2) $\forall \underset{\text{prime}}{I} \lhd K[\bar{x}]\ \exists \bar{a} \subseteq \mathcal{M}\ I(\bar{a}/K) = I$.

Proof 1) "$\Rightarrow$" $tp(\bar{a}/K) = tp(\bar{a}'/K) \Rightarrow \exists f \in \operatorname{Aut}(\mathcal{M}/K)$ $f(\bar{a}) = \bar{a}'$

$\underline{I(\bar{a}/K) = I(\bar{a}'/K)}$

$\left[\text{alternatively: } f \in I(\bar{a}/K) \iff \text{"} f(\bar{x}) = 0 \text{"} \in tp(\bar{a}/K)\right]$

"$\Leftarrow$" Assume $I(\bar{a}/K) = I(\bar{a}'/K) = I$.

$$\Downarrow$$

$$K[\bar{a}] \underset{K}{\cong} K[\bar{x}]/I \cong K[\bar{a}']$$

$$\Downarrow$$

$$\exists f \in Aut(\mathcal{M}/K)\ f(\bar{a}) = \bar{a}'$$

$$\Downarrow$$

$$tp(\bar{a}/K) = tp(\bar{a}'/K).$$

2) $K \subseteq K[\bar{x}]/I = K[\bar{a}] \supseteq \bar{a} = \bar{x}/I$ and $I(\bar{a}/K) = I$.

$$\vdots$$

$\square$

$S_1(K) = \{ tp(a/K) : a \in \mathcal{M} \}$: a top. space.

$$\Downarrow$$

$$p(x) = tp(a/K),\ I_p = I(a/K) \lhd K[x]$$

a) $I_p \neq \{0\}$, i.e. a is algebraic $/K$, so

$0 \neq f$ (irreducible over K) "$f(x) = 0$" $\in p(x)$. In fact,

"$f(x) = 0$" $\vdash p(x)$ (isolates)

Let $a' \in \mathcal{M}$ s.t. $f(a')=0 \Rightarrow I(a'/K) \ni f \Rightarrow$

f generates $I(a'/K)$

$p = tp(a/K) = tp(a'/K) \Leftarrow I(a/K) = I(a'/K) \Leftarrow$

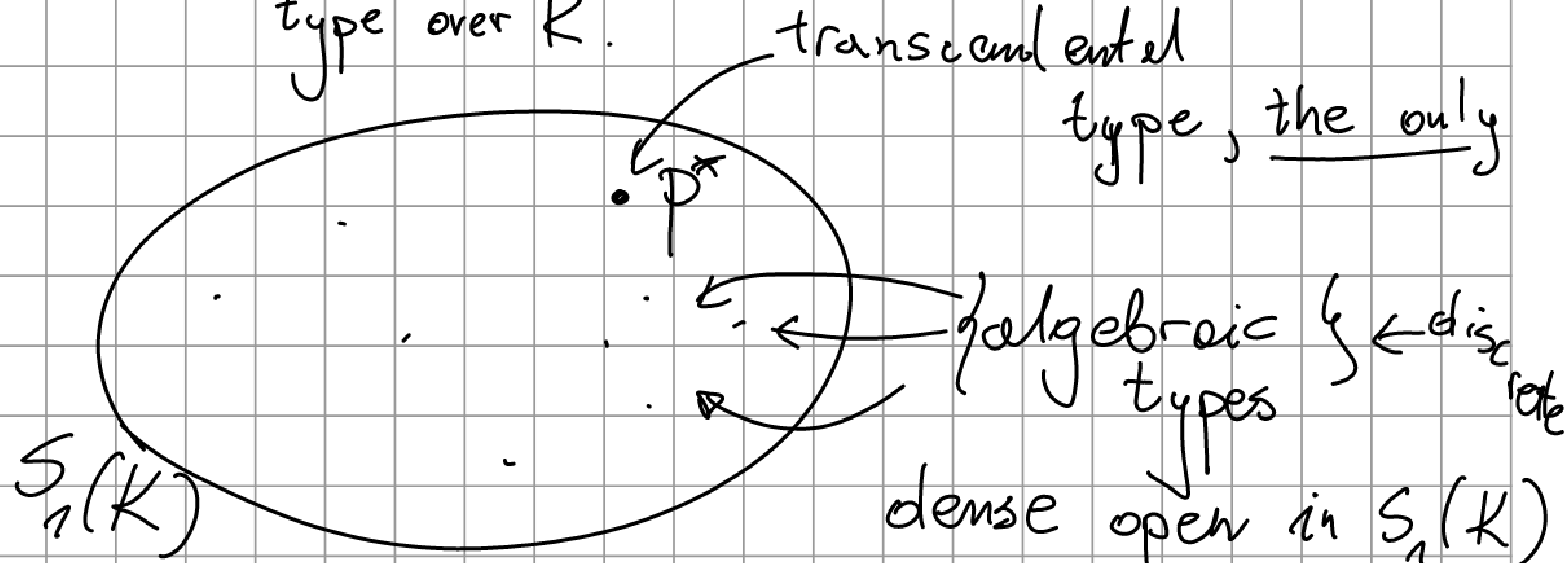
$p(x)$ here is called <u>algebraic</u>.

More generally (T: arbitrary)

<u>Def</u> $\varphi(\bar{x}) \in L(A)$ is called <u>algebraic</u>, if $0 < |\varphi(\mathcal{M})| < \aleph_0$, similarly for q: a type.

b) $I_p = \{0\}$: $p = tp(a/K)$ s.t. $a \in \mathcal{M}$ transcendental over K

↑ the transcendental type over K.



If K ctble then $S_1(K)$ ctble

Corollary ACF_p is $\aleph_0$-stable [recall: T is $\varkappa$-stable $\iff \forall A \subseteq \mathcal{M}, |A| \leq \varkappa$ $|S_1(A)| \leq \varkappa$]

Proof Let $\underset{\text{ctble}}{A} \subseteq \mathcal{M}$.

$$A \subseteq \underset{\substack{\uparrow \\ \text{ctble} \\ \text{subfield}}}{K} \subseteq \mathcal{M} \quad |S_1(A)| \leq |S_1(K)| = \aleph_0$$

Remark T is totally transcendental $\iff T$: $\aleph_0$-stable

Proof "$\Rightarrow$" from def, "$\Leftarrow$": (A.a.) Let $\varkappa > \aleph_0$.

Suppose $|A| \leq \varkappa < |S_1(A)|$ for some $A \subseteq \mathcal{M}$.

Shall find $\underset{\text{ctble}}{A_0} \subseteq A$ with $|S_1(A_0)| \geq 2^{\aleph_0}$.

(def) $\varphi(x) \in L(A)$ is <u>large</u> iff $|S_1(A) \cap [\varphi]| > \varkappa$

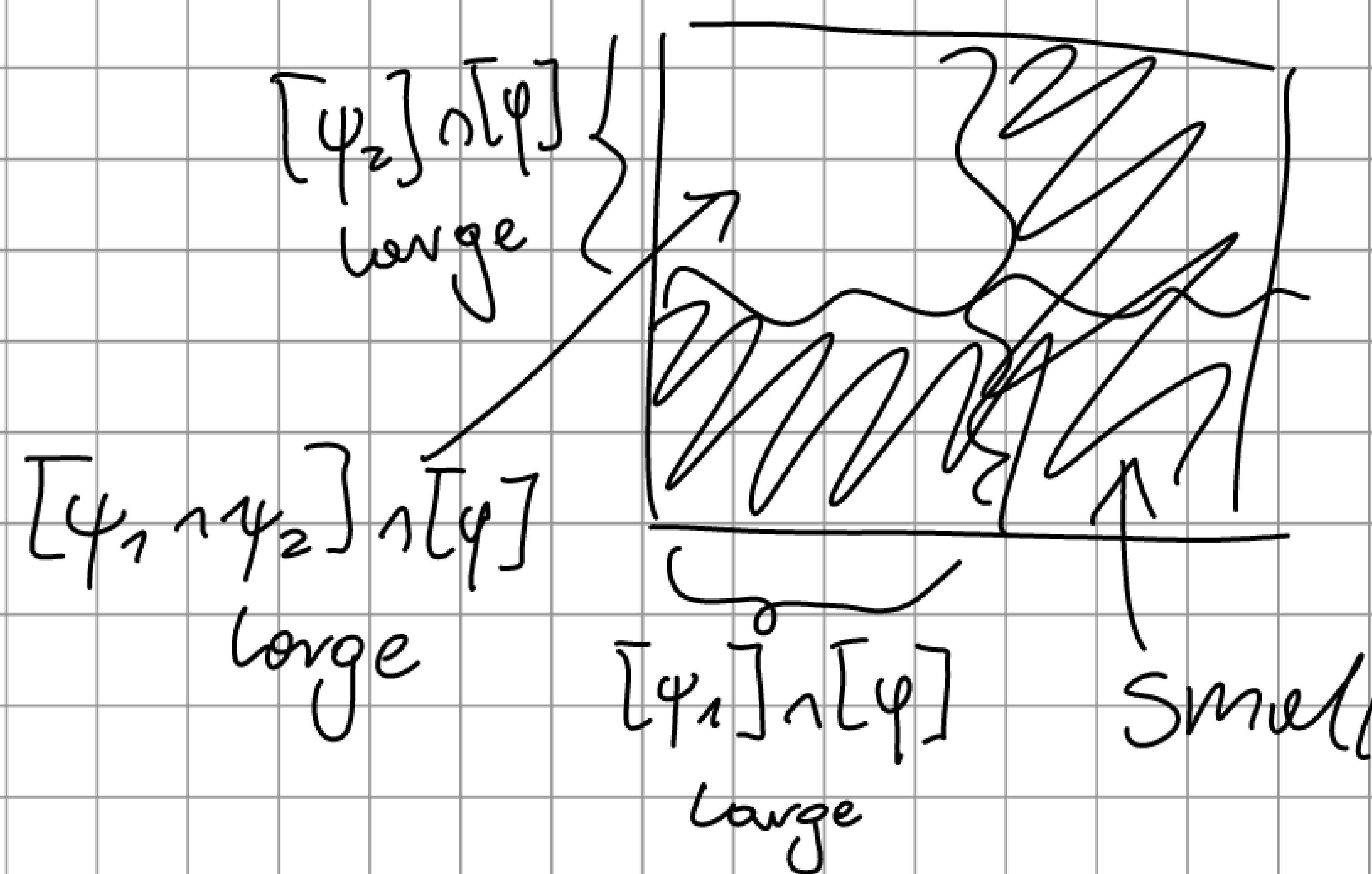
otherwise: $\varphi(x)$ small.

(a) "$x = x$" is large

(b) if $\varphi(x)$: large, then $\exists \psi_1, \psi_2 \in L(A)$ large s.t. $\varphi(\mathcal{M}) = \psi_1(\mathcal{M}) \,\dot\cup\, \psi_1(\mathcal{M})$

Pf. (b) <u>if not</u> then $p^* = \{\psi \in L_1(A) : \psi \wedge \varphi$ is large$\}$ is a complete type is $S_1(A) \cap [\varphi]$

1° p^*: consistent: $\psi_1, \psi_2 \in p^* \Longrightarrow \psi_1 \wedge \psi_2 \in p^*$



2° p^*: complete <u>OK</u>

$p^* \in S_1(A)\cap[\varphi]$: the only large type.

$$\underbrace{S_1(A)\cap[\varphi]}_{>\varkappa} = \underbrace{\{p^*\} \cup \bigcup_{\substack{\psi\in L_1(A)\\ \text{small}\\ \underbrace{\psi\vdash\varphi}_{\leq\varkappa}}} \underbrace{\left(S_1(A)\cap[\psi]\right)}_{\leq\varkappa}}_{\leq\varkappa}$$

↯

c) a tree of large formulas in $L_1(A)$ $\varphi_\eta(x)$,

$\eta \in 2^{<\omega}$ s.t. $\varphi_\eta(\mathcal{M}) = \varphi_{\eta 0}(\mathcal{M}) \dot\cup \varphi_{\eta 1}(\mathcal{M})$

by (b)

Let $A_0 \subseteq A$: the set of all params of $\varphi_\eta, \eta \in 2^{<\omega}$

Then $|A_0| \leqslant \aleph_0$.

For $\eta = 2^{\omega}$: $p^0_\eta = \{\varphi_{\eta|n}(x) : x < \omega\}$: a consistent 1-type over A_0

$\cap$

When $\nu \neq \eta$ then $p_\eta \neq p_\nu$ $\quad$ $p_\eta \in S_1(A_0)$

$\Downarrow$

$$|S_1(A_0)| \geqslant 2^{\aleph_0} > \aleph_0 \quad \lightning$$

21.03.2022

CONSTRUCTION OF SPECIAL MODELS: $N, M \models T$

Def. (1) M is atomic if $\forall \bar{a} \subseteq M$ (finite) $tp(\bar{a}/\emptyset) = tp(\bar{a})$ is isolated.

(2) M is prime if $\forall N \models T \; \exists f: M \overset{\equiv}{\Rightarrow} N$

Example $T = ACF_p$, F_p: prime field of char p

(a) $\hat{F}_p$: atomic (exercise)

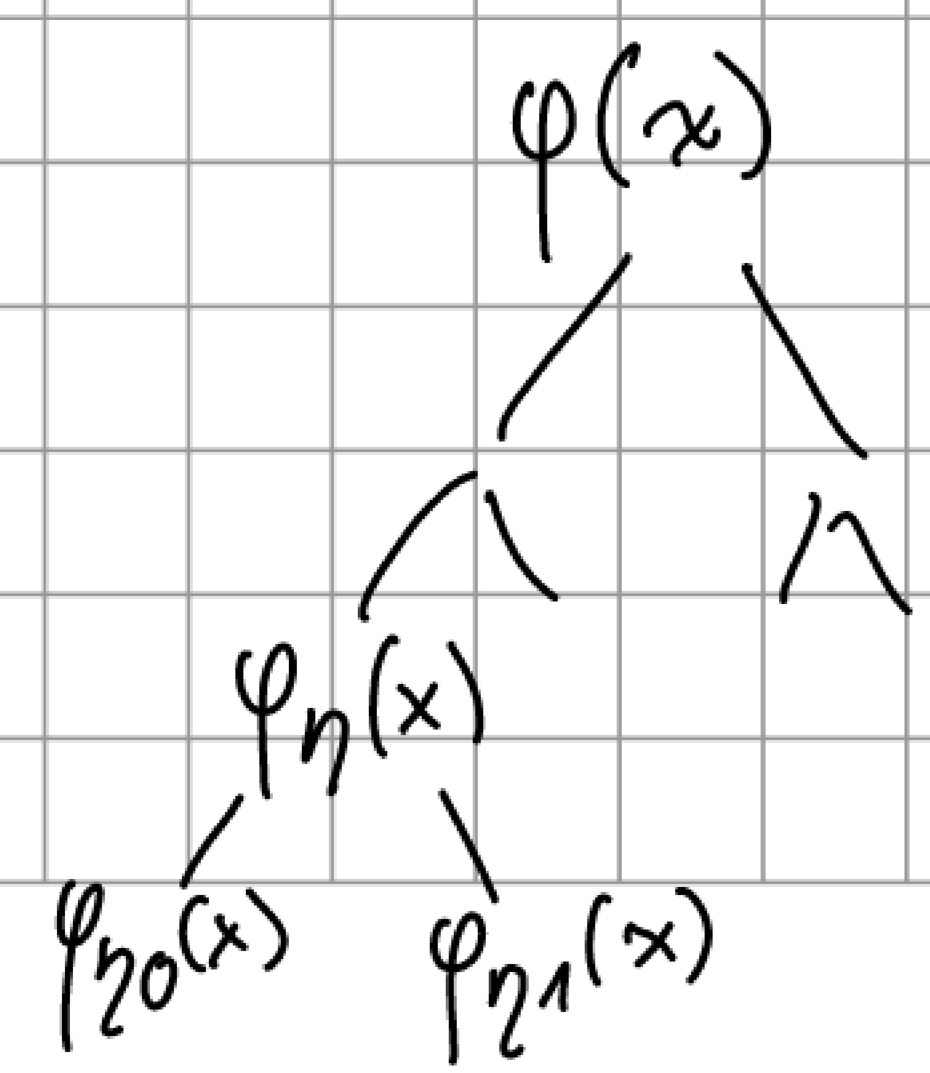
(b) $\hat{F}_p$: prime (exercise)

Thm. T: $\aleph_0$-stable $\Rightarrow$ T has a prime model.

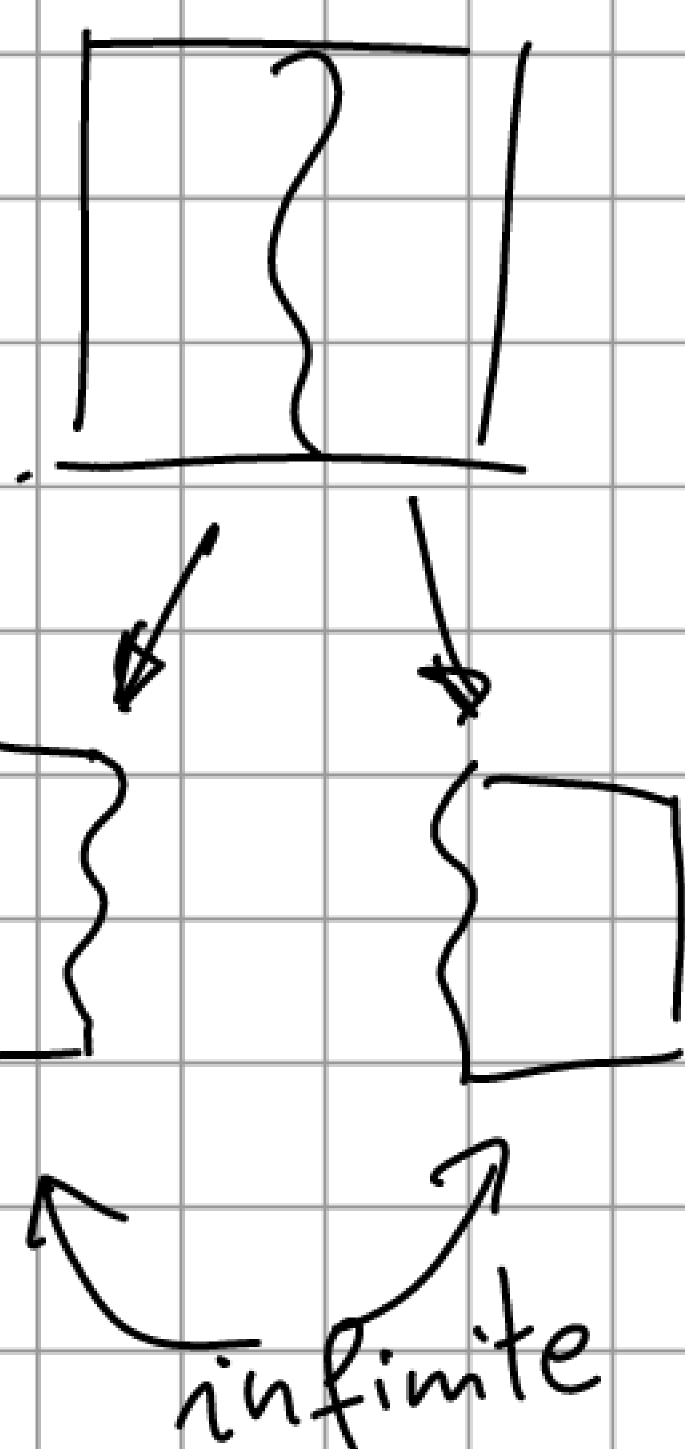
Lemma 1 T: $\aleph_0$-stable $\Rightarrow$ $\forall A \subseteq \mathcal{M}$ $\{$isolated types$\} \subseteq S_1(A)$ dense

Pf. Suppose $\varphi(x) \in L_1(A)$ (consistent with T) s.t. in $S_1(A) \cap [\varphi]$

there is no isolated types $\Rightarrow$ a tree of formulas $\varphi_\eta(x) \in L_1(A)$, $\eta \in 2^{<\omega}$



$\rightsquigarrow$ contradicts $\aleph_0$-stability of T

clopen 

$\subseteq S_1(A) \cap [\varphi]$

$\square$

<u>Lemma 2</u> $(a, b \in \mathcal{M})$ $tp(a)$ isolated and $tp(b/a)$ isolated $\iff$ $tp(ab)$ isolated

Pf „$\Rightarrow$": $\varphi(x) \vdash tp(a)$, $\psi(a,y) \vdash \underbrace{tp(b/a)(y)}_{= p_a(y) \subseteq S_y(a)}$

Then: $\varphi(x) \wedge \psi(x,y) \vdash tp(ab)$

Let $a', b' \in \mathcal{M}$ satisfy $\varphi(x) \wedge \psi(x,y)$

$\models \varphi(a') \Rightarrow tp(a') = tp(a) \Rightarrow \psi(a',y) \vdash \underset{\substack{\cap \\ S_y(a')}}{p_{a'}(y)}$

$\models \psi(a',b') \Rightarrow \models p_{a'}(b')$

$\Downarrow$

$ab \xrightarrow{\equiv} a'b'$

and $tp(ab) = tp(a'b')$

"$\Leftarrow$": $\theta(x,y) \vdash tp(a,b)$.

(a) "$\exists y\, \theta(x,y)$" $\vdash tp(a)$.
$\quad\uparrow$
$tp(a)$

<u>because</u> Let $a' \in \mathcal{M}$ satisfy "$\exists y\, \theta(x,y)$".

So there is b' s.t. $\models \theta(a',b')$

$\Rightarrow tp(ab) = tp(a',b') \Rightarrow tp(a) = tp(a')$.

(b) $\theta(a,y) \vdash tp(b/a)(y)$

<u>because</u>: similar to (a) $\blacksquare$

<u>Proof of the thm.</u> Construction of a prime model of T:

$A = \{a_n : n < \omega\} \subseteq \mathcal{M}$ so that:

1) A satisfies TV-test

2) $\forall n$ $tp(a_n / a_{<n})$ is isolated

At step n choose a_n: $\quad [a_{<n} = \{a_k : k < n\}]$

Let $\varphi(x) \in L_1(a_{<n})$ consistent.

Let $a_n \in \varphi(\mathcal{M})$ s.t. $tp(a_n / a_{<n})$ is isolated
(lemma 1)

Suitable choice of φ's ensures (1).

$M \models T$ is prime. Let $\underset{\text{arbitrary}}{N} \models T$. We find $f(a_n) \in N$ for $n < \omega$ s.t. $\exists f : M \xrightarrow{\equiv} N$.

At step n $f[a_{<n}] \subseteq N$ with $f : a_{<n} \xrightarrow{\equiv} N$

Let $p(x) = tp(a_n / a_{<n})$ (isolated)

$$\Downarrow$$

$$\underset{\text{isolated}}{f^*(p)(x)} \in S(f(a_{<n}))$$

too, hence realised by $f(a_n)$

$$\Downarrow$$

$$f : a_{\leq n} \xrightarrow{\equiv} N$$

$\blacksquare$

<u>Remark</u> (1) A prime model $M \models T$ is atomic.

(2) If $M \models T$ ctble atomic, then M prime

<u>Corrolary</u> $\hat{F}_p$ is atomic.

Proof of remark

(1) Let $p(\bar{x}) \in S_n(\emptyset)$ non-isolated. Will show $p(M) = \emptyset$. Let $N \models T$ be ommiting p.

$\exists f : M \xrightarrow{\equiv} N \Rightarrow p(M) = \emptyset$.

(2) Let $\underset{\overset{\shortparallel}{T}}{M} = \{a_n : n < \omega\}$ atomic.

Then $\forall n \; tp(a_{\leq n})$ is isolated

$$\Downarrow \text{lemma 2}$$

$\forall n \; tp(a_n / a_{<n})$ is isolated

$$\Downarrow \text{pf of thm}$$

M prime. ▨

Corollary A prime model of T is unique (up to $\cong$)

Proof Let $M, N \models T$ both prime $\underset{\text{remark}}{\Rightarrow}$ M, N are ctble and atomic, so we have embeddings in both direction, using back-and-forth we get the iso.

Def $M \models T$ is <u>minimal</u> if $\neg \exists N \precneqq M$

<u>Example</u> $\widehat{\mathbb{F}_p}$ is minimal.

<u>Fact</u> T has a prime model $\iff \forall n \ \{\text{isolated types}\} \underset{\text{dense}}{\subseteq} S_n(\emptyset)$

<u>Proof</u> „$\Rightarrow$": Let $\underset{\text{prime}}{M} \models T \Rightarrow M$: atomic

what we need

$N \models T$, then $\forall n \ \{p \in S_n(\emptyset) : p(N) \neq \emptyset\}$ is dense in $S_n(\emptyset)$
any
exercise

„$\Leftarrow$": <u>Claim</u> Assume $\underset{\text{finite}}{\bar{a}} \subseteq \mathcal{M}$ and $tp(\bar{a})$ is isolated.

Then $\{\text{isolated types}\} \underset{\text{dense}}{\subseteq} S_1(\bar{a})$.

<u>Proof of claim</u> Let $n = |a|$, $\varphi(\bar{x}) \vdash tp(\bar{a})$.

Let $\psi(\bar{x}, y) \in L_{n+1}(\emptyset)$ s.t. $\psi(\bar{a}, y)$ is consistent.

We seek $q(y) \in S_1(A) \cap [\psi(\bar{a}, y)]$ isolated.

Let $\chi(\bar{x}, y) = \varphi(\bar{x}) \wedge \psi(\bar{x}, y)$.

By assumptions of $\Leftarrow$ $\exists \underset{\text{isolated}}{p(\bar{x}, y)} \in S_{\bar{x}, y}(\emptyset) \cap [\chi(\bar{x}, y)]$.

Let $\bar{a}', b' \models p(\bar{x}, y)$. Then $\bar{a}' \models \underbrace{p(\bar{x}, y) \restriction_{\bar{x}}}_{\supseteq \varphi(\bar{x})} = tp(\bar{a})$

Let $f \in Aut(\mathcal{M}) : f(\bar{a}') = \bar{a}$
$b = f(b')$

Then $\bar{a}'b' \xrightarrow[f]{\equiv} \bar{a}b \Rightarrow \bar{a}b \models p(\bar{x}, y)$

so $\underset{\overset{\parallel}{p}}{tp(\bar{a}b)}$ is isolated $\underset{\text{lemma 2}}{\Rightarrow}$ $tp(b/\bar{a})$ isolated

$$\begin{array}{c} \psi \\ \psi(\bar{a}, y) \end{array}$$

So $q(y) = tp(b/\bar{a})$

Claim $\rightsquigarrow$ we construct a model $M = \{a_n : n < \omega\} \prec \mathcal{M}$

s.t. $\forall n$ $tp(a_n / a_{<n})$ is isolated

$\Downarrow$ lemma 2

M atomic ctble $\Rightarrow$ M prime.

<u>Corollary</u> If $\forall n$ $|S_n(\emptyset)| \leq \aleph_0$, then T has a prime model.

<u>Corollary</u> A prime model (of a ctble T) is homogeneous (exercise).

The number of countable models of T: $I(T, \aleph_0)$, $n(T)$.

<u>Remark</u> $1 \leqslant n(T) \leqslant 2^{\aleph_0}$

$$\underset{\text{ctble}}{M} \models T \Rightarrow M \cong \underbrace{(\mathbb{N}, \ldots)}_{\leqslant 2^{\aleph_0}}$$
$\leqslant 2^{\aleph_0}$ L-structures like that

<u>Recall</u> $n(T) = 1 \Leftrightarrow \forall n \ |S_n(\emptyset)| < \aleph_0$

(T: $\aleph_0$-categorical)

<u>Vaught conjecture (1961)</u>

$$n(T) > \aleph_0 \Rightarrow n(T) = 2^{\aleph_0}$$

<u>Thm</u> (M. Morley, 1971) $\aleph_0 < n(T) < 2^{\aleph_0} \Rightarrow n(T) = \aleph_1$

<u>Thm</u> (Vaught, 1961) $n(T) \neq 2$

<u>Proof</u> (A.a) suppose $n(T) = 2$.

$$n(T) < 2^{\aleph_0} \Rightarrow T \text{ small } (\text{i.e. } \forall n \ |S_n(\emptyset)| \leqslant \aleph_0)$$

$\vdots$

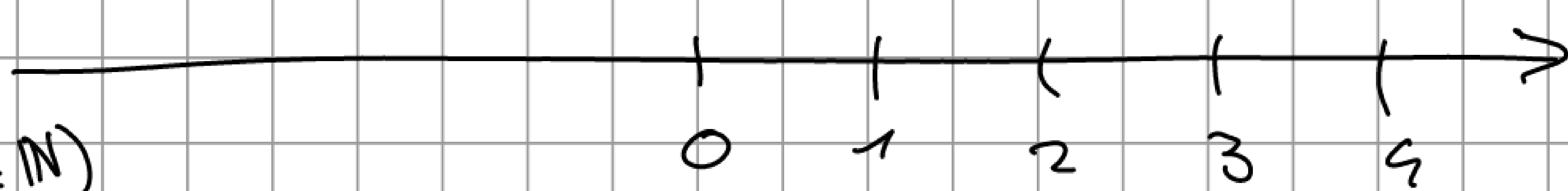
25.03.2022

Example (Andrzej Ehrenfeucht) Theory with exactly 3 ctble theories.

$T_0 = Th(\mathbb{Q}, \le)$. Look at $T = T_0(\mathbb{N}) = Th(\mathbb{Q}, \le, n)_{n\in\mathbb{N}}$.

T_0 is q.e. $\Rightarrow$ T is also q.e.

$S_1^T(\emptyset) = S_1^{T_0}(\mathbb{N})$. The types in $S_1^T(\mathbb{N})$:



realised in $(\mathbb{Q}, \le \mathbb{N})$

isolated

- $p_i(x) \equiv \{x = i\}$, $i \in \mathbb{N}$
- $r_i(x) \equiv \{i-1 \le x \le i\}$, $i \in \mathbb{N}$, $-1 \approx -\infty$
- $s(x) \equiv \{x > i : i \in \mathbb{N}\}$

ommited

We will point 3 ctble models $N \models T$.

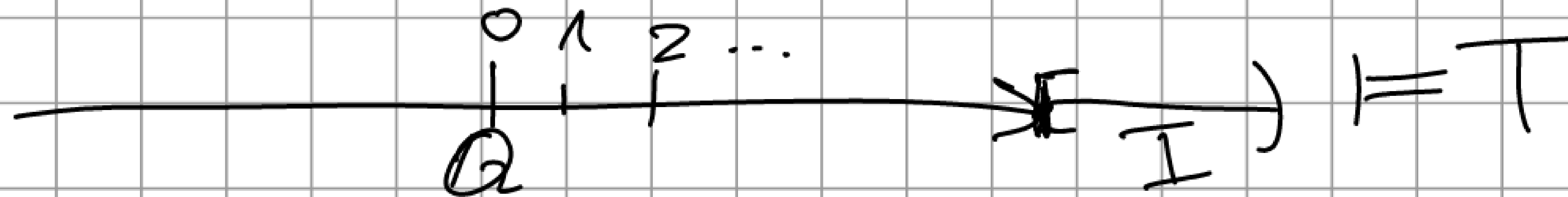
1° $N = M$: prime model of T.

2° $s(N)$ has the minimal element.

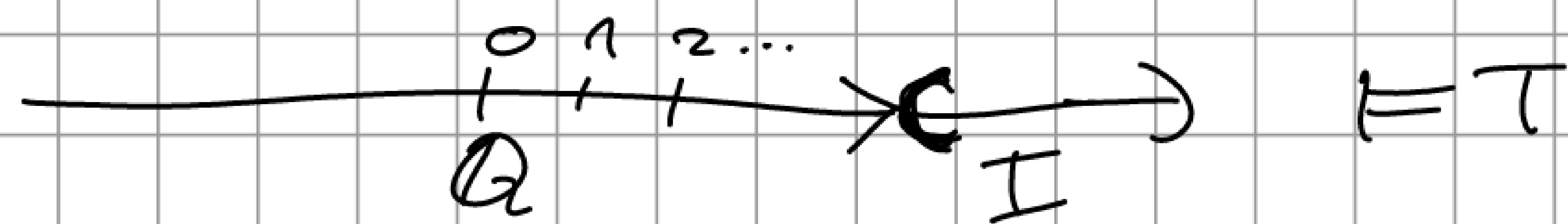
3° $s(N)$ hasn't minimal element.

2°: "$\mathbb{Q} \cup \big([0,1) \cap \mathbb{Q}\big)$":

$\mathbb{N}$:



3°:



Variants: 3, 4, 5, 6, ...

<u>Problem</u> Does there exist a stable T with

$1 < n(T) < \aleph_0$?

<u>Def</u> ($A \subseteq_{\text{small}} \mathfrak{M}$). $M \prec \mathfrak{M}$ is <u>prime over A</u> if:

(1) $A \subseteq M$

(2) $\forall \underset{A}{\underset{\cup\!|}{N}} \prec \mathfrak{M} \ \exists f : M \xrightarrow{\equiv} N : f\restriction_A = \mathrm{id}_A$

<u>Equivalently</u> M is a prime model of $T(A)$.

- If A is ctble $\longrightarrow$ full description of prime models of $T(A)$.
- If A is unctble $\longrightarrow$ in general not much can be said

Thm T $\aleph_0$-stable $\Longrightarrow$ $\forall \underset{\mathfrak{M}}{\overset{\cap}{A}}\ \exists \underset{\text{prime}}{M} \models T(A)$

Proof $M = A \cup \{a_\alpha : \alpha < ?\}$

Construction of a_α's s.t.:

(i) $A \cup \{a_\alpha : \alpha < ?\}$ satisfies the TV-test

(ii) $\forall \alpha < ?$ $tp(a_\alpha / A a_{<\alpha})$ is isolated.

At some point it has to terminate

(i.e. we cannot add more elements).

Claim Assume $\underset{A}{\overset{\shortparallel}{N}} \prec \mathfrak{M}$. Then $\forall \alpha\ \exists f : A \cup a_{<\alpha} \xrightarrow{\equiv} N$

s.t. $f\restriction_A = id_A$.

Proof We define $f(a_\beta)$ for all $\beta < \alpha$

by ind. on β so that $f\restriction_A = id_A$ and $f : A a_{\leq \beta} \xrightarrow{\equiv} N$.

Take $\beta < \alpha$ and suppose $\forall \beta' < \beta$ $f(a_{\beta'})$

so that the condition holds.

$p(x) = tp(a_\beta / A a_{<\beta})$ is isolated.

$f : A a_{<\beta} \xrightarrow{\equiv} f[A a_{<\beta}] \subseteq N$

$\underset{\text{isolated}}{p(x)} \in S_1(A a_{<\beta}) \overset{f^*}{\rightsquigarrow} \underset{\text{isolated}}{f(p)} \in S_1(f[A a_{<\beta}])$

$f(p)$ is realised by c

Now we put $f(a_\beta) = c$.

Claim $\blacksquare$

By the claim after some time we cannot get any more elements.

Additional property of the construction:

At the step α we consider a $\underbrace{\ }_{\text{consistent}}$ formula $\varphi(x) \in L(A a_{<\alpha})$ with no realisation in $A a_{<\alpha}$, choose a_α s.t. $\models \varphi(a_\alpha)$.

Problems Is a prime model over A unique up to isomorphism over A?

Answer: not always. However the prime model M over A constructed by the previous construction is unique up to $\underset{A}{\cong}$ and it's called primary over A.

Thm M, N: primary over $A \Rightarrow M \underset{A}{\cong} N$.

<u>Proof</u> $M = A \cup \{a_\alpha : \alpha < \gamma\}$: an „isolated construction" of M over A, i.e. $tp(a_\alpha / A a_{<\alpha})$ is isolated by a formula $\varphi_\alpha(x)$ over $A a_{C_\alpha}$ s.t. $\underset{\text{finite}}{C_\alpha} \subseteq \gamma$

<u>Def.</u> $X \subseteq \gamma$ is <u>closed</u> if $\forall \alpha \in X \; C_\alpha \subseteq X$

<u>Remark</u> (1) $\alpha \in \gamma \Rightarrow \exists$ minimal $\underset{\text{finite}}{X \subseteq \alpha}$ s.t. $\underbrace{X \cup \{\alpha\}}_{cl(\alpha)}$ is closed.

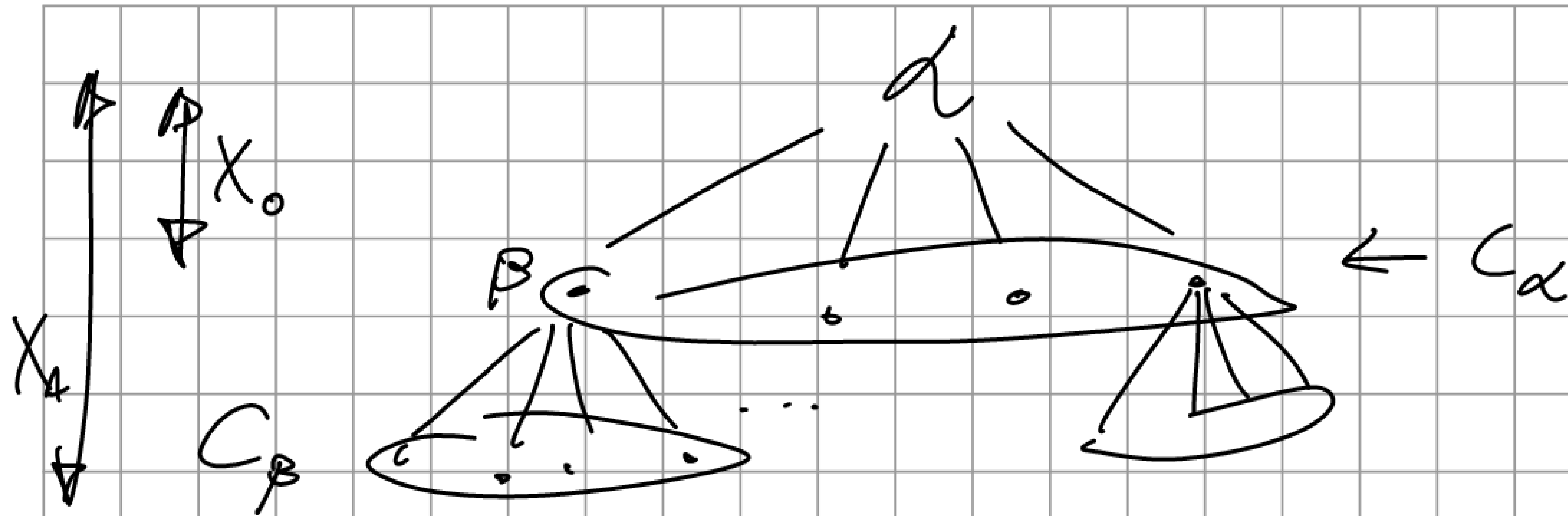
(2) A union of family of closed subsets of γ is closed.

<u>Proof</u> (remark) (2) is obvious.

(1): Take $X_0 = C_\alpha$, then $X_n = X_{n+1} \cup \bigcup_{\beta \in X_n} C_\beta$

finite because C_β finite

Then $X = \bigcup_n^{\nearrow} X_n$. Then $X \cup \{\alpha\}$ is finite and closed!



This tree has no infinite branch (because there are no infinite decreasing sequence of ordinals).

remark $\square$

<u>Remark</u> Assume X : closed. Then
$X \subseteq \gamma$

$$A \cup \langle a_\alpha : \alpha \in X \rangle ^\frown \langle a_\alpha : \alpha \in \gamma \setminus X \rangle$$

concatenation

is an isolated construction over A.

<u>Pf. (remark)</u> (1) $A \cup \langle a_\alpha : \alpha \in X \rangle$ is an i-construction over A (by the fact that X is closed)

(2) Suppose that $\alpha < \gamma$ and $\alpha \notin X$. Will show that $tp(a_\alpha / A a_X a_{X^c \cap (<\alpha)})$ is isolated.

$\varphi_\alpha(a) \vdash tp(a_\alpha / A a_{<\alpha}) \vdash tp(a_\alpha / A a_X a_{X \cap \alpha < \alpha})$

will show

Let $X_0 \subseteq X$ finite s.t. $X_0 \cap \alpha \neq \emptyset$.

Enough to show that $tp(a_\alpha / A a_{<\alpha}) \vdash tp(a_\alpha / A a_{<\alpha} a_{X_0})$

Wlog By the remark $X_0 \cup \alpha$ is closed.

So: $A \cup \langle a_\beta : \beta < \alpha \rangle ^\frown \langle a_\beta : \beta \in X_0 \rangle$ is

an i-construction over A, but for $\beta \in X_0$

$tp(a_\beta / A a_{<\alpha} a_{(<\beta) \cap X_0}) \vdash tp(a_\beta / A a_{<\beta})$

Because $\varphi_\beta(x)^\vdash$ and $\varphi_\beta{}^\in$.

So it implies also $tp(a_\beta / A a_{\leq \alpha} a_{(<\beta) \cap X_0})$.

$\Rightarrow$ $tp(a_\alpha / A a_{<\alpha} a_{(<\beta) \cap X_0}) \vdash tp(a_\alpha / A a_{<\alpha} a_{(\leq\beta) \cap X})$

the proof of e1 l3

(we switch a_α with a_β)

We just continue with induction on β

(start with $\beta = \min X_0$).

$\boxed{}$

Claim M: primary $/A \Rightarrow$ atomic $/A$

Pf. Let $\bar{m} \subseteq M = A \cup \{a_\alpha : \alpha < \gamma\}$.

$\bar{m} \subseteq A \cup a_X$, $X \subseteq \gamma$ finite closed

$A \cup a_X$: a partial i-construction.

$\Downarrow$

$tp(a_X/A)$ is isolated

$\Downarrow$

$tp(\bar{m}/A)$ —— " ——

claim $\square$

Pf (of thm) $M = A \cup \{a_\alpha : \alpha < \gamma\}$,

$N = A \cup \{b_\alpha : \alpha < \delta\}$: i-constructions $/A$.

We construct $f : M \xrightarrow{\cong} N$, $f = \bigcup_\alpha f_\alpha$: elementary.

(i) $\mathrm{Dom} f_\alpha \supseteq A$, $\mathrm{Rng} f_\alpha \supseteq A$, $f_\alpha \restriction_A = id_A$

(ii) $|\mathrm{Dom} f_\alpha \setminus A|$, $|\mathrm{Rng} f_\alpha \setminus A| \leq |\alpha| \cdot \aleph_0$

(iii) $\beta \in \mathrm{Lim}$ $f_\beta = \bigcup_{\alpha < \beta} f_\alpha$

(iv) $a_\alpha \in \mathrm{Dom} f_{\alpha+1}$, $b_{\alpha+1} \in \mathrm{Rng} f_{\alpha+1}$.

(v) $\mathrm{Dom} f_\alpha \setminus A = a_X$, $\mathrm{Rng} f_\alpha \setminus A = b_Y$, where $X \subseteq \gamma$, $Y \subseteq \delta$ are closed.

The recursive step from f_α to $f_{\alpha+1}$.

Let $A' = A \cup \operatorname{Dom} f_\alpha$: an i-construction over A,

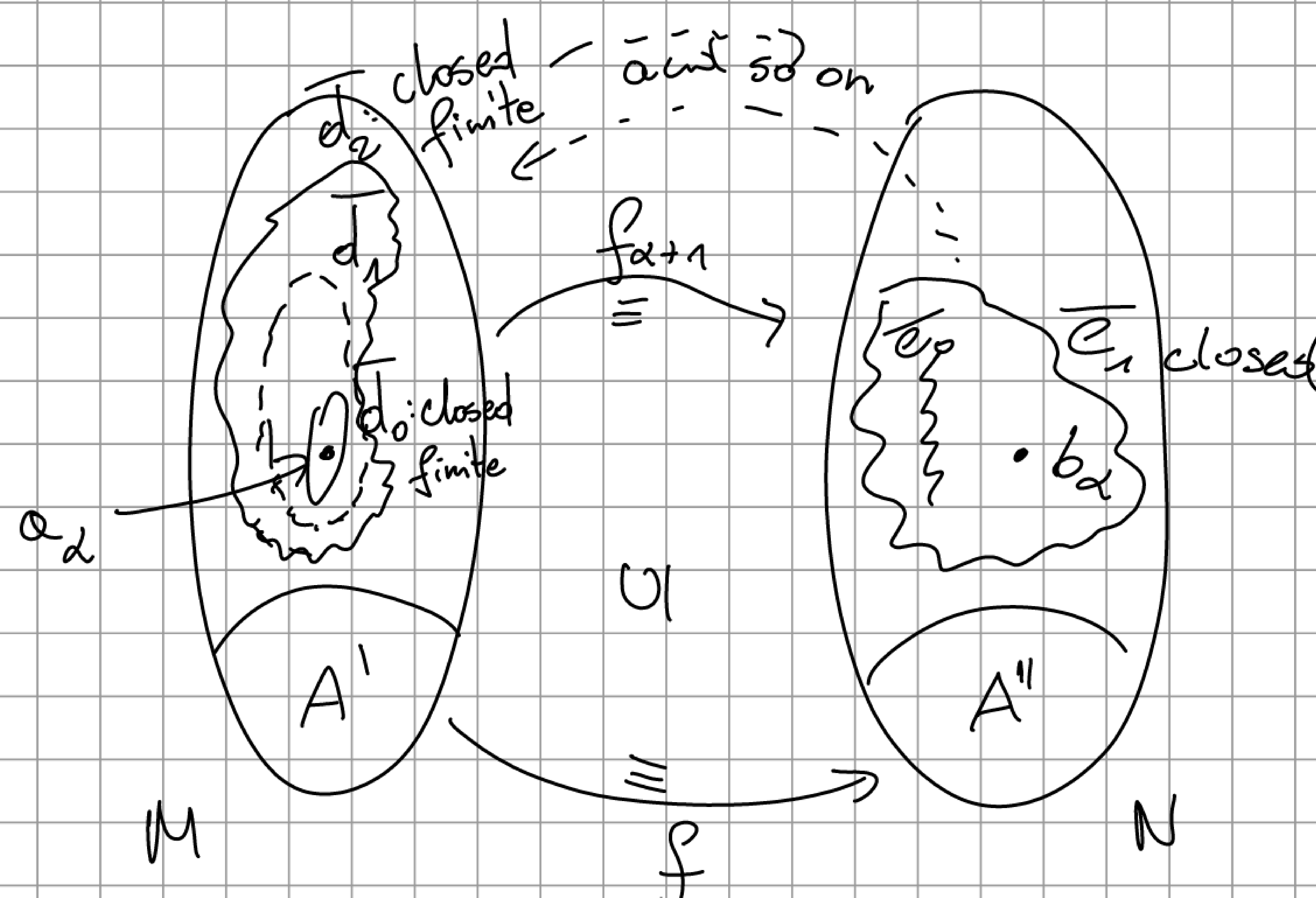
likewise $A'' = A \cup \operatorname{Rng} f_\alpha$: —— " ——

and M is primary over A' (by the remark)

and N is primary over A''.

$p(x) = tp(a_\alpha / A')$ is isolated, so $f_\alpha(p)$ is isolated

too, therefore $\exists b \in N$ s.t. $f_\alpha(p) = tp(b / A'')$.



4.04.22

Def. (1) $\varphi(\bar{x}, \bar{a}) \in L_n(\bar{a})$ is algebraic if $0 < |\varphi(\mathcal{M})| < \aleph_0$

(2) a type $p(\bar{x})$ (over $\mathcal{M}$) is algebraic if $0 < |p(\mathcal{M})| < \aleph_0$

(3) $a \in \underset{\text{algebraic closure}}{acl(A)}$, if $tp(a/A)$ is algebraic

(4) $a \in \underset{\text{definable closure}}{dcl(A)}$, if $a \in \mathcal{M}$ is the only realisation of $tp(a/A)$

Remark (1) $p(\bar{x})$: an algebraic type $\Leftrightarrow p(\bar{x}) \vdash \varphi(\bar{x})$ for some algebraic formula $\varphi(\bar{x})$

(2) $|p(\mathcal{M})| = 1 \Leftrightarrow \exists \varphi \, (p \vdash \varphi$ and $|\varphi(\mathcal{M})| = 1)$

Proof. 1 $(\Leftarrow)$ $p(\mathcal{M}) \subseteq \varphi(\mathcal{M})$

$(\Rightarrow)$ (a.a.) let $n \in \mathbb{N}$ arbitrary. Will show: $|p(\mathcal{M})| \geq n$.

Let $\bar{x}_1, \ldots, \bar{x}_n$: disjoint tuples of variables, $|\bar{x}_i| = |\bar{x}|$.

$\{\varphi(\bar{x}_i) : \varphi(\bar{x}) \in p, \, i = 1, \ldots, n\} \cup \{``\bar{x}_i \neq \bar{x}_j" : 1 \leq i < j \leq n\}$: a consistent type.

It is realised in $\mathcal{M}$ so it has $\geq n$ realisations. ↯

<u>Fact</u> $acl(A) = \bigcup\{\varphi(\mathcal{M}) : \varphi(x) \in L_1(A) \text{ algebraic}\}$

$dcl(A) = \bigcup\{\varphi(\mathcal{M}) : \varphi(x) \in L_1(A) \wedge |\varphi(\mathcal{M})| = 1\}$

<u>Remark</u> Let $\varphi(\bar{x}) \in L_n(M)$. Then $\varphi(\bar{x})$ algebraic $\iff 0 < |\varphi(M)| < \aleph_0$

<u>Proof</u> $M \prec \mathcal{M}$, $|\varphi(M)| = k \iff M \models (\exists!^k \bar{x})\varphi(\bar{x})$

$\iff \mathcal{M} \models (\exists!^k \bar{x})\varphi(\bar{x}) \iff |\varphi(\mathcal{M})| = k$

<u>Remark</u> Let $A \subseteq \mathcal{M}$, $a, b \in \mathcal{M}$, then: $tp(ab/A)$ is algebraic

$\iff tp(a/A)$ is algebraic and $tp(b/Aa)$ is algebraic

<u>Pf.</u> ($\Rightarrow$) $p(x,y) = tp(ab/A)$. Let $q(x) = p\restriction_x$ $= tp(a/A)$. Let $f : \mathcal{M}^2 \to \mathcal{M}$ projection to the first coord. Then $f : p(\mathcal{M}) \twoheadrightarrow q(\mathcal{M})$

Why?

Take $a' \in q$, choose $g \in Aut(\mathcal{M}/A)$, $g(a) = a'$, then $b' = g(b) \Rightarrow tp(ab/A) = tp(a'b'/A)$.

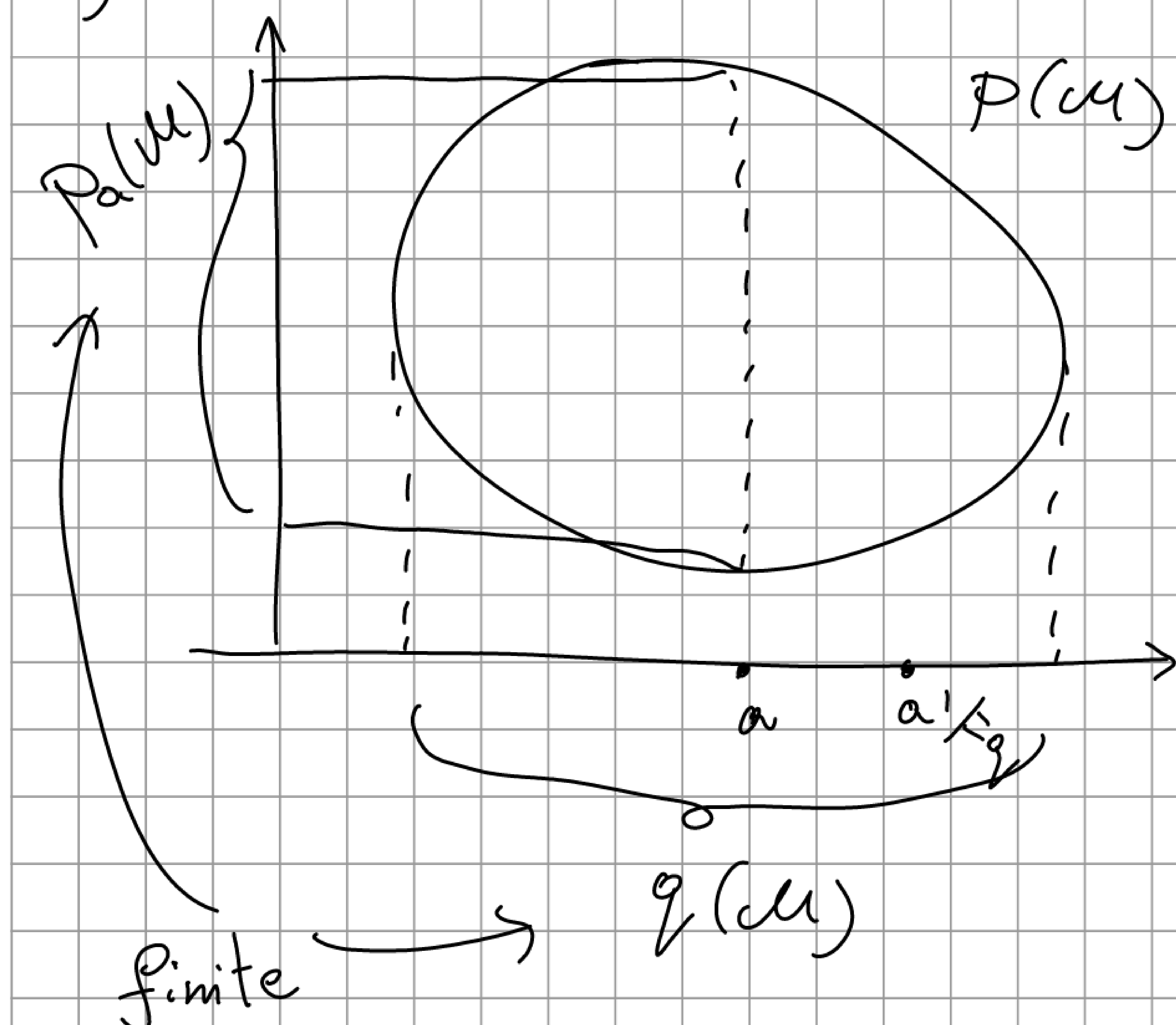
Then $|p(\mathcal{M})| < \aleph_0 \Rightarrow |q(\mathcal{M})| < \aleph_0$ and $|p_a(\mathcal{M})| < \aleph_0$

Let $p_a(y) = tp(b/Aa) = \{\varphi(a,y) : \varphi(x,y) \in p\}$

Then $p_a(\mathcal{M}) = p(\mathcal{M})_a$ ← "verticalization"

this is finite

(⇐)



$$p(\mathcal{M})_{a'} = g(p(\mathcal{M})_a) \text{ for any } g \in Aut(\mathcal{M}/A)$$

with $g(a) = a'$.

$$p(\mathcal{M}) = \bigcup_{a' \models q} (\{a'\} \times p_{a'}(\mathcal{M})) : \text{finite} \qquad \square$$

Remark (1) (i) $A \subseteq \operatorname{acl}(A)$, (ii) $\operatorname{acl}(\operatorname{acl}(A)) = \operatorname{acl}(A)$,

(iii) $A \subseteq B \Rightarrow \operatorname{acl}(A) \subseteq \operatorname{acl}(B)$ (a closure operator)

(iv) $\operatorname{acl}(A) = \bigcup_{\substack{A_0 \subseteq A \\ \text{finite}}} \operatorname{acl}(A_0)$ (finite character)

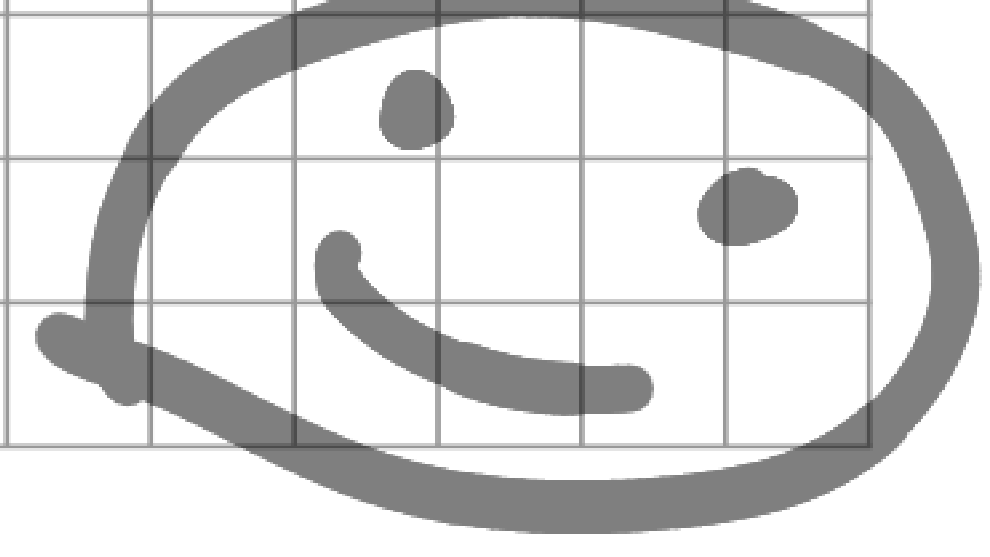
(2) The same for dcl

Pf. 1 • $a \in \operatorname{acl}(A) \Leftrightarrow a \in \varphi(\mathcal{M})$ finite, $\varphi(x) \in L_1(A)$, so $\varphi \in L_1(A_0)$ for some $A_0 \subseteq A$ finite, so (iv).

(ii): $a \in \operatorname{acl}(\operatorname{acl}(A)) \Rightarrow a \in \operatorname{acl}(A \cup b)$, for some $b \subseteq \operatorname{acl}(A) \Rightarrow tp(b/A)$ algebraic and $tp(a/Ab)$ algebraic $\overset{\text{remark}}{\Longrightarrow} tp(ab/A)$ is algebraic

$\Rightarrow tp(a/A)$ algebraic ∎

Example $T = ACF_p$, $A \subseteq \mathcal{M} \models T$. $\operatorname{acl}(A) =$ the algebraic closure (in $\mathcal{M}$) of the field generated by A.



Measuring definable sets and types.

The Cantor–Bendixson rank.

Def. Let X: compact T_2 space.

$$\underbrace{X'}_{\text{CB-derivative}} = X \setminus \underbrace{\{\text{isolated points}\}}_{\text{open in } X} \Rightarrow \underbrace{X' \subseteq X}_{\text{closed}}$$

Iteration: $X^{(\alpha+1)} = (X^{(\alpha)})'$, $X^{(\delta)} = \bigcap_{\alpha<\delta} X^{(\alpha)}$ where $\delta \in Lim$,

$$X^{(\infty)} = \bigcap_{\alpha \in Ord} X^{(\alpha)} = X^{(\beta)} \text{ for some } \beta < |w(X)|^+$$

the perfect core of X

minimal cardinality of basis of X

Def $\underline{CB}: X \to Ord \cup \{\infty\}$

$$\underset{\text{(CB-rank)}}{CB(p)} = \begin{cases} \min\{\alpha \in Ord : p \notin X^{(\alpha+1)}\} & \text{if } p \notin X^{(\infty)} \\ \infty & p \in X^{(\infty)} \end{cases}$$

Now: $X = S(A)$: 0-dimensional (extremly disconnected).

Let: $Clopen(X) = \{V \subseteq X : V \text{ clopen}\}$.

Def. $CB: \mathrm{Clopen}(X) \to \{-1\} \cup \mathrm{Ord} \cup \{\infty\}$: the smallest function (value-wise) s.t.: $CB(V) \geq \alpha+1 \iff$

$\forall n < \omega \; \exists \underset{\substack{\text{clopen} \\ \text{disjoint}}}{V_1, \dots, V_n} \subseteq V \quad CB(V_i) \geq \alpha$.

(also we define "$\geq \delta$")

Then $CB(V) := \min\{\alpha : \neg CB(V) \geq \alpha+1\}$

Properties Let $U, V \subseteq X$ clopen.

(0) $CB(U) = -1 \iff U = \emptyset$

(1) $CB(U) = 0 \iff 0 < |U| < \aleph_0$

(2) $U \subseteq V \Rightarrow CB(U) \leq CB(V)$

very easy

(3) $CB(U \cup V) = \max\{CB(U), CB(V)\}$

Pf. 3 Obv. $CB(U \cup V) \geq \max\{CB(U), CB(V)\}$.

Now assume $CB(U \cup V) \geq \alpha \Rightarrow \max\{CB(U), CB(V)\} \geq \alpha$.

Pf by ind on α. Base and limit easy.

Succesor step $\alpha \to \alpha+1$. Assume $CB(U \cup V) \geq \alpha+1$.

So $\forall n \; \exists \underset{\substack{\text{clopen} \\ \text{disjoint}}}{W_1, \dots, W_n} \subseteq U \cup V \quad \bigwedge_{i=1}^{n} CB(W_i) \geq \alpha$.

By ind hyp.: $\max\{CB(W_i \cap U), CB(W_i \cap V)\} \geq \alpha$.

So $\forall n \left(\exists W_1, \dots, W_n \subseteq U \ \bigwedge_{i=1}^{n} CB(W_i) \geq \alpha\right)$

(or the same for V)

$\Downarrow$

$CB(U) \geq \alpha+1$ or $CB(V) \geq \alpha+1$.

(4) $CB(V) \geq \alpha+1 \iff \left(\exists V_n \underset{\substack{\text{clopen}\\ \text{disjoint}}}{\subseteq} V, n<\omega\right) \bigwedge_n CB(V_n) \geq \alpha$

(5) For $p \in X$

$$CB(p) = \min\{CB(U) : p \in U \underset{\text{clopen}}{\subseteq} X\}$$

Notation

In model theory: $X = S(A)$.

$CB(a/A) := CB(tp(a/A))$

$CB(a/A) = 0 \iff tp(a/A)$ is isolated

$p \in S(A) \rightsquigarrow CB(p) = CB_A(p)$

$\varphi \in L(A) \rightsquigarrow [\varphi] \underset{\text{clopen}}{\subseteq} S(A) \rightsquigarrow CB_A(\varphi)$

$\qquad = CB_A([\varphi] \cap S(A))$

Morley rank "CB on $L(\mathcal{M})$, in $S(\mathcal{M})$".

Def. $\mathrm{RM}: L(\mathcal{M}) \to \{-1\} \cup \mathrm{Ord} \cup \{\infty\}$:

the minimal function s.t. for $\mathcal{U} \subseteq \mathcal{M}^n$ definable

$[\varphi(\bar{x}) \in L_n(\mathcal{M})$ identified with $\mathcal{U} = \varphi(\mathcal{M}) \subseteq \mathcal{M}^n]$

(1) $\mathcal{U} = \emptyset \Rightarrow \mathrm{RM}(\mathcal{U}) = -1$

(2) If $\mathcal{U} \neq \emptyset$, then $\mathrm{RM}(\mathcal{U}) \geqslant \alpha + 1$

$\iff \forall n < \omega \ \exists V_1, \dots, V_n \underset{\substack{\text{def.}\\ \text{disjoint}}}{\subseteq} \mathcal{U} \ \bigwedge_{i \leq n} \mathrm{RM}(V_i) \geqslant \alpha$

Def. • For $\varphi(\bar{x}) \in L(\mathcal{M})$, $\mathrm{RM}(\varphi) = \mathrm{RM}(\varphi(\mathcal{M}))$

• For a type $p(\bar{x})$ over $\mathcal{M}$ (not necesserly completed)

$$\mathrm{RM}(p) = \min\{\mathrm{RM}(\varphi) : p \vdash \varphi\}$$
$$= \min\{\mathrm{RM}(\mathcal{U}) : p(\mathcal{M}) \subseteq \mathcal{U}\}$$

Remark (1) $\mathrm{RM}(\varphi) = \mathrm{CB}_{\mathcal{M}}(\varphi)$

(2) For $p \in S(\mathcal{M})$ $\mathrm{RM}(p) = \mathrm{CB}_{\mathcal{M}}(p)$

Here $\mathcal{M}$ may be replaced with any $\aleph_0$-saturated $M \prec \mathcal{M}$.

Properties (1) $\varphi \vdash \psi \Rightarrow RM(\varphi) \leq RM(\psi)$

(2) $p \vdash q \Rightarrow RM(p) \leq RM(q)$

(3) $RM(\varphi \vee \psi) = \max\{RM(\varphi), RM(\psi)\}$

(4) $RM(\varphi) \geq \alpha+1 \iff \left(\exists \underset{\text{pairwise contradictory}}{\varphi_n \vdash \varphi}, n<\omega\right) \bigwedge_n RM(\varphi_n) \geq \alpha$

(5) If $\delta \in Lim$, then $RM(\varphi) \geq \delta \iff (\forall \alpha < \delta) RM(\varphi) \geq \alpha$

Thm T is $\aleph_0$-stable $\iff RM(\text{"}x=x\text{"}) < \infty$.

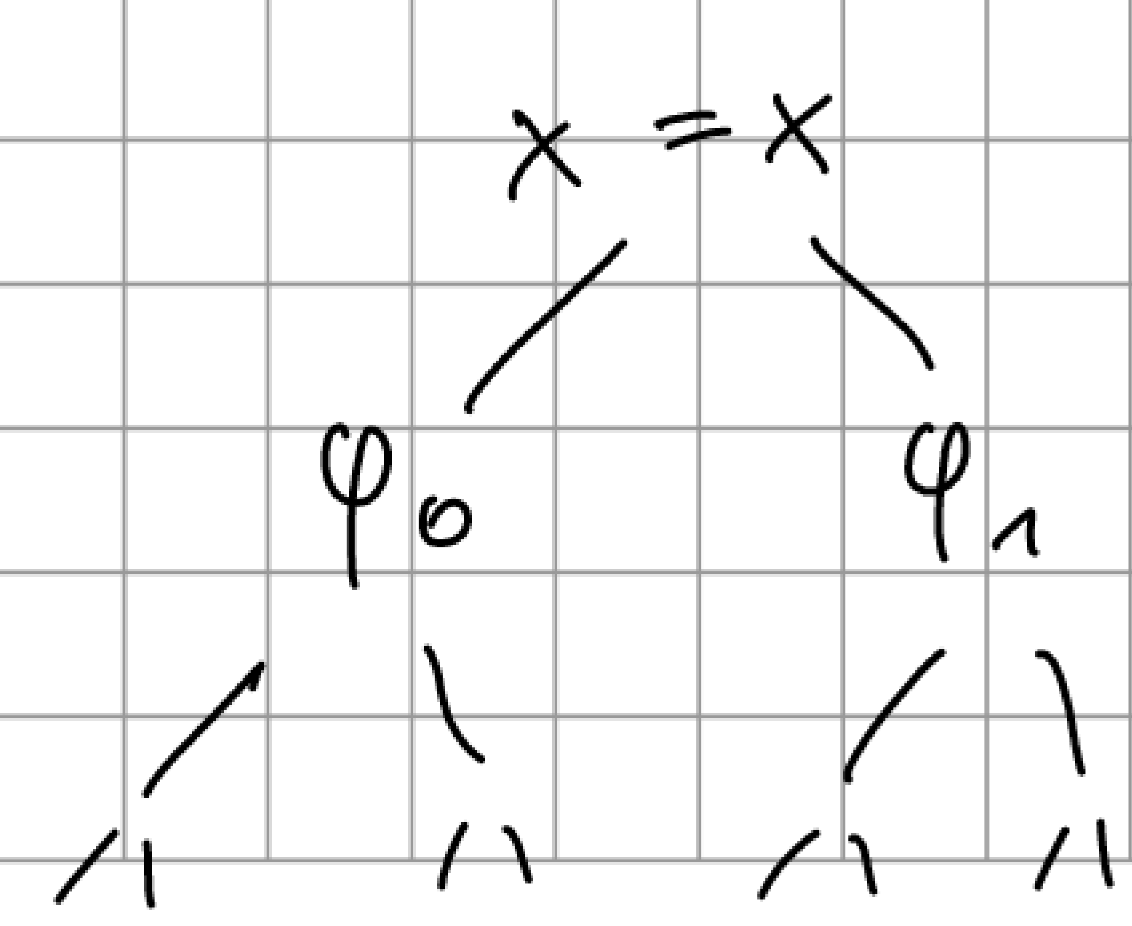
Pf. $(\Rightarrow)$ (a.a.) If $tp(\bar{a}) = tp(\bar{b})$, then

$RM(\varphi(\bar{x},\bar{a})) = RM(\varphi(\bar{x},\bar{b}))$

$\exists f \in Aut(\mathcal{M})\ f(\bar{a}) = \bar{b} \Rightarrow f(\varphi(\mathcal{M},\bar{a})) = \varphi(\mathcal{M},\bar{b})$

So $|Rng(RM)| \leq 2^{|T|} \Rightarrow \exists \alpha \in Ord\ \forall \varphi [RM(\varphi) \geq \alpha \Rightarrow RM(\varphi) = \infty]$.

Suppose $RM(\text{"}x=x\text{"}) = \infty \Rightarrow \geq \alpha+1$



: pairwise contradictory, $RM \geq \alpha \Rightarrow \geq \alpha+1$

Proceed like that

We get $2^{\aleph_0}$ many types over ctble set A

$\Rightarrow T$ is not $\aleph_0$-stable.

11.04.2022

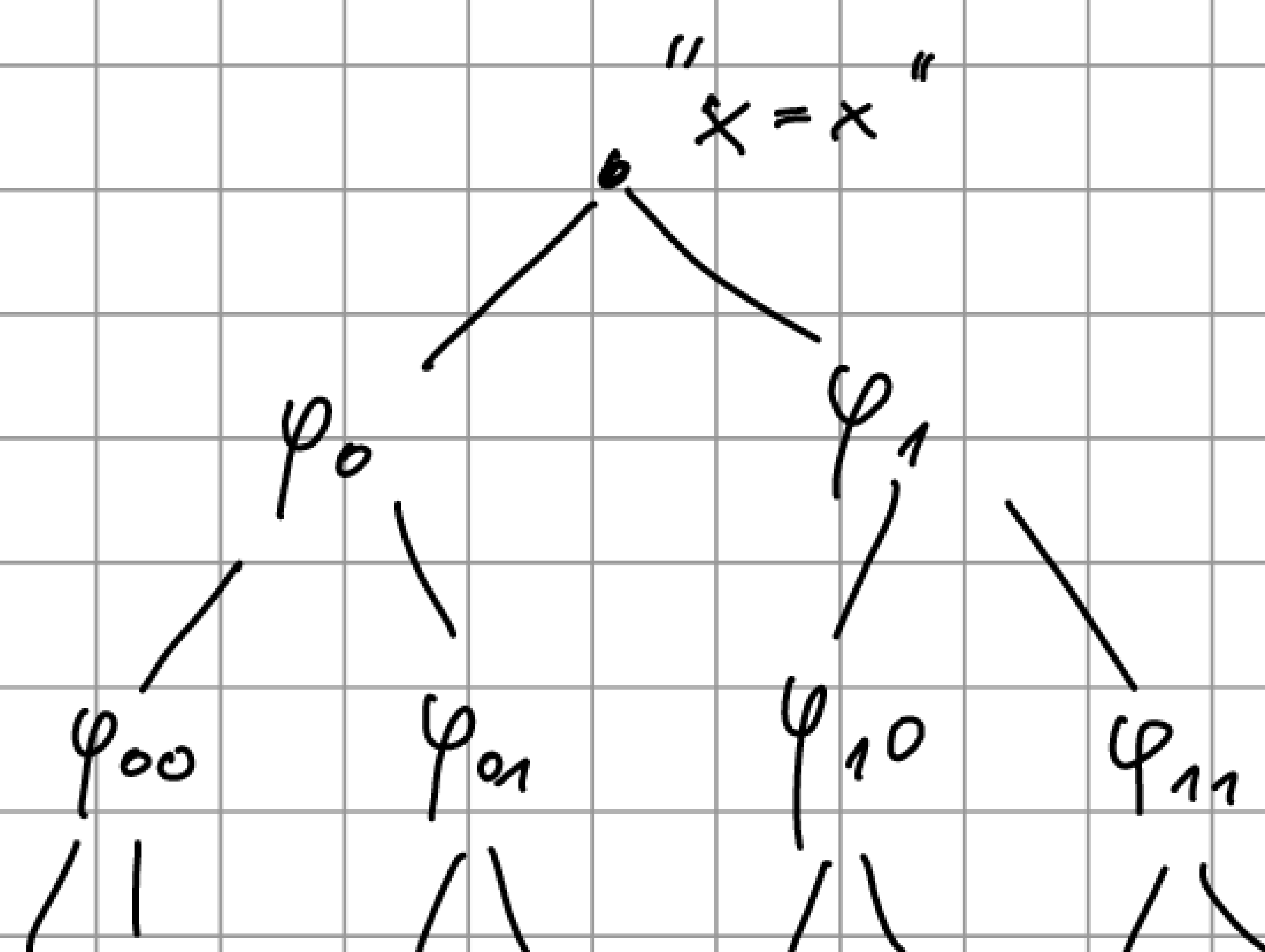
Thm T is $\aleph_0$-stable $\iff RM(x=x)\in \mathrm{Ord}$

Pf. "$\Rightarrow$" last time. "$\Leftarrow$" Suppose T is not $\aleph_0$-stable.

$|S(A)| = 2^{\aleph_0}$, $S(A)$: a Polish space

$\Rightarrow S(A)^{(\infty)} \neq \emptyset \Rightarrow$ get a binary tree of formulas

$$\{\varphi_\eta(x)^{\in L_1(A)} : \eta \in 2^{<\omega}\}$$



Let $\alpha = \min\{RM(\varphi_\eta) : \eta\in 2^{<\omega}\}$.

If $\eta \subseteq \nu \in 2^{<\omega}$, then $RM(\varphi_\nu) = \alpha$.

$RM(\varphi_\eta) \not\geq \alpha+1$, so:

$$(*)\ \exists m\in\mathbb{N}\ \neg\exists \psi_1 \dots \psi_m \vdash \varphi_\eta \ \bigwedge_{i=1}^{m} RM(\psi_i)\geq \alpha$$

Take n s.t. $2^n > m$. φ_ν, $\eta\subseteq \nu \in 2^{<\omega}$, $|\nu| = |\eta|+n$

contradicts $(*)$. ∎

Def. <u>Multiplicity</u> of $\varphi(\bar{x}) \in L(\mathcal{M})$ s.t. $\underbrace{RM(\varphi(\bar{x}))}_{\alpha \in Ord} < \infty$:

$Mlt(\varphi) =$ the largest $m \in \mathbb{N}$ s.t.

$$\exists \psi_1 \dots \psi_m \ \bigwedge_{i=1}^{m} RM(\psi_i) \geq \alpha$$

Properties • $RM(\varphi_1) = RM(\varphi_2)$ and $\varphi_1(\mathcal{M}) \cap \varphi_2(\mathcal{M}) = \emptyset$, then $Mlt(\varphi_1 \vee \varphi_2) = Mlt(\varphi_1) + Mlt(\varphi_2)$

• If $RM(\varphi_1) < RM(\varphi_2) < \infty$, then $Mlt(\varphi_1 \vee \varphi_2) = Mlt(\varphi_2)$

Example If $\underbrace{\varphi(\bar{x}) \text{ is algebraic}}_{\Updownarrow \; RM(\varphi) = 0}$, then $Mlt(\varphi) = |\varphi(\mathcal{M})|$

Def. Assume $p(\bar{x})$: a type with $RM(p(\bar{x})) < \infty$.

$$Mlt(p(\bar{x})) = \min\{Mlt(\varphi(\bar{x})) : p \vdash \varphi \text{ and } RM(\varphi) = RM(p)\}$$

Def. $p(\bar{x})$ is <u>stationary</u>, if $Mlt(p(\bar{x})) = 1$

Remark Assume $p(\bar{x})$: a type over A. Then $\exists q \in S(A)$, $p(\bar{x}) \subseteq q(\bar{x})$ s.t. $RM(p) = RM(q)$

Pf. Let $q_0 = \{\varphi(\bar{x}) \in L(A) : RM(p \cup \{\neg\varphi\}) < RM(p)\}$

• $q_0 \supseteq p$, if $\varphi \in p$, then $RM(p \cup \{\neg\varphi\})$
$= -1 < RM(p)$

$\nwarrow$ consistent

$RM(p)$
$\|$

• q_0: a consistent type: $\varphi_1, \dots, \varphi_n \in q_0$, $RM(p \cup \{\neg\varphi_i\}) < \alpha$

Choose ψ with $p \vdash \psi$ and $RM(\psi) = \alpha$,

ψ_i with $p \vdash \psi_i$ and $RM(\psi_i \wedge \neg\varphi) < \alpha$.

Wlog $\psi = \bigwedge_{i=1}^{n} \psi_i$

$\vdots$

Let $q_0 \subseteq q \in S(A)$, then $RM(q) = \alpha$. Clearly, $RM(q) \leq \alpha$. If $RM(q) < \alpha \Rightarrow q \vdash \varphi$ with some $RM(\varphi) < \alpha$. By compactness $\exists$ finite $q' \subseteq q$ $q' \vdash \varphi$ $\Leftrightarrow \underbrace{\bigwedge q'}_{\chi(\bar{x}) \in q(\bar{x})} \vdash \varphi$, $\chi(\bar{x}) \vdash \varphi(\bar{x})$ so $RM(\chi(\bar{x})) < \alpha$,

then $RM(p \cup \{\chi(\bar{x})\}) < \alpha$

$$\Downarrow$$

$$\begin{cases} \neg\chi(\bar{x}) \in q_0(\bar{x}) \subseteq q(\bar{x}) \\ \chi(\bar{x}) \in q(\bar{x}) \end{cases} \quad \lightning$$

Example $\overbrace{RM(p)=0}^{\Leftrightarrow p:\text{ algebraic}}$, then $Mlt(p) = |p(\mathcal{M})|$

Example $T = ACF_p$, $\underset{\text{subfield}}{K} \subseteq \mathcal{M}$, $p \in S_n(K)$, $p = tp(a^{\in \mathcal{M}}/K)$

a) p algebraic $\Leftrightarrow$ $RM(p)=0$,

$\underset{K[x]}{\overset{\cap}{(W)}} = I(a/K) \neq \{0\}$, "$W(x)=0$" $\in p(x)$,

$Mlt(p) = Mlt(W(x)=0) = \#\{\text{roots of } W \text{ in } \mathcal{M}\}$.

b) p: transcendental, then $I(a/K) = \{0\}$,

$RM(p) = 1 = RM(x=x)$

$Mlt(p) = Mlt(x=x) = 1$

So p: stationary.

c) $p = tp(\bar{a}/K)$, $\bar{a} = \langle a_1, \dots, a_n \rangle \subseteq \mathcal{M}$. By q.e. it is determined by $I(\bar{a}/K) \trianglelefteq K[X_1, \dots, X_n]$

$\{W(\bar{x}) \in K[\bar{x}] : W(\bar{a}) = 0\}$.

Let $V_p = V(I(\bar{a}/K)) = \bigcap_{W \in I(\bar{a}/K)} Z(W) =$

$Z(W) = \{\bar{b} \in \mathcal{M}^n : W(\bar{b}) = 0\}$

$= \bigcap_{i=1}^{l} Z(W_i)$ ← definable on $\mathcal{M}$ over K, $\quad I(\bar{a}/K) = (W_1, \dots, W_l)$

$\bar{a} \subseteq V_p \Rightarrow p(\bar{x}) \vdash$ "$\bar{x} \in V_p$"

$RM(p) = RM(V_p) = \dim V_p$

$Mlt(p) = Mlt(V_p) =$ The number of irreducible components of V_p of $\dim = \dim V_p$

In $T = Th(ACF_p)$: $RM(\bar{x}=\bar{x}) < \omega$

(Order) indiscernible sets:

Let $(I, \leq)$: a linear ordered set of indicies.

<u>Def.</u> $\{\bar{a}_i : i \in I\} \subseteq \mathcal{M}$: <u>order indiscernible over</u> <u>$A \subseteq \mathcal{M}$</u>, if $\forall k \, \forall i_1 < \ldots < i_k \in I$ $tp(\bar{a}_{i_1} \ldots \bar{a}_{i_k} / A) = tp(\bar{a}_{j_1} \ldots \bar{a}_{j_k} / A)$
$j_1 < \ldots < j_k \in I$

<u>Recall</u>: 1) Assume $p(\bar{x})$: a non-algebraic type over $A \subseteq \mathcal{M}$. Then $\exists \{\bar{a}_n : n < \omega\} \subseteq p(\mathcal{M})$
infinite order indiscernible

2) (stretching) Assume $\{a_i : i \in I\}$ order indiscernible $/A$, I: infinite, $(J, \leq)$: a linear ordering. Then $\exists \{b_j : j \in J\}$: order ind. $/A$ s.t. $\forall k \, \forall i_1 < \ldots < i_k \in I$
$\forall j_1 < \ldots < j_k \in I$

$tp(a_{i_1} \ldots a_{i_k} / A) = tp(b_{j_1} \ldots b_{j_k} / A)$

Pf. (1) by Ramsey thm.

(2) Let $b_j, j \in J$: new constant symbols.

$$T^* = T(A) \cup \{ \varphi(b_{j_1}, \dots, b_{j_k}) : \varphi(\bar{x}) \in L_k(A),\ j_1 < \dots < j_k \in J$$
$$\text{and } \forall i_1 < \dots < i_k \in I \models \varphi(a_{i_1}, \dots, a_{i_k}) \}.$$

T^* : consistent. $b_{j_1}, \dots, b_{j_k} \overset{\text{interpret}}{\underset{\text{as}}{\rightsquigarrow}} a_{i_1}, \dots, a_{i_k}$

$\Downarrow$

has a model M for any $i_1 < \dots < i_k \in I$

$$M = (\underbrace{M, a^M}_{\models T(A)}, b_j^M)_{\substack{a \in A \\ j \in J}} \models T^*$$

$$T(A) \Rightarrow \exists f : (M, a^M)_{a \in A} \xrightarrow{\equiv} (\mathcal{M}, a)_{a \in A}$$

$$a^M \overset{f}{\longmapsto} a$$

Let $b_j = f(b_j^M)$. $\{ b_j : j \in J \} \subseteq \mathcal{M}$ is good.

Example (1) $M = (\mathbb{Q}, \leq) \leftarrow \mathbb{Q}$ is order indisc (indexed by itself)

(2) $T = ACF_p$, $\mathcal{M} \models T$, $\{ a_i : i \in I \} \subseteq \mathcal{M}$, acl-indepen. over $K \subseteq \mathcal{M}$ (subfield),

then it is indisc $/K$ in $\mathcal{M}$

$\Updownarrow$

$I(a_{i_1} \dots a_{i_k}/K) = \{0\}$

Def. $\{\bar{a}_i : i \in I\} \subseteq \mathcal{M}$ is <u>indiscernible over $A \subseteq \mathcal{M}$</u>

if $\forall k \; \forall i_1, \dots, i_k \subseteq I$ $\quad tp(a_{i_1} \dots a_{i_k}/A) = tp(b_{i_1} \dots b_{i_k}/A)$
$j_1, \dots, j_k \subseteq I$

(3) $T = Th(V, +, 0, k)_{k \in K}$: infinite vector space.

$V \supseteq \{a_i : i \in I\}$ indiscernible $/\emptyset \iff$ linear independ.

Thm. T: stable, $\{a_i\}_{i \in I}$ order indisc., then
$\{a_i\}_{i \in I}$ indiscernible. $\quad$ infinite

Pf. (A.a.) T: $\varkappa$-stable. Wlog $(I, \leq)$ is (by stretching)

a dense linear ordering with $J \subseteq_{\text{dense}} I$, $|J| = \varkappa$

s.t. every nonempty interval in I has power $> \varkappa$.

$\{a_i\}_{i \in I}$: ord-indisc. but not indisc.:

$\exists k \; \exists i_1 < \dots < i_k \in I \; \exists j_1, \dots, j_k \in I \quad tp(a_{i_1} \dots a_{i_k}) \neq tp(a_{j_1} \dots a_{j_k})$

$\Rightarrow$ for some φ: $\models \varphi(a_{i_1} \dots a_{i_k}) \wedge \neg \varphi(a_{i_{\sigma(1)}} \dots a_{i_{\sigma(k)}})$

for some $\sigma \in Sym(k)$

25.04.22 Pf. c.d. $\sigma = \tau_1 \circ \tau_2 \circ \ldots \circ \tau_l$: product of transpositions of consecutive numbers.

Let $\sigma_t = \tau_1 \circ \ldots \circ \tau_t$, $t = 1, \ldots, l$, $\sigma_0 = id$. Then $\models \varphi(\bar{a}_{i_{\sigma_0}})$ and $\models \varphi(\bar{a}_{i_{\sigma_l}})$. For some $0 \leq t < l$:

$$\models \varphi(\bar{a}_{i_{\sigma_t}}) \wedge \neg \varphi(\bar{a}_{i_{\sigma_{t+1}}})$$

Let $\sigma' = \sigma_t$, $\sigma'' = \sigma_{t+1} = \sigma' \circ \overset{\tau}{\overset{\shortparallel}{\tau_{t+1}}}$

Then $\models \underbrace{\varphi(a_{i_{\sigma'(1)}}, \ldots, a_{i_{\sigma'(k)}})}_{\psi(a_{i_1}, \ldots, a_{i_k})}$ (by renaming)

But $\models \neg \varphi(a_{i_{\sigma'(\tau(1))}}, \ldots, a_{i_{\sigma'(\tau(k))}})$,

so $\models \neg \psi(a_{i_{\tau(1)}}, \ldots, a_{i_{\tau(k)}}) \wedge \psi(a_{i_1}, \ldots, a_{i_k})$

$(\forall i_1 < i_2 < \ldots < i_k \in I)$

e.g. $\tau = (3,4)$ and $k > 4$. Choose $i_1 < i_2 < i_5 < \ldots < i_k$.

Let $\chi(x_3, x_4) = \psi(a_{i_1}, a_{i_2}, x_3, x_4, a_{i_5}, \ldots, a_{i_k}) \in L(\bar{a}_J)$ $\overset{\cap}{J}$ dense $\cap$ I

Will show: $|S(\bar{a}_J)| > \varkappa$, $|\bar{a}_J| \leq \varkappa$.

Namely: let $i < i' \in (i_2, i_5)_I$. Then

$$tp(a_i / \bar{a}_J) \neq tp(a_{i'} / \bar{a}_J) \leftarrow \text{enough}$$

$\exists$ $i_1 < i_2 < i < j < i' < i_5 < \ldots < i_k$.

Then $\models \varphi(a_i, a_j)$, $\models \neg\varphi(a_{i'}, a_j)$

because $i < j$ because $i' > j$

Then $\varphi(x, a_j) \in tp(a_i / \bar{a}_J)$,

$\neg\varphi(x, a_j) \in tp(a_{i'} / \bar{a}_J)$. $\lightning$

<u>Remark</u> (T: stable, $\varphi(x, \bar{y}) \in L$). There is $n < \omega$ $\forall \underset{\substack{\text{indiscernible}\\ \text{infinite}}}{I \subseteq \mathcal{M}}$ $\forall \bar{a} \subseteq \mathcal{M}$ one of the sets

$$I^+_{\bar{a}} = \{c \in I : \models \varphi(c, \bar{a})\}$$

$$I^-_{\bar{a}} = \{c \in I : \models \neg\varphi(c, \bar{a})\}$$

has $\leq n$ elements.

Pf. (a.c.) If there's no such n, then $\forall n\, \exists I_n\, \exists \bar{a}_n\ |I^+_{n,\bar{a}_n}|, |I^-_{n,\bar{a}_n}| > n$. Let

$\{c_i, i<\omega\}$: new constant symbols.

$\bar{d}$: , $|\bar{d}| = |\bar{y}|$.

Let $T' = T \cup \{ \text{"}\underbrace{\{c_i : i<\omega\}}_{I} \text{ is indiscernible in } T\text{"}\}$

$\cup \{ \text{"}I^+_{\bar{d}} = \{c_{2i} : i<\omega\}\text{"}\} \cup \{\text{"}I^-_{\bar{d}} = \{c_{2i+1} : i<\omega\}\text{"}\}$.

wlog $|I| = \varkappa$

T' is consistent, so it has a model M'.

Wlog $M' \restriction_L \prec \mathcal{M}$. So $I \subseteq \mathcal{M}$ indiscernible,

$\underset{I,\bar{d}}{\cup}\quad I^+_{\bar{d}}, I^-_{\bar{d}} \subseteq \mathcal{M} \Rightarrow |S(I)| = 2^{\varkappa} > \varkappa$.

Pf. $|I^+_{\bar{d}}|, |I^-_{\bar{d}}| = \varkappa$. For any $I' \subseteq I$ with $|I'| = |I \setminus I'| = \varkappa$ $\exists f \in Aut(\mathcal{M})\, [f[I] = I,$ $f(I') = I^+_{\bar{d}}, f(I \setminus I') = I^-_{\bar{d}}]$. Let $\bar{a}_{I'} = f^{-1}(\bar{d})$.

If $I' \neq I''$, then $tp(\bar{a}_{I'}/I) \neq tp(\bar{a}_{I''}/I)$.

$\aleph_1$ - categorical theories

Examples. 1. ACF_p,

2. $Th(V, +, k)_{k \in K}$ ← ctble field

inf. vec. space

3. $Th(\mathbb{N}, S)$, $Th(\mathbb{Z}, S)$, $Th(\mathbb{N}, =)$ ← "no structure"

4. $Th(G, +)$, G: torsion free divisible abelian

5. $Th(\mathbb{Z}_p^{\aleph_0}, +)$

6. $Th(G)$ where G: an algebraic group

Theorem If $\kappa > \aleph_0$ and T is κ-categorical, then T is $\aleph_0$-stable.

Lemma $\forall \kappa \geq \aleph_0 \ \exists M \models T$, $\|M\| = \kappa$ $\forall A \subseteq M$, $|A| < \aleph_0$

$$|\{p \in S(A) : p(M) \neq \emptyset\}| \leq \aleph_0 .$$

Pf. thm (lemma $\Rightarrow$ thm) (A.c.) Suppose T is not $\aleph_0$-stable. $\exists_{\text{ctble}} N \models T$ $|S(N)| > \aleph_0$

But T: κ-categorical. $N \prec N_1$ s.t. $\|N_1\| = \aleph_1$,

$$|\{p \in S(N) : p(N_1) \neq \emptyset\}| = \aleph_1 .$$

$N_1 \prec N_\varkappa \leftarrow$ of power $\varkappa$. Let $M_\varkappa$: a model from the lemma. Then $M_\varkappa \cong N_\varkappa$ ↯.

<u>Pf.</u> (lemma) $T \subseteq T^S$: the skolemization in $L^S \supseteq L$.

Let $I = \{a_n : n < \omega\}$: an infinite order indisc. set in T^S. $I \subseteq \mathcal{J} = \{a_\alpha : \alpha < \varkappa\}$ (stretching).

Then $\mathcal{J} \subseteq N^S \models T^S$. Let $M^S = \mathcal{H}(\mathcal{J}) \prec N^S$, i.e.

$M^S = \{ t^{N^S}(\vec{j}) : t(\bar{x}) : \text{a term in } L^S, \vec{j} \subseteq \mathcal{J} \text{ increasing} \}$

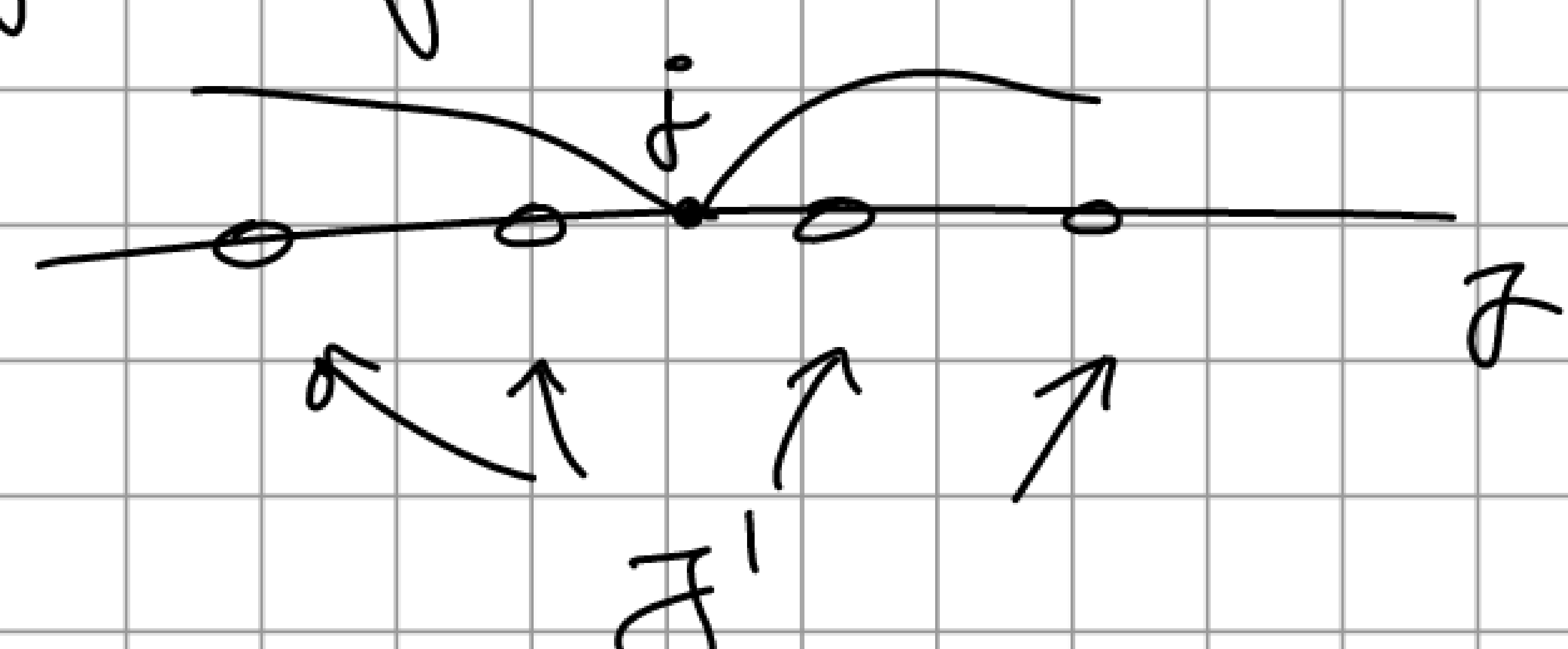
Will show that M^S satisfies the conditions on the lemma for T^S (then $M := M^S \restriction_L$ satisfies conditions for T)

Let $\underset{\text{ctble}}{A} \subseteq \mathcal{H}(\mathcal{J})$. Wlog $A = \mathcal{H}(\mathcal{J}')$ for some $\mathcal{J}' \subseteq \mathcal{J}$.

$M^S \ni a = t(j_1, \dots, j_k)$ for some $j_1 < \dots < j_k \in \mathcal{J}$.

$\sim$ eq. rel. on $\mathcal{J}$:

$j \sim j' \overset{\text{def.}}{\iff} j$ and j' determine the same cut in $\mathcal{J}'$.



$|\mathcal{J}/_\sim|$: ctble as $|\mathcal{J}'|$ ctble. Now

$$\underset{< \; <}{(j_1, \dots, j_k)} \sim \underset{< \; <}{(j_1', \dots, j_k')} \overset{def}{\iff} \bigwedge_{1 \le i \le k} j_i \sim j_i'$$

again ctbly many classes

Let $\underset{t(j_1,\dots,j_k)}{\overset{\shortparallel}{a}}, \underset{t'(j_1',\dots,j_k')}{\overset{\shortparallel}{a'}} \in M^s$. $(*)$ If $t = t'$ and $(j_1, \dots, j_k) \sim (j_1', \dots, j_k')$ then $tp^{M^s}(a/A) = tp^{M^s}(a'/A)$.

(it obvs. implies the lemma). $\mathcal{J}'$

Pf $(*)$: $\underset{\substack{\shortparallel \\ \varphi(x, b),\, b \subseteq A \\ \varphi(x, \bar{y}) \in L}}{\varphi(x)} \in L(A)$, $b = \bar{t}(\underset{\cup}{\bar{j}''})$.

$\bar{j} \sim \bar{j}'$ + order indiscernibility of $\mathcal{J}$

$$\models \varphi(a, b) \iff \models \varphi(t(\bar{j}), \bar{t}(\bar{j}'')) \iff \models \varphi(t(\bar{j}'), \bar{t}(\bar{j}''))$$

$\iff \models \varphi(a', b)$ ∎

<u>Remark</u>

Let T: $\aleph_0$-stable, $p \in S(A)$, $RM(p) < \infty$, $Mlt(p) = 1$.

Then $\forall \underset{\mathfrak{M}}{\overset{\cap\!\shortparallel}{B}} \supseteq A$ $\exists! \, p_B \in S(B)$ $RM(p_B) = RM(p)$

Def. $I = \{a_\alpha : \alpha < \beta\} \subseteq \mathcal{M}$ is a Morley sequence in $p \in S(A)$ if $\forall \alpha < \beta$ $\; a_\alpha \models p_{Aa_{<\alpha}} \in S(Aa_{<\alpha})$

$$\begin{array}{c} p_{Aa_{<\alpha}} \\ \supseteq \\ p \,;\ \mathrm{Mlt}(1) \\ \in S(A) \end{array}$$

Remark A Morley sequence in p is indiscernible over A.

Pf. Enough to show order-indiscernibility. Wlog $I = \{a_\alpha : \alpha < \beta\}$, $\beta = \underset{\in CN}{\kappa} \geq \aleph_0$. Induction on $k < \omega$:

$$\forall \begin{array}{l}\alpha_1 < \dots < \alpha_k < \kappa \\ \beta_1 < \dots < \beta_k\end{array} \quad tp(a_{\alpha_1} \dots a_{\alpha_k}/A) = tp(a_{\beta_1} \dots a_{\beta_k}/A)$$

Step $k \to k+1$, $\alpha_{k+1} > \alpha_k$, $\beta_{k+1} > \beta_k$.

$$p \subseteq tp(a_{\alpha_{k+1}}/Aa_{\alpha_1} \dots a_{\alpha_k}) \subseteq p_{Aa_{<\alpha_{k+1}}}$$

the same

RM, Mlt$=1$ $\Longrightarrow tp(a_{\alpha_{k+1}}/Aa_{\alpha_1} \dots a_{\alpha_k}) = p_{Aa_{\alpha_1} \dots a_{\alpha_k}}$

Likewise $tp(a_{\beta_{k+1}}/Aa_{\beta_1} \dots a_{\beta_k}) = p_{Aa_{\beta_1} \dots a_{\beta_k}}$

Consider $f : Aa_{\alpha_1 \dots \alpha_k} \longrightarrow Aa_{\beta_1} \dots a_{\beta_k}$ s.t.

$f|_A = id$, $f(a_{\alpha_i}) = a_{\beta_i}$, $i = 1, \dots, k$. Then f: element.

and $f(\underbrace{p_{Aa_{\alpha_1} \dots A_{\alpha_k}}}_{\supseteq\, p}) = p_{Aa_{\beta_1} \dots a_{\beta_k}}$

and has the same RM as $RM(p)$

$\Downarrow$

$f \cup \{\langle a_{\alpha_{k+1}}, a_{\beta_{k+1}} \rangle\}$ elementary

So $tp(a_{\alpha_1} \dots a_{\alpha_{k+1}} / A) = tp(a_{\beta_1 \dots \beta_{k+1}} / A)$. ∎

<u>Thm</u> (Morley, Shelah) If T: $\aleph_0$-stable and $\kappa \geq \aleph_0$ then T has a saturated model of power κ.

9.05.2021 **Thm (Morley, Shelah)** T: $\aleph_0$-stable $\Rightarrow$

T has a saturated model of power $\varkappa$

Pf. $\bullet$ $M = \bigcup\limits_{\alpha<\varkappa}^{\uparrow} M_\alpha \leftarrow$ elementary chain of models of T of power $\varkappa$

$\|M_\alpha\| = \varkappa$

- $M_\delta = \bigcup\limits_{\alpha<\delta}^{\uparrow} M_\alpha$ when $\delta \in \mathrm{Lim}$
- $\alpha \to \alpha+1$: $M_{\alpha+1} \succ M_\alpha$ s.t.

  (a) $\forall \underset{\text{finite}}{A \subseteq M_\alpha}\ \forall p \in S(A)\ p(M_{\alpha+1}) \neq \emptyset$

  (b) $\forall \underset{\text{finite}}{A \subseteq M_\alpha}\ \forall \underset{\text{stationary}}{p \in S(A)}\ \exists I \subseteq M_{\alpha+1}$ ($|I| = \varkappa \wedge$ I is a Morley sequence in p)

Claim M is saturated

- M is $\aleph_0$-saturated (easy)
- M is $\varkappa$-saturated: (a.c.) let $A \subseteq M$, $|A| < \varkappa$, $p \in S(A)$, $p(M) = \emptyset$. Choose A and p so that $(RM(p), Mlt(p))$ is lexicographically minimal. Then $Mlt(p) = 1$.

~~pf.~~ let choose $\varphi \in p$ with $RM(p) = RM(\varphi)$, $Mlt(p) = Mlt(\varphi)$.

- If $Mlt(\varphi) > 1$, then

$(*)$ $\exists \psi(x) \in L(\mathcal{M})$ $RM(\varphi \wedge \psi) = RM(\varphi \wedge \neg\psi) = RM(\varphi)$

So $\psi(x) = \psi(x, \underset{\mathcal{M}}{\underset{\cap}{\bar{c}}})$, $\varphi(x) = \varphi(x, \underset{M}{\underset{\cap}{\bar{a}}})$. Choose $\bar{c}' \subseteq M$ s.t. $tp(\bar{c}'/\bar{a}) = tp(\bar{c}/\bar{a}) \in S_k(\bar{a})$, $(k = |\bar{c}|)$

(we can choose it by (a))

$(*)$ holds for $\psi'(x, \bar{c}')$ in place of ψ

Let $A' = A \cup \bar{c}' \subseteq M$, $|A'| < \kappa$

$$\begin{array}{ccc} S(A) & & S(A') \\ \Psi & & \\ & & p_1 \\ p & & p_2 \\ & & \vdots \\ & & p_l \end{array}$$

s.t. $RM(p_i) = RM(p)$

$(l \leq Mlt(p))$

Also $(*) \Rightarrow Mlt(\varphi) = Mlt(\varphi \wedge \psi') + Mlt(\varphi \wedge \neg\psi')$

Look at p_1: either $\varphi \wedge \psi \in p_1$ or $\varphi \wedge \neg\psi \in p_1$

$\Downarrow$

$(RM(p_1), Mlt(p_1)) <_{lex} (RM(p), Mlt(p))$

$p_1(M) \subseteq p(M) = \emptyset \Rightarrow p_1(M) = \emptyset$ ↯

Therefore $\mathrm{Mlt}(\varphi) = 1$. Choose a finite $B \subseteq A$

s.t. $RM(p) = RM(p|B)$, $\mathrm{Mlt}(p) = \mathrm{Mlt}(\underbrace{p|B}_{p'})$

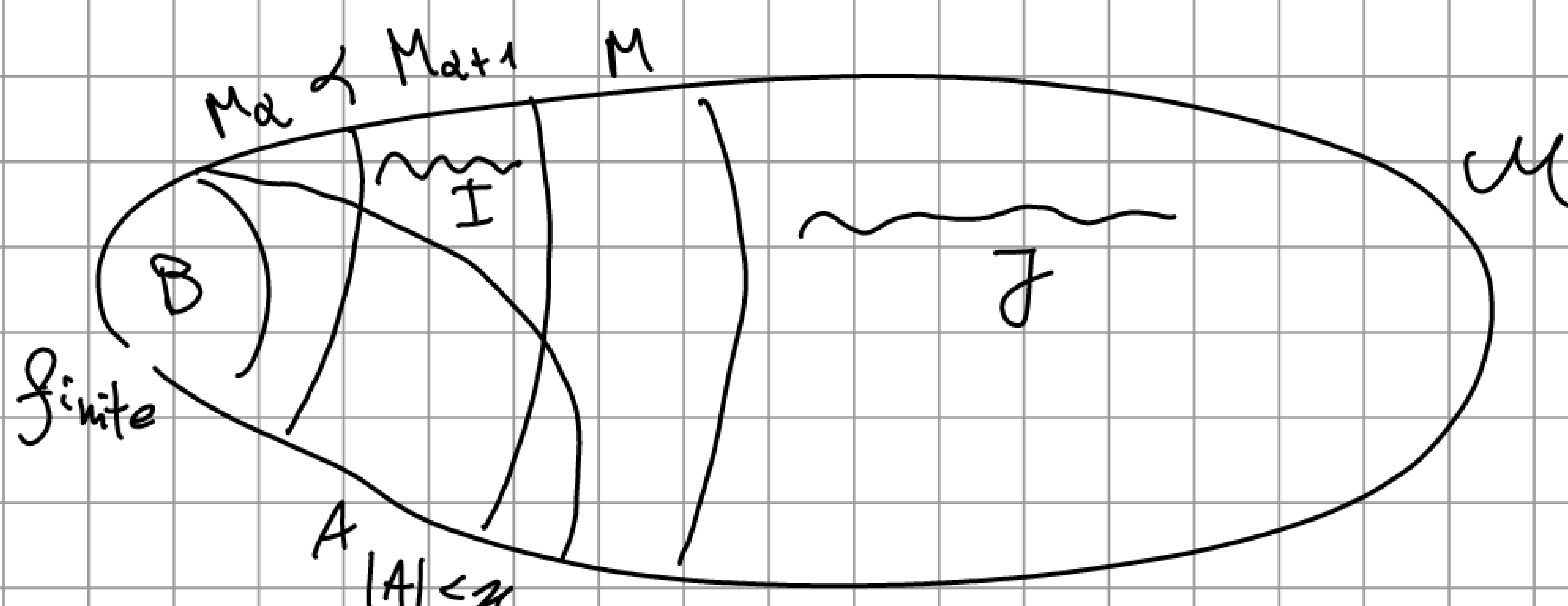
(enough that $\varphi \in L(B)$)

By (b) $\exists I \subseteq M$: a Morley sequence in p',

$|I| = \kappa$. Let $\underset{\mathcal{M}}{\overset{?}{J}} = \{a_\alpha : \alpha < \kappa\}$: a M. sequence in $p'_{AI} \in S(AI)$

$p \subseteq$

Then $I \cup J$ is a Morley sequence in p'_{AI}.



Let $\chi(x) \in p \subseteq p'_{AI} \Rightarrow \underset{\text{infinite}}{J} \subseteq \chi(\mathcal{M})$,

However $I \cup J$ indiscernible $\Rightarrow \{i \in I : \models \chi(i)\}$ is also cofinite in I (by the lemma from prev. lecture)

$I \cup J = (I \cup J)^+_\chi \cup \underset{\text{finite}}{(I \cup J)^-_\chi}$

$$\left|\bigcup_{\chi\in p} I_\chi^-\right| < \kappa \Rightarrow \left|\bigcap_{\chi\in p} I_\chi^+\right| = \kappa \Rightarrow \left|\bigcap_{\chi\in p} I_\chi^+\right| \neq \emptyset.$$

$(|p|<\kappa)$ Any $c \in \bigcap_{\chi\in p} I_\chi^+$ realises p in M ■

So: if $\kappa > \aleph_0$, T: κ-categorical

$$\Downarrow$$

T: $\aleph_0$-stable

$$\Downarrow$$

$\exists \underset{\text{saturated}}{M} \models T$ $\|M\| = \kappa$

S-isolation (s = "set")

Def. (1) $p \in S(A)$ is s-isolated if $\exists \underset{\text{finite}}{B \subseteq A}$ $p|_B \vdash p$.

(2) M is s-atomic over A if $\forall \underset{\text{finite}}{\bar{a} \subseteq M}$ $tp(\bar{a}/A)$ is s-isolated

(3) M is an s-model if $\forall \underset{\text{finite}}{A \subseteq M}$ $\forall p \in S(A)$ $p(M) \neq \emptyset$

$\qquad \qquad \qquad \qquad \equiv$ $\aleph_0$-saturated

(4) M is s-prime over A if M is s-model and $\forall \underset{\hat{M}}{N} \supseteq A$ ($\aleph_0$-saturated) $\exists f: M \xrightarrow[A]{\equiv} N$ $(f|_A = id_A)$

Remark T: $\aleph_0$-stable, $p \in S(B)$, $B \subseteq A$ finite

$\Rightarrow \exists q \in S(A)$, $p \subseteq q$, q: s-isolated

Proof (A.c.) Suppose there's no such q.

(1) $p \nvdash$ a type in $S(A)$. So there's $B \subseteq B_0 \subseteq A$ and $p_0 \neq p_1 \in S(B_0)$ extending p.

(2) $p_i \nvdash$ a type in $S(A)$. So: there is $B_0 \subseteq B_1 \subseteq A$
$i = 0, 1$

and $p_{00}, p_{01}, p_{10}, p_{11} \in S(B_1)$, $p_0 \subseteq p_{00}, p_{01}$, $p_1 \subseteq p_{10}, p_{11}$
pairwise distinct

(3) ....

$B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$

$$
\begin{array}{ccccccccc}
 & & & p \in S(B) & & & & \\
 & & \swarrow & & \searrow & & & \\
 & p_0 & & & & p_1 & & \in S(B_0) \\
 \swarrow & & \searrow & & \swarrow & & \searrow & \\
 p_{00} & & p_{01} & & p_{10} & & p_{11} & \in S(B_1) \\
 \swarrow\ \downarrow & & \swarrow\ \searrow & & \swarrow\ \downarrow & & \swarrow\ \searrow &
\end{array}
$$

Let $B_\omega = \bigcup\limits_n^{\uparrow} B_n \subseteq A$: ctble set, but $|S(B_\omega)| \geq \aleph_0$ ∎

__Property of s-isolation:__

$tp(ab/A)$ s-isolated $\Leftrightarrow$ $tp(a/A)$ is s-isolated and $tp(b/Aa)$ is s-isolated.

(exercise)

__Corollary__ T: $\aleph_0$-stable $\Rightarrow \forall A \subseteq \mathcal{M}\ \exists M \supseteq A$

M s-prime over A

__Proof__ (sketch) $M = A \cup \underbrace{\{a_\alpha : \alpha < ?\}}_{\text{s-construction}}$ s.t.:

(1) $tp(a_\alpha / A a_{<\alpha})$ is s-isolated

(2) M is $\aleph_0$-saturated

it terminates at some point

Then M is s-prime

The model M constructed this way is s-constructible $/A$ and s-primary $/A$, it's unique up to $\underset{A}{\cong}$.

Corollary T: $\aleph_0$-stable, M: s-prime $/A$

$\Rightarrow M$: s-atomic

Proof Let N: s-primary $/A$

$\Downarrow$ property of s-isolation

N: s-atomic

$$M\text{: s-prime }/A \Longrightarrow \exists f: M \xrightarrow[A]{\equiv} N$$

$\Downarrow$

M: s-atomic

Def (M, N) is a Vaughtian pair for T, if

$N \precneqq M \models T$ and for some $\varphi(x) \in L_1(N)$ non-algebraic consistent

$\varphi(N) = \varphi(M)$

(here (M, N, φ): Vaughtian triple)

Lemma 1 Assume T: $\aleph_0$-stable (enough T: small). If T has a Vaughtian pair, then there's Vp (M, N) s.t. M, N: ctble and saturated.

Proof Let (M_0, N_0, φ): Vaughtian triple.

$\varphi(\bar{x}, \bar{a})$

$\bar{a} \subseteq N_0$

Let $L' = L \cup \{\bar{a}\} \cup \{P(x)\}$

$\bar{a}$: new constant symbols; $P(x)$: new predicate symbol

T': a complete theory in L' s.t.

(0) $T' \supseteq Th(N_0, \bar{a}) = Th(M_0, \bar{a})$ and for any $M' \models T'$:

(1) $N := P^{M'}(M')\restriction_L \prec M := M'\restriction_L$

(2) $\bar{a}^{M'} \subseteq P(M')$: $\bigwedge_i P(a_i) \in T'$

(3) (M, N, φ) : a V. triple : $\varphi = \varphi(x, \bar{a}^{M'})$

- $[\forall x (\varphi(x) \longrightarrow P(x))] \in T'$
- $[\exists x \neg P(x)] \in T$

Ad(1): Let $\psi(\bar{x}, y) \in L$.

$T' \ni [\forall \bar{x} [\bigwedge_i P(x_i) \wedge \exists y\, \psi(\bar{x}, y) \longrightarrow \exists y (P(y) \wedge \psi(\bar{x}, y)]]$

<u>Fact</u> $\exists M' \models T'$ ($M := M'\restriction_L$ and $N : P(M')\restriction_L$ are both ctble and saturated)
ctble

<u>Pf</u> (fact) $M' = \bigcup_{n<\omega} M'_n$: models of T' elem. chain

- M_0' : arbitrary
- $n \rightsquigarrow n+1$: $\underset{\text{ctble}}{M'_{n+1}} \succ M_n$ such that:

(i) $\forall \underset{\text{finite}}{A \subseteq M_n'}\ \forall p \in S^L(A)\ p(M'_{n+1}) \neq \emptyset$

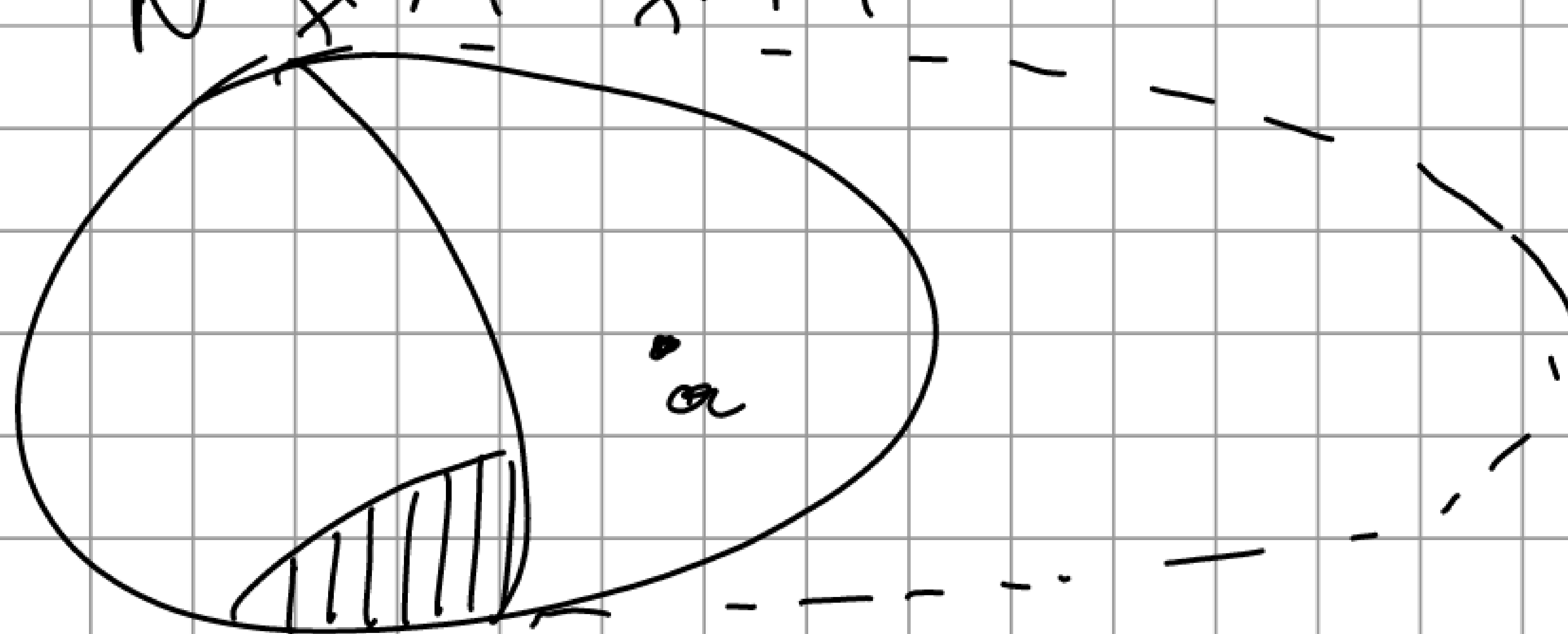
(ii) $\forall \underset{\text{finite}}{A \subseteq P(M_n')}\ \forall p \in S^L(A)$ $p(x) \cup \{P(x)\}$ is realised in M'_{n+1}.

Fact, Lemma 1 ∎

<u>Lemma 2</u> (stretching Vaughtian pair)

Let (M, N, φ): V. triple, M, N: $\aleph_0$-saturated, T: $\aleph_0$-stable. Then $\exists M' \succneq M$ ($\aleph_0$-saturated) (M', N, φ) is a V. triple

$N \precneq M \precneq M'$



$\varphi(N) = \varphi(M) = \varphi(M')$

<u>Proof</u> Let $a \in M \setminus N$, $p = tp(a/N)$, $MR(p) = 1$.

So $p \subseteq q \in S(M)$, $RM(q) = RM(p)$
unique

⋮

16.05.2022

Lemma 2 (Stretching Vaughtian pair)

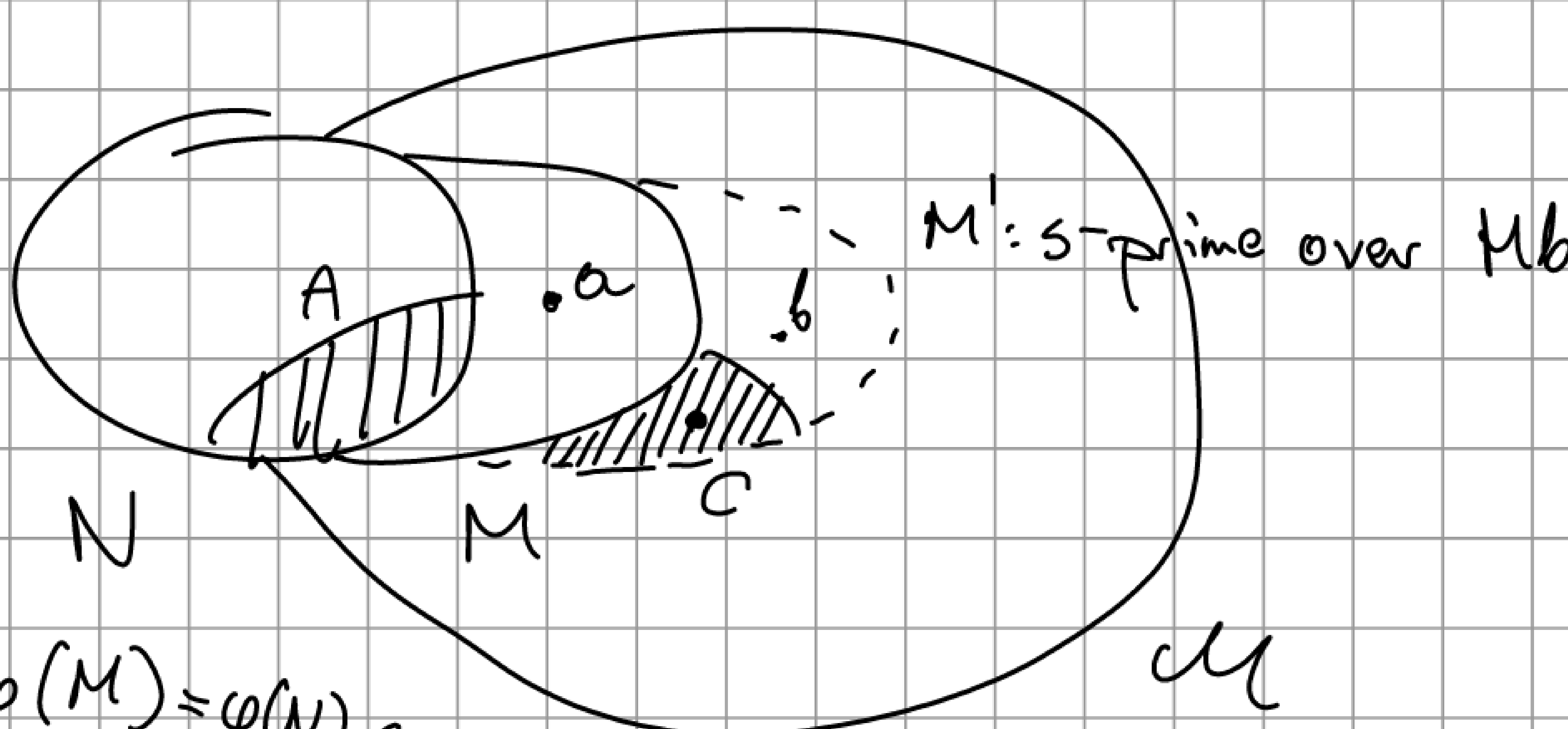
T: $\aleph_0$-stable, (M, N, φ): Vaughtian triple

$\aleph_0$-saturated

$\Rightarrow \exists M' \succneq M \quad (M', N, \varphi)$: Vt.

$\aleph_0$-saturated

Proof



$A = \varphi(M) = \varphi(N) \subseteq \varphi(M')$

$tp(b/M) \supseteq tp(a/M)$, $RM(b/A) = RM(b/M) = RM(a/N)$

Claim: $\varphi(M) = \varphi(M')$

$\blacksquare$

Corollary Suppose $\varkappa > \aleph_0$, T: $\varkappa$-categorical.

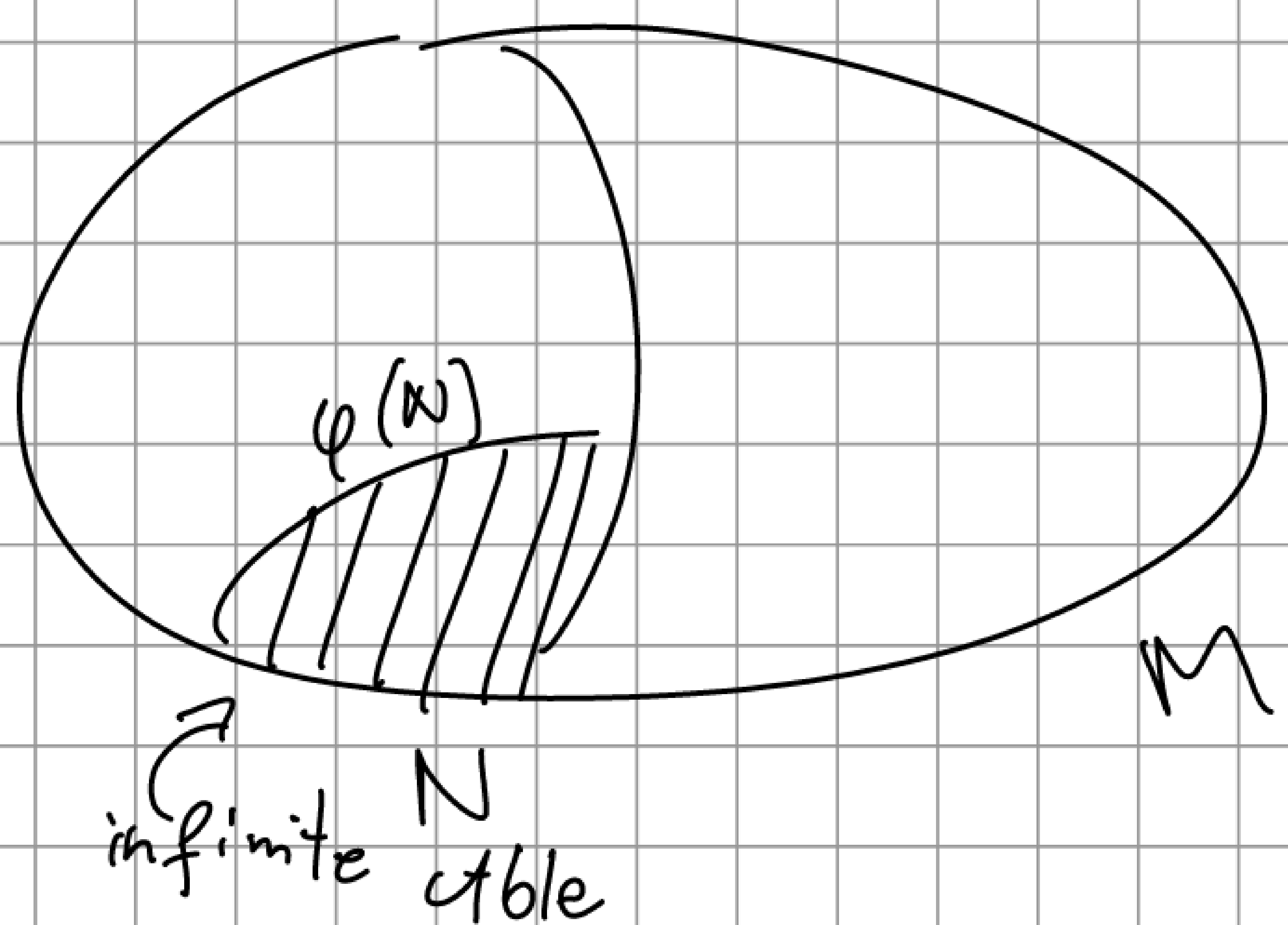
T has no Vaughtian pair and T: $\aleph_0$-stable.

Proof T: $\aleph_0$-stable: proved earlier.

No Vaughtian pair: (A.c.) suppose (M, N, φ): Vt.

By lemma 1 wlog M, N are $\aleph_0$-saturated, N: ctble

by lemma 2 wlog $\|M\| = \kappa$, so M is not saturated



$p(x) = \{\varphi(x)\} \cup \{x \neq a\}_{a \in \varphi(N)}$ ⟹ not realised in M.

But T: $\aleph_0$-stable ⇝ $\exists M'$ $\|M'\| = \kappa$ saturated

$M \not\cong M'$ ↯

<u>Theorem</u> If T: $\aleph_0$-stable without a Vaughtian pair.

Then $\forall \kappa > \aleph_0$ T: κ-categorical.

<u>Corollary</u> (Morley thm, 1964) Let $\kappa > \aleph_0$, ⊃

(1) T: κ-categorical

(2) T: $\aleph_1$-categorical

Lemma 3 (T: $\aleph_0$-stable with no Vaughtian pair).

$\exists$ a $\underbrace{\text{strongly minimal}}_{RM(\varphi)=Mdt(\varphi)=1}$ formula $\varphi(x, \bar{c})$, $\bar{c} \subseteq \underset{\text{prime}}{M} \models T$

Proof Problem from list 6 ∎

Lemma 4 (T: $\aleph_0$-stable, no V. pair) Assume

$M \models T$, $\underbrace{\varphi(x)}_{s.m.} = \varphi(x, \underset{\underset{M}{\cap}}{\bar{a}})$, then:

(a) M is prime and minimal $/ \varphi(M) \cup \{\bar{a}\}$

(b) If $\|M\| > \aleph_0$, then $\dim(\varphi(M)/\bar{a}) = \|M\|$

Explanation 1. $A \subseteq M \prec \mathfrak{M}$, M: minimal $/A$

$\Leftrightarrow \neg \exists \underset{\underset{A}{\cup}}{M'} \precneq M$

2. acl-dimension: Assume $\mathcal{U} = \varphi(M)$, where

$\varphi(x, \bar{a})$: s.m., $acl_{\bar{a}}: P(\mathcal{U}) \to P(\mathcal{U})$,

$acl_{\bar{a}}(\underset{\underset{\mathcal{U}}{\cap}}{A}) = acl(A\bar{a}) \cap \mathcal{U}$

3. $(\mathcal{U}, \underbrace{acl_{\bar{a}}}_{cl})$ is a pregeometry, i.e.

- $cl(cl(A)) = cl(A) \supseteq A$

- $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$
- $\mathrm{cl}(A) = \bigcup_{\substack{A_0 \subseteq A \\ \text{finite}}} \mathrm{cl}(A_0)$
- $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A) \Rightarrow b \in \mathrm{cl}(Aa)$

$\rightsquigarrow$ Basis $\mathcal{B} \subseteq \mathfrak{U}$ cl-independent set, $\dim(\mathfrak{U}) = |\mathcal{B}|$.

In the case of $\mathrm{acl}_{\bar{a}}$ on $\varphi(M)$ additionally:

Let $\underset{\text{s.m.}}{p(x)} \in S(\bar{a}) \cap [\varphi(x)]$ i.e. $RM(p) = MH(p) = 1$

Then an $\mathrm{acl}_{\bar{a}}$-independent set $I \subseteq \varphi(M)$ is a Morley sequence in $p \Rightarrow$ indiscernible over $\bar{a}$.

Pf (lemma 4)

(a) Suppose M prime and is not minimal $/ \varphi(M) \cup \bar{a}$.

So $\varphi(M) \cup \bar{a} \subseteq \underset{\substack{\uparrow \\ \text{prime } / \varphi(M) \cup \bar{a}}}{N} \not\cong M$.

But then (M, N, φ): a V. triple $\lightning$

(b) Let $\underset{\text{an } \mathrm{acl}_{\bar{a}}\text{-basis of } \varphi(M)}{I \subseteq \varphi(M)}$. Then $\bar{a} \cup \varphi(M) \subseteq \mathrm{acl}_{\bar{a}}(I) \cup \bar{a} \subseteq M$

$\|M\| \overset{(a)}{=} |\bar{a} \cup \varphi(M)| \underset{\substack{\uparrow \\ \varphi(M) \\ \text{uncountable}}}{=} |I| = \dim(\varphi(M)/\bar{a})$

<u>Proof of Morley thm</u>

Let $\varkappa > \aleph_0$, $\underset{\curlyvee}{M}, \underset{\curlyvee}{N} \models T$ of power $\varkappa$.

$\underset{\substack{\cup\!\!\!\mid \\ \bar{a}}}{M_0}$: prime

$\varphi(x, \bar{a})$: s.m. (lemma 3)

By lemma 4 : M prime & minimal $/ \varphi(M, \bar{a}) \cup \bar{a}$

N : —— $\|$ —— $/ \varphi(N, \bar{a}) \cup \bar{a}$

Let $I \subseteq \varphi(M)$ bases.
$J \subseteq \varphi(N)$

Let $p \in S(\bar{a}) \cap [\varphi(x)]$: a s.m. type

I, J : Morley sequences in p. By L46

$|I| = |J| = \varkappa$. $f : I\bar{a} \to J\bar{a}$ s.t.

$f = \begin{cases} f \restriction \bar{a} = \mathrm{id}_{\bar{a}} \\ f : I \xrightarrow[\text{onto}]{1-1} J \end{cases}$ $\quad$ f is elementary

$\Downarrow$

$f : \underset{\substack{\cup\!\!\!\mid \\ \varphi(M)_{\bar{a}}}}{\mathrm{ad}(I\bar{a})} \xrightarrow{\cong} \underset{\substack{\cup\!\!\!\mid \\ \varphi(N)_{\bar{a}}}}{\mathrm{ad}(J\bar{a})}$

$$f\restriction_{\varphi(M)\bar a} : \underset{\substack{\rotatebox{90}{$\subseteq$}\\ M}}{\varphi(M)\bar a} \xrightarrow{\equiv} \underset{\substack{\rotatebox{90}{$\subseteq$}\\ N}}{\varphi(N)\bar a}$$

$$\rotatebox{-90}{$\subseteq$}$$

$$f' : M \xrightarrow{\cong} N$$

(lemma 4a)

∎

<u>Thm</u> (Baldwin, Lachlan, 1971) Assume T: $\aleph_1$-categorical not $\aleph_0$-categorical. Then:

(1) T has $\aleph_0$ many ctble models

(2) Every model of T is homogenous.

$\aleph_1$-categorical theories:

- $\aleph_0$-stable
- have prime model
- no V. pairs
- s.m. sets
- M. sequences, "bases"

Stable formulas

T: fixed complete theory with infinite models

$$\underset{\text{ctble}}{L} \ni \varphi = \varphi(\overset{x}{\bar{x}}, \overset{y}{\bar{y}})$$

distinguished variables

parameters variables

<u>Def</u> Let $\delta(x, y) \in L$.

(1) $\delta(x, c)$, $c \subseteq \mathcal{M}$: an <u>instance</u> of δ

(2) $\varphi(x)$ is <u>positive δ-formula</u> if

$\models \varphi \leftrightarrow$ [positive] boolean combination of instances of δ

(3) ($A \subseteq \mathcal{M}$) <u>δ-type</u> = a type consisting of δ-formulas,

$L_\delta(A) = \{\delta\text{-formulas over } A\}$

(4) $S_\delta(A) = \{\text{complete } \delta\text{-types over } A\}$ (ultrafilters in $L_\delta(A)$)

<u>Remark</u> Let $\varphi \in L(A)$, $A \subseteq M$. Then

φ is a δ-formula $\Leftrightarrow \models \varphi \leftrightarrow$ boolean comb. of instances of δ over M

<u>Pf</u> ($\Rightarrow$) $\mathcal{M} \models \varphi(\bar{x}) \leftrightarrow \bigwedge_i \bigvee_j \delta(x, c_{ij})^{\varepsilon_{ij}}$ $\varepsilon_{ij} \in \{0,1\}$

$c_{ij} \in \mathcal{M}$

$\mathcal{M} \models (\exists y_{ij})\left(\varphi(\bar{x}) \leftrightarrow \bigwedge_i \bigvee_j \delta(x, y_{ij})\right)$

$\succ$

$M \models$ ——— " ———

So c_{ij} may be taken from M.

<u>Example</u> T: the theory of a simple equivalence relation E with two infinite classes. Let $\delta(x,y) = E(x,y)$.

$\models \underbrace{x = x}_{} \leftrightarrow E(x,a) \vee E(x,b)$

is a δ-formula $/\emptyset$, where $a,b \in \mathcal{M} \models T$, $\models \neg E(a,b)$

<u>Def</u> (1) $\delta(x,y)$ has <u>order property</u>, if

$\exists \underset{i<\omega}{a_i, b_i} \subseteq \mathcal{M} \quad \forall i,j<\omega \models \delta(a_i, b_j) \Leftrightarrow i \leq j$

(2) $\delta(x,y)$ is <u>stable</u> if it does not have the order property

<u>Lemma 1</u> (1) $\varphi(x,y), \psi(x,z)$: stable

$\Rightarrow \neg\varphi(x,y), (\varphi \vee \psi)(x,yz), (\varphi \wedge \psi)(x,yz)$

are stable

(2) Let $\psi(y,x) = \varphi(x,y)$, then

φ stable $\Leftrightarrow \psi$ stable

(3) φ : stable $\Leftrightarrow \exists n<\omega \ \neg\exists a_i b_i \ (i \leq n)$

$\bigwedge_{i,j} \models \varphi(a_i, b_j) \leftrightarrow i \leq j$

Pf. excercise

Def Let $p \in S(M)$ or $p \in S_\delta(M)$.

A δ-definition of p: a formula $\psi(y) \in L(M)$ s.t. $\forall c \in M$ $(\delta(x,c) \in p \Leftrightarrow \models \psi(c))$

(i.e. $\{c \subseteq M : \delta(x,c) \in p\} = \psi(M)$)

Lemma 2 Assume $\delta(x,y)$ is stable, $p \in S(M)$ or $p \in S_\delta(M)$. Then:

(1) $p(x)$ has a δ-definition $\psi(y)$ that is a positive δ^*-formula, where $\delta^*(y,x) = \delta(x,y)$

(2) $A \subseteq M$ and M is $|A|^+$-saturated, then

$\exists c_1, c_2, \ldots \subseteq M$ $c_i \models p|_{Ac_{<i}}$

and the δ-definition of p is equivalent to a positive boolean combination of formulas $\delta(c_i, y)$, $i < \omega$.