Preliminaries

T : a complete consistent theory, in language L (countable) with infinite models.

that is, $T = Th(M)$, M : L-structure. infinite

L denotes also the set of formulas of language L

$M = (|M|; \ldots)$, but M also denotes $|M|$ (for brevity)

$\underset{\emptyset \neq \text{ universe of } M}{\overset{\uparrow}{|M|}}$

usually we omit $|\cdot|$ in $|M|$.

$M \supseteq A$: a set of parameters.

$L_n(A) = \{\varphi(x_1, \ldots, x_n, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in L,\ \bar{a} \subseteq A\}$

$L(A) = \bigcup_n L_n(A)$, also $L(A)$: language L extended by names for elements of A.

$L_n(A)$: Lindenbaum algebra.

[formally: on $L_n(A)$: $\varphi \sim \psi \Leftrightarrow T(A) \vdash \varphi \leftrightarrow \psi$

$\Leftrightarrow M \models \varphi \leftrightarrow \psi$

here: $\underbrace{T(A)}_{} = Th(M, a)_{a \in A}$

a complete theory in language $L(A)$.

$L_n(A)/_\sim$: a Boolean algebra

(Lindenbaum algebra)

$[\varphi]_\sim \wedge [\psi]_\sim = [\varphi \wedge \psi]_\sim$ etc.

shortly: $L_n(A)$ denotes also $L_n(A)/_\sim$.

$S_n(A) = \{$complete n-types over A, in $M\}$.
in variables $x_1, \dots, x_n$

consistent n-type over A $\rightsquigarrow$ proper filter in $L_n(A)$

An n-type $p(\bar{x})$ over A is <u>complete</u> if

$\cap\!|$
$L_n(A)$

- $p(\bar{x})$: consistent type
- $\forall \varphi(\bar{x}) \in L_n(A)$ $(\varphi(\bar{x}) \in p$ or $(\neg\varphi(\bar{x})) \in p)$

$S(A) := S_1(A)$
(default)

$S_n(A)$: topological space:

for $\varphi(\bar{x}) \in L_n(A)$

$$[\varphi] = \{p \in S_n(A) : \varphi \in p\}$$

basic open set [clopen]
$\shortparallel$
closed and open

$S_n(A)$: compact Hausdorff space, 0-dimensional
(i.e. basis of clopen sets)

complete n-types /A $\leftrightsquigarrow$ ultrafilters in $L_n(A)$

So $S_n(A) = S(L_n(A))$, the Stone space of ultrafilters in $L_n(A)$

- the ~~topo~~ topology on $S_n(A)$ = the Stone space topology.

For $p(\bar{x}) \in S_n(A)$

$$p(M) = \{\bar{a} \in M^n : \underbrace{\overset{\text{realizes } p}{\bar{a} \text{ satisfies } p}}_{\bar{a} \models p \text{, i.e. } M \models \varphi(\bar{a}) \text{ for every } \varphi(\bar{x}) \in p(\bar{x})}\}$$

- The same notation for arbitrary type (also incomplete)
- A formula $\varphi(\bar{x}) \in L(M)$: a special case of a type $\{\varphi(\bar{x})\}$.

$\varphi(M) = \ldots$

- When $p \in S_n(A)$, $\bar{a} \subseteq M$ and $\bar{a} \models p$, then

$$p = tp^M(\bar{a}/A) = tp(\bar{a}/A) = \{\varphi(\bar{x}) \in L_n(A) : M \models \varphi(\bar{a})\}.$$

<u>Example</u> Assume $p(\bar{x})$: a consistent type over M.

Then $\exists N \succcurlyeq M$, $\underbrace{p \text{ is realized in } N}_{\text{i.e. } p(N) \neq \emptyset.}$

From now on "a type" means "a consistent type".

Def A type $p(\bar{x})$ over A is _isolated_, if:

$\exists\, \varphi(\bar{x}) \in L_n(A)$ ① $\varphi(\bar{x})$ is consistent (with T), i.e.

$\varphi(M) \neq \emptyset$ ($\Leftrightarrow T(A) \vdash \exists \bar{x}\, \varphi(\bar{x})$)

(2) ~~$\varphi(M)$~~

symbolically: $\varphi(\bar{x}) \vdash p(\bar{x})$ $\longrightarrow$ $\forall \psi(\bar{x}) \in p(\bar{x})$ $\varphi(M) \subseteq \psi(M)$

$$\Updownarrow$$

$$T(A) \vdash \varphi(\bar{x}) \to \psi(\bar{x})$$

- When $p(\bar{x})$: a complete type over A, then:

  $p(\bar{x})$ is isolated $\Leftrightarrow$ p is isolated in $S_n(A)$ in the topological sense (i.e. $\{p\}$ is open)

Tarski – Vaught test

Assume $A \subseteq M$. Then $A = |N|$ for some $N \prec M$ _iff_

$$\forall \varphi(x) \in L_1(A)\ [\varphi(M) \neq \emptyset \Rightarrow \varphi(M) \cap A \neq \emptyset]$$

Construction of an elementary submodel of M containing A:

- $A_n \subseteq M$, $n < \omega$, increasing chain of sets

recursive construction:

$A_0 = A$

$A_n \subseteq A_{n+1} \subseteq M$ such that $\forall \psi(x) \in L_1(A_n)$

$$[\psi(M) \neq \emptyset \Rightarrow \psi(M) \cap A_{n+1} \neq \emptyset]$$

$A_\infty = \bigcup_{n<\omega} A_n$ satisfies TV-test.

Omitting types theorem

Assume $p_n(\bar{x}_n)$, $n<\omega$: a family of non-isolated types in theory T, over $\emptyset$. Then:

$(\exists M \models T)$ M omits every p_n [i.e. $p_n(M)=\emptyset$]

Assume $\underset{\substack{\cup\!| \\ A}}{M}, N \models T$

Def. $f : A \to N$ is elementary ($f : A \xrightarrow{\equiv} N$) if:

$$\forall \bar{a} \subseteq A \ \forall \varphi(\bar{x}) \in L \ (M \models \varphi(\bar{a}) \iff N \models \varphi(f(\bar{a})))$$

$(\iff tp^M(\bar{a}) = tp^N(f(\bar{a})))$

Elementary diagram of $A \subseteq M$:

$D_e(A) = T(A) = Th(M, a)_{a \in A}$

Remark $f : A \to N$ is elementary $\iff (N, f(a))_{a \in A} \models T(A)$

Atomic diagram of $A \subseteq M$:

$D_{at}(A) = \{\varphi \in D_{el}(A) : \varphi$ is a quantifier free sentence$\}$

$= \{\varphi(\bar{a}) \in L(A) : M \models \varphi(\bar{a})$ and $\varphi(\bar{a})$: q.f.-sentence$\}$

Remark $f : M \to N$ is a monomorphism (i.e.: $f : M \xrightarrow{\cong} f(M) \underset{\uparrow \text{substructure}}{\subseteq} N$)

$\iff (N, f(a))_{a \in M} \models D_{at}(M)$.

Here always $f: M \longrightarrow N$ denotes a monomorphism.

$M \subseteq N$: M is a submodel (substructure) of N

$M \prec N$: M is an elementary submodel of N, i.e:

$$M \subseteq N \text{ and } id_M : M \xrightarrow{\equiv} N$$

Remark Assume $M \prec N$, $A \subseteq M$.

(1) Assume $p(\bar{x}) \subseteq L_n(A)$. Then

$p(\bar{x})$ is a consistent type in $M \Leftrightarrow p(\bar{x})$ is a consistent type in N

(2) Assume $A \subseteq B \subseteq M$

- If $p(\bar{x})$: a type over B, then $\underbrace{p\restriction_A \overset{def}{=} p(\bar{x}) \cap L(A)}_{\text{a type over } A}$
- Let $r : S_n(B) \longrightarrow S_n(A)$, $r(p) \overset{def}{=} p\restriction_A$.

  Then r: continuous and "onto".

(3) If $p(\bar{x})$: a type over A, then $\exists q(\bar{x}) \in S_n(A)\ p(x) \subseteq q(\bar{x})$

Saturation, universality, (strong) homogeneity.

Let $\varkappa \in CN$, $\varkappa \geqslant \aleph_0$.

Def. (1) M $\varkappa$-saturated (nasycony) if $\underset{|A|<\varkappa}{\forall A \subseteq M}\ \forall p \in S_1(A)\ p(M) \neq \emptyset$

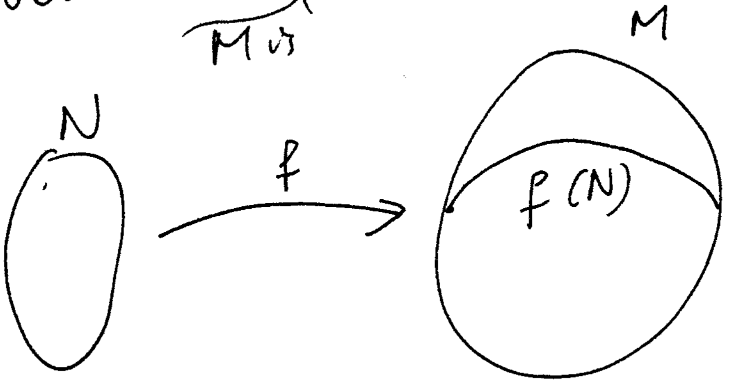
M is saturated if M is $\|M\|$-saturated

(2) M is $\varkappa$-universal if $\forall N \equiv M\ (\|N\| \leqslant \varkappa \Rightarrow \exists f : N \xrightarrow{\equiv} M)$

elementarily equivalent

i.e. $Th(N) = Th(M)$

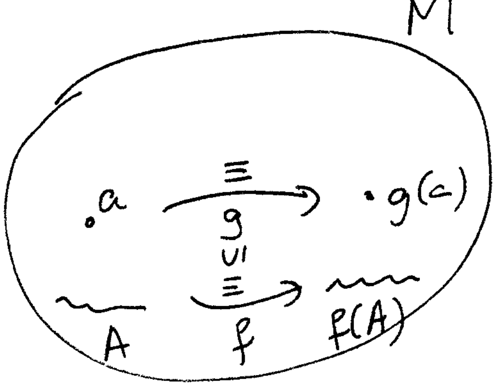
M: universal $\underset{M \text{ is}}{\Longleftrightarrow}$ $\|M\|$-universal



$f: N \xrightarrow{\cong} f(N) \prec M$

(3) M : $\varkappa$-homogeneous if $\forall \underset{|A|<\varkappa}{A \subseteq M}\ \forall a \in M\ \forall f : A \xrightarrow{\equiv} M$
$\exists\, \overset{\cap\mid}{g} : A \cup \{a\} \xrightarrow{\equiv} M$

homogeneous = $\|M\|$-homogeneous.



4. M strongly $\varkappa$-homogeneous if $\forall \underset{|A|<\varkappa}{A \subseteq M}\ \forall f : A \xrightarrow{\equiv} M$
$\exists\, \overset{\cap\mid}{g} : M \xrightarrow{\cong} M$

strongly homogeneous =: strongly $\|M\|$-homogeneous.

5. M is $\varkappa$-compact if ($\forall$ 1-type $p(x)$ over M)
($|p| < \varkappa \Rightarrow p(M) \neq \emptyset$)

Elementary chains of structures

Def $\langle M_\alpha : \alpha < \mu \rangle$, $\mu \in \mathrm{Ord}$, : an ~~e~~ elementary chain of structures if $(\forall \alpha < \beta < \mu)\ M_\alpha \prec M_\beta$.

Union of chain (when $\mu \in \mathrm{Lim}$)

$M_\mu = \bigcup_{\alpha < \mu} M_\alpha$:

- $|M_\mu| := \bigcup^{\nearrow}_{\alpha<\mu} |M_\alpha|$
- $c \in L$ constant symbol $\quad c^{M_\mu} = c^{M_\alpha}$ for $\alpha<\mu$
- P : relation symbol

$$P^{M_\mu}(\underset{\substack{\cap \\ |M_\mu|}}{a_1,\dots,a_n}) \iff M_\alpha \models P(a_1,\dots,a_n) \text{ for } \alpha<\mu$$

sufficiently large
[so that $\bar{a} \subseteq M_\alpha$]

- $f^{M_\mu}(\bar{a}) = b \iff M_\alpha \models f(\bar{a}) = b$ for $\alpha<\mu$ sufficiently large

Fact (Tarski) $M_\alpha \prec M_\mu$ for all $\alpha<\mu$.

Proof (1) $M_\alpha \subseteq M_\mu$ (substructure) : exercise

(2) $\forall \varphi(\bar{x}) \in L \ \forall \alpha<\mu \ \forall \bar{a} \subseteq M_\alpha \ (M_\alpha \models \varphi(\bar{a}) \iff M_\mu \models \varphi(\bar{a}))$

(a) φ atomic : $M_\alpha \subseteq M_\mu$ ✓

(b) $\varphi = \psi_1 \wedge \psi_2$, $\varphi = \neg\psi$: easy

(c) $\varphi(\bar{x}) = \exists y\, \psi(\bar{x}, y)$

$M_\alpha \models \varphi(\bar{a}) \Rightarrow M_\alpha \models \psi(\bar{a}, b)$ for some $b \in M_\alpha$

$\Downarrow$ ind. assumption for ψ

$M_\mu \models \psi(\bar{a}, b)$

$\Downarrow$

$M_\mu \models \varphi(\bar{a})$

$$\begin{tikzcd}
M_\mu \models \varphi(\bar a) \arrow[r, Rightarrow] \arrow[d, equal] & M_\mu \models \psi(\bar a, b) \text{ for some } \arrow[d, Rightarrow, "\text{ind. assumption}"] & b \in M_\mu \arrow[d, Rightarrow] \\
\exists y \, \psi(\bar a, y) & M_\beta \models \psi(\bar a, b) \arrow[d, Rightarrow] & b \in M_\beta \text{ for some } \alpha \leq \beta < \mu \\
& M_\beta \models \varphi(\bar a) \arrow[d, Rightarrow, "M_\alpha \prec M_\beta"] & \\
& M_\alpha \models \varphi(\bar a) &
\end{tikzcd}$$

Elementary directed systems of structures:

Let $(I, \leq)$: a directed set, i.e.:

(1) $\leq$: partial order on I

(2) $(\forall a, b \in I)(\exists c \in I)(a \leq c \wedge b \leq c)$

Example J : a set $\rightsquigarrow$ $([J]^{<\omega}, \subseteq)$: directed set.

Directed system of structures:

- $\mathcal{M} = (M_i, f_{ij})_{i \leq j \in I}$

  f_{ij} ↑ connecting functions $f_{ij} : M_i \longrightarrow M_j$, $f_{ii} = id_{M_i}$ such that

  $(\forall\, i \leq j \leq t \in I)$ $f_{it} = f_{jt} \circ f_{ij}$
  (compatibility)

$$\begin{tikzcd}
& M_t \\
M_i \arrow[ur, "f_{it}"] \arrow[r, "f_{ij}"'] & M_j \arrow[u, "f_{jt}"'] \arrow[ul, phantom, "\#"]
\end{tikzcd}$$

- System $\mathcal{M}$ is elementary if all f_{ij} are elementary.

Example Elementary chain $(M_\alpha)_{\alpha<\mu}$

$\mathcal{M} = (M_\alpha, f_{\alpha\beta})_{\alpha\le\beta<\mu}$ $\quad f_{\alpha\beta} = id_{M_\alpha} : M_\alpha \xrightarrow{\equiv} M_\beta$

elementary directed system of structures

Direct limit of a directed system $\mathcal{M}$: $M_\infty = \varinjlim \mathcal{M}$

$(M_\infty, f_{i\infty})_{i\in I}$, where $f_{i\infty}: M_i \longrightarrow M_\infty$ such that

(1) $\forall i \le j \in I \quad f_{i\infty} = f_{j\infty} \circ f_{ij}$ [compatible with connecting functions]

$$\begin{tikzcd}
M_j \arrow[dr, "f_{j\infty}"] \arrow[d, phantom, "\#"] & \\
 & M_\infty \\
M_i \arrow[uu, "f_{ij}"] \arrow[ur, "f_{i\infty}"'] &
\end{tikzcd}$$

(2) $(\forall (M'_\infty, f'_{i\infty})_{i\in I}$ satisfying (1)$)\ \exists! f : M_\infty \to M'_\infty$

(universality) $\quad (\forall i \in I)\ f'_{i\infty} = f \circ f_{i\infty}$

$$\begin{tikzcd}
 & M_\infty \arrow[dd, "f"] \\
M_i \arrow[ur, "f_{i\infty}"] \arrow[dr, "f'_{i\infty}"'] \arrow[r, phantom, "\#"] & \\
 & M'_\infty
\end{tikzcd}$$

Fact M_∞ exists (and is unique up to $\cong$).

If $\mathcal{M}$ is elementary, then $f_{i,\infty} : M_i \xrightarrow{\equiv} M_\infty$.

Proof 1. Construction of M_∞:

$S := \dot{\bigcup}_{i\in I} |M_i|$: formally disjoint union.

$\sim$ on S : an equivalence relation

$$\overset{M_i}{\overset{\Downarrow}{x}} \sim \overset{M_j}{\overset{\Downarrow}{y}} \overset{\text{def}}{\iff} f_{it}(x)=f_{jt}(y) \text{ for some } (\equiv \text{every}) \; t \geqslant i,j$$

exercise : $\sim$ is transitive.

$$|M_\infty| := S/\sim$$

- $\sim\restriction_{|M_i|}$: the equality (because f_{ij} : 1-1 (monomorphism)
- $f_{i\infty}(\underset{\underset{M_i}{\Uparrow}}{x}) = x/\sim$, $\quad f_{i\infty} : |M_i| \xrightarrow{1-1} |M_\infty|$.

L-structure on $|M_\infty|$:

- $c^{M_\infty} = c^{M_i}/\sim$
- $p^{M_\infty}(a_{i_1}/\sim, \ldots, a_{i_n}/\sim) \iff M_t \models P(f_{i_1 t}(a_{i_1}), \ldots, f_{i_n t}(a_{i_n}))$

  $a_{i_j} \in M_{i_j}$ $\qquad$ for $t \geqslant i_1, \ldots, i_n$
- f^{M_∞} : similarly

the rest is an exercise.

How to extend elementary mappings?

~~Bat.~~ BAlg : Category of Boolean algebras

Comp$_0$: —//— of compact Hausdorff 0-dimensional spaces

$$\begin{array}{rcl} F : BAlg & \longrightarrow & Comp_0 \\ & & \rotatebox{90}{$\in$} \\ F(A) & = & S(A) \end{array} \qquad \begin{array}{rcl} G : Comp_0 & \longrightarrow & BAlg \\ & & \rotatebox{90}{$\in$} \\ G(X) & = & Clopen(X) . \end{array}$$

F, G : contravariant functors "inverse" to each other

~~(F, G) is a duality of categories... (look it up)~~

Categories BAlg and Comp$_0$ are dually equivalent.

A, B : Boolean algebras

$$f : A \longrightarrow B \text{ homomorphism} \Rightarrow \overset{\overset{f^*}{\shortparallel}}{F(f)} : S(B) \longrightarrow S(A)$$

$$F(f)(p) = f^{-1}[p]$$

continuous.

Assume $\begin{array}{c} f : A \xrightarrow{\equiv} B \\ \quad\ \rotatebox{90}{$\subseteq$} \qquad \rotatebox{90}{$\subseteq$} \\ T \Dashv M \ , \ N \models T \end{array}$. Then $\hat{f} : L_n(A) \longrightarrow L_n(B)$

$$\hat{f}(\varphi(\bar{x}, \bar{a})) = \varphi(\bar{x}, f(\bar{a}))$$

homomorphism of Boolean algebras.

even: monomorphism.

We skip ^ in $\hat{f}$, so:

$$f : L_n(A) \longrightarrow L_n(B) \text{ monomorphism}$$

$$\Downarrow$$

$$f^* : S_n(B) \longrightarrow S_n(A) \text{ epimorphism in } Comp_0$$

i.e: continuous onto

Lemma (on extensions of elementary mappings)

Assume $M, N \models T$, $A \subseteq M$, $B \subseteq N$, $f : A \xrightarrow{\equiv} B$ "onto".

Assume $\underset{a}{\overset{\rotatebox{90}{$\in$}}{\cancel{\bar{a}}}}, \underset{b}{\overset{\rotatebox{90}{$\in$}}{\cancel{\bar{b}}}}$, $p = tp(a/A)$, $q = tp(b/B)$.

Then $f \cup \{\langle a, b\rangle\}$ is elementary $\iff f^*(q) = p$.

[here $f^* : S(B) \xrightarrow[\text{homeomorphism}]{\approx} S(A)$].

Proof exercise.

Def. M is $(<\aleph_0)$-universal $\iff \forall n\ \forall p \in S_n(\emptyset)\ p(M) \neq \emptyset$.

Remark M : κ-universal $\Rightarrow$ M : $(<\aleph_0)$-universal.

Proof Let $p \in S_n(\emptyset)$.

Choose a countable $N \models T$ with $p(N) \neq \emptyset$.

M : κ-universal $\Rightarrow$ $\exists f : N \xrightarrow{\equiv} M$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \bar{a} \models p & \overset{f}{\mapsto} & f(\bar{a}) \models p. \end{array}$$

Thm. (1) M : κ-saturated $\Rightarrow$ M : κ-homogeneous and κ-universal.

(2) M : κ-~~universal~~ homogeneous and $(<\aleph_0)$-universal $\Rightarrow$ M : κ-saturated.

Proof. (1) κ-homogeneity of M:

Assume $f : A \xrightarrow{\equiv} M$, $A \subseteq M$, $|A| < \kappa$, $a \in M$.

We seek $b \in M$ s.t. $g = f \cup \{\langle a, b \rangle\}$ elementary

$$\Updownarrow \text{ Lemma}$$

$$f^*(tp(b/B)) = tp(a/A).$$

Let $\underset{\substack{\uparrow \\ S_1(A)}}{p} = tp(a/A)$, $q = (f^*)^{-1}(p) \in S_1(B)$

$\shortparallel$

Let $\underset{\substack{\uparrow \\ \text{good.}}}{b} \in M$ (exists by $\varkappa$-saturation of M)

- $\varkappa$-universality of M:

Assume $N \equiv M$, $\|N\| \le \varkappa$.

We seek $f : N \xrightarrow{\equiv} M$.

Let $\{a_\alpha : \alpha < \mu\}$: an enumeration of N, $\overset{\substack{\varkappa \\ \vee\!|}}{\mu} = \|N\|$.

We define $f(a_\alpha)$ by induction on $\alpha < \mu$:

- Suppose $f(a_\beta)$ defined for all $\beta < \alpha$ so that

  ~~and~~ $f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} M$

  Want to find $f(a_\alpha)$ so that

  $f : \{a_\beta : \beta \le \alpha\} \xrightarrow{\equiv} M$.

~~By the Lemma it is enough that t~~

Let $p = tp(a_\alpha / \{a_\beta : \beta < \alpha\})$.

By the lemma it is enough to find $f(a_\alpha) \in M$ so that $f^*(tp(f(a_\alpha)/\{f(a_\beta) : \beta < \alpha\})) = p$.

(MT2/4)

So let $q = (f^*)^{-1}(p) \in S_1(\underbrace{\{f(a_\beta) : \beta < \alpha\}}_{\text{power} < \kappa})$

M κ-saturated $\Rightarrow$ q realized in M.

Let $f(a_\alpha) \in M$ s.t. $f(a_\alpha) \models q$.

(2) Assume M is κ-homogeneous & $(<\aleph_0)$-~~saturated~~ universal.

Want: M: κ-saturated.

So: Let $A \subseteq M$, $|A| < \kappa$, $p \in S_1(A)$. Show: $p(M) \neq \emptyset$.

<u>Induction on $|A|$.</u>

Case (a): $|A| < \aleph_0$.

$\exists N \succ M$ $p(N) \neq \emptyset$. So let $\overset{N}{b} \models p$.

Let $A^* = \underset{\{a_1,\ldots,a_k\}}{\underset{\shortparallel}{A}} \cup \{b\}$

Let $q = tp^N(a_1,\ldots,a_k,b) \in S_{k+1}(\emptyset)$

q is realized in M (($\overset{\text{by}}{<}\aleph_0$)-universality),

by $\langle \underbrace{a'_1,\ldots,a'_k}_{A'}, b' \rangle$

Let $g : A \cup \{b\} \longrightarrow A' \cup \{b'\}$, $g(a_i) = a_i'$, $g(b) = b'$.

g: elementary.

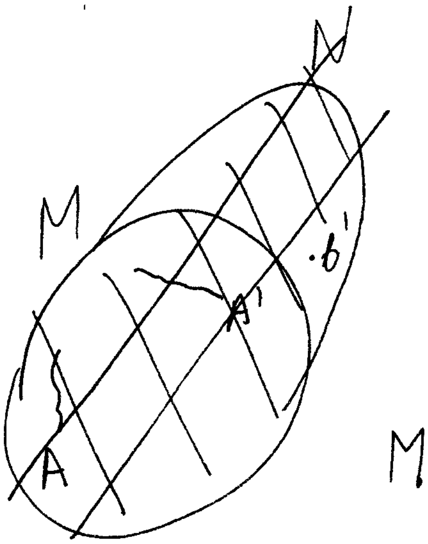
$\Rightarrow \; g\restriction_A : A \xrightarrow{\equiv} A'$

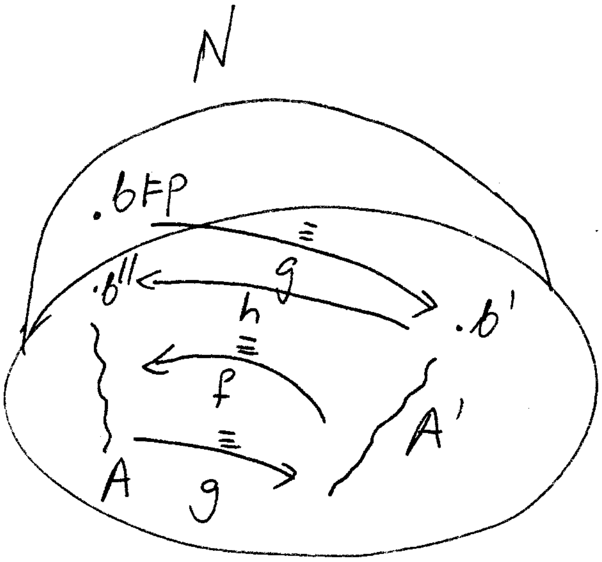
$\Downarrow \; f := (g\restriction_A)^{-1} : A' \xrightarrow{\equiv} A$

$\cap|$

M: κ-homogeneous $\Rightarrow \; \exists \; h : A' \cup \{b'\} \xrightarrow{\equiv} A \cup \{b''\}$

for some $b'' \in M$.





$$\begin{tikzcd}
Ab \arrow[rr, "\equiv", "h\circ g"'] \arrow[dr, "g"', "\equiv"] & & Ab'' \\
& A'b' \arrow[ur, "h"', "\equiv"] &
\end{tikzcd}$$

Let $s = h \circ g$

$s\restriction A = \underbrace{(h\restriction_{A'})}_{\overset{\shortparallel}{(g\restriction_A)^{-1}}} \circ (g\restriction_A) = id_A$.

$\Downarrow$

~~$s^*(tp(b''/A)) = tp(b$~~

$s\restriction_A = id_A \Rightarrow \underset{\overset{\shortparallel}{id_{S(A)}}}{s^*} : S(A) \xrightarrow{\simeq} S(A)$

hence: $p = tp(b/A) \underset{\overset{\uparrow}{\text{Lemma}}}{=} s^*(tp(b''/A)) \underset{\overset{\uparrow}{s^* = id_{S(A)}}}{=} tp(b''/A)$

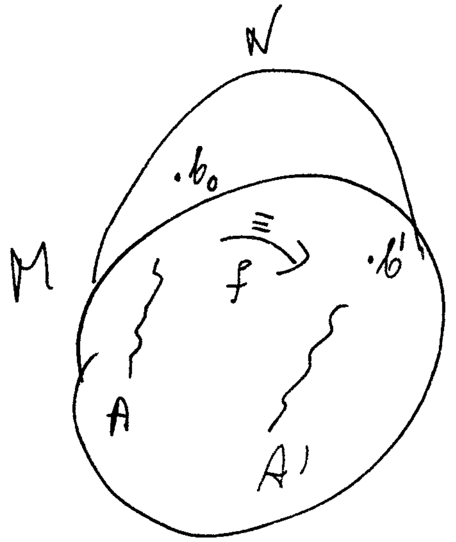
and $b'' \models p$

Case (b) $|A| = \mu$, $\aleph_0 \leq \mu < \kappa$.

$A = \{a_\alpha : \alpha < \mu\}$, $p \in S_1(A)$.

$p\restriction\emptyset \in S_1(\emptyset) \underset{M:(<\aleph_0)\text{-universal}}{\Longrightarrow} \exists b' \in M \ b' \models p\restriction\emptyset$.

$\exists N \succ M \ \exists \cancel{b} \underset{\substack{\models \\ p}}{b_0} \in N$



Will find $A' = \{a'_\alpha : \alpha < \mu\} \subseteq M$

s.t. $f : Ab_0 \longrightarrow A'b'$

given by $f(a_\alpha) = a'_\alpha$

$f(b_0) = b'$

is <u>elementary</u>!

We find a'_α, $\alpha < \mu$ by induction on $\alpha < \mu$.

So suppose $\alpha < \mu$ and a'_β already defined for all $\beta < \alpha$

so that $\underbrace{f\restriction_{\{a_\beta : \beta < \alpha\} b_0}}_{=: f_0} : \{a_\beta : \beta < \alpha\} b_0 \xrightarrow{\equiv} \{a'_\beta : \beta < \alpha\} b'$

We look for a'_α.

Let $q = tp(a_\alpha / \{a_\beta : \beta < \alpha\} \cup \{b_0\})$

Then $(f_0^*)^{-1}(q) \in S(\underbrace{\{a'_\beta : \beta < \alpha\} \cup \{b'\}}_{\text{Power } < \mu \leq |A|})$

By the lemma it is enough that $a'_\alpha \underset{M}{\overset{?}{\models}} (f_0^*)^{-1}(q)$.

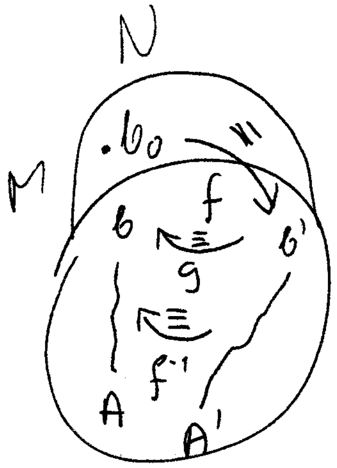
~~But M: κ-~~

~~Let A~~ By the inductive assumption on A:

$(f_0^*)^{-1}(q)$ is realized in M, so we are done with constructing A'.

Now: $f^{-1} : A' \xrightarrow{\equiv} A$ in $M \leftarrow \kappa$-homogeneous, so

$\exists\ \overset{\frown\prime}{g} : A'b' \xrightarrow{\equiv} Ab$ for some $b \in M$.



~~Let~~ Let $s = g \circ f$

$s : Ab_0^{\#} \xrightarrow{\equiv} Ab$

$s\restriction_A = \underset{\underset{(f\restriction_A)^{-1}}{\parallel}}{(g\restriction_{A'})} \circ (f\restriction_A) = id_A$,

so $p = tp(b_0^{\#}/A) = tp(b/A)$

and p is realized in M.

<u>Corollary</u>

M is κ-saturated $\iff$ M is κ-homogeneous and ~~κ-universal~~. and κ-universal.

<u>Proof</u> $\Rightarrow$ by Thm (1).

$\Leftarrow$ κ-homogeneous + κ-universal $\Rightarrow$

κ-homogeneous + $(<\aleph_0)$-universal $\underset{\text{Thm (2)}}{\Rightarrow}$ κ-saturated

Properties of saturated models.

Thm. Assume $M, N \models T$ saturated models of the same power. Then $M \cong N$.

Proof $M = \{m_\alpha : \alpha < \varkappa\}$, $N = \{n_\alpha : \alpha < \varkappa\}$,

$\varkappa = \|M\| = \|N\|$. We find $f : M \xrightarrow{\cong} N$

back-and-forth method:

$$f = \bigcup_{\alpha<\varkappa}^{\nearrow} f_\alpha \qquad f_\alpha : M \xrightarrow{\overset{\equiv}{\circ}} N \quad \text{s.t.:}$$

partial, elementary

(1) $m_\alpha \in \mathrm{Dom}\, f_{\alpha+1}$

$n_\alpha \in \mathrm{Rng}\, f_{\alpha+1}$, $|f_\alpha| \leq 2 \cdot |\alpha|$

(2) $f_0 = \emptyset$

(3) For $\delta \in Lim$, $f_\delta = \bigcup_{\alpha<\delta}^{\nearrow} f_\alpha$.

(4) $f_{\alpha+1} = f_\alpha \cup \{\langle m_\alpha, \underset{\underset{N}{\uparrow}}{n}\rangle, \langle \underset{\underset{M}{\uparrow}}{m}, n_\alpha\rangle\}$

Inductive step:

Suppose we have f_α. Want: $f_{\alpha+1}$.

Let $A_\alpha = \mathrm{Dom}\, f_\alpha \subseteq M$, $B_\alpha = \mathrm{Rng}\, f_\alpha \subseteq N$.

$$\begin{array}{c} f_\alpha : A_\alpha \xrightarrow{\equiv} B_\alpha \\ \Downarrow \\ f_\alpha^* : S(B_\alpha) \xrightarrow{\cong} S(A_\alpha). \end{array}$$

"forth": Find $n \in N$ st. $f_\alpha \cup \{\langle m_\alpha, n\rangle\}$ elementary

$$\Updownarrow$$

$$(f_\alpha^*)^{-1}(tp(m_\alpha/A_\alpha)) = tp(n/B_\alpha).$$

Let $p = tp(m_\alpha/A_\alpha)$.

So $(f_\alpha^*)^{-1}(p) \in S(B_\alpha)$ is realized in N by some n.

"~~back~~": similarly.
back

Thm Assume $M, N \models T$ are homogeneous, of the same power and $\forall n<\omega\ \forall p \in S_n(\emptyset)\ (p(M) \neq \emptyset \Leftrightarrow p(N) \neq \emptyset)$.
Then $M \cong N$.

Lemma Under the assumptions of the Thm,

$$\forall A \subseteq M\ \exists f : A \xrightarrow{\equiv} N.$$

Proof. Induction on $|A|$.

Case (a) $|A| < \aleph_0$. $A = \{a_1, \dots, a_n\}$.

Let $p = tp(\langle a_1, \dots, a_n\rangle) \in S_n(\emptyset)$. realized in M

$$\Downarrow$$

realized in N

by some $\langle b_1, \dots, b_n\rangle \subseteq N$.

$f(a_i) = b_i$ is good.

Case (b) $|A| = \mu \geq \aleph_0$, $A = \{a_\alpha : \alpha < \mu\}$

We find $f(a_\alpha)$ by induction on $\alpha < \mu$.

Inductive step.

Suppose $\alpha < \mu$ and for every $\beta < \alpha$ we have $f(a_\beta)$

s.t. $f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} N$.

We shall find $f(a_\alpha) \in N$ s.t. $f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} N$.

Let $a_{<\alpha} := \{a_\beta : \beta < \alpha\}$. Likewise $a_{\leq\alpha}$.

$|a_{\leq\alpha}| < \mu = |A|$

By inductive assumption: $\exists g : a_{\leq\alpha} \xrightarrow{\equiv} N$.

Then
$$f \circ g^{-1} : \begin{array}{c} g(a_{<\alpha}) \\ \cap \\ N \end{array} \xrightarrow{\equiv} \begin{array}{c} f(a_{<\alpha}) \\ \cap \\ N \end{array}$$

By homogeneity of N: $\exists f(a_\alpha) \in N$ s.t.
$$f \circ g^{-1} : \underset{\substack{\| \\ g(a_{\leq\alpha})}}{\underline{g(a_{<\alpha})\, g(a_\alpha)}} \xrightarrow{\equiv} f(a_{<\alpha})\, f(a_\alpha) = f(a_{\leq\alpha})$$

Then $f = (f \circ g^{-1}) \circ g : a_{\leq\alpha} \xrightarrow{\equiv} N$.

Proof of the theorem

$\kappa := \|M\| = \|N\|$

$f : M \xrightarrow{\cong} N$ constructed by back-and-forth method

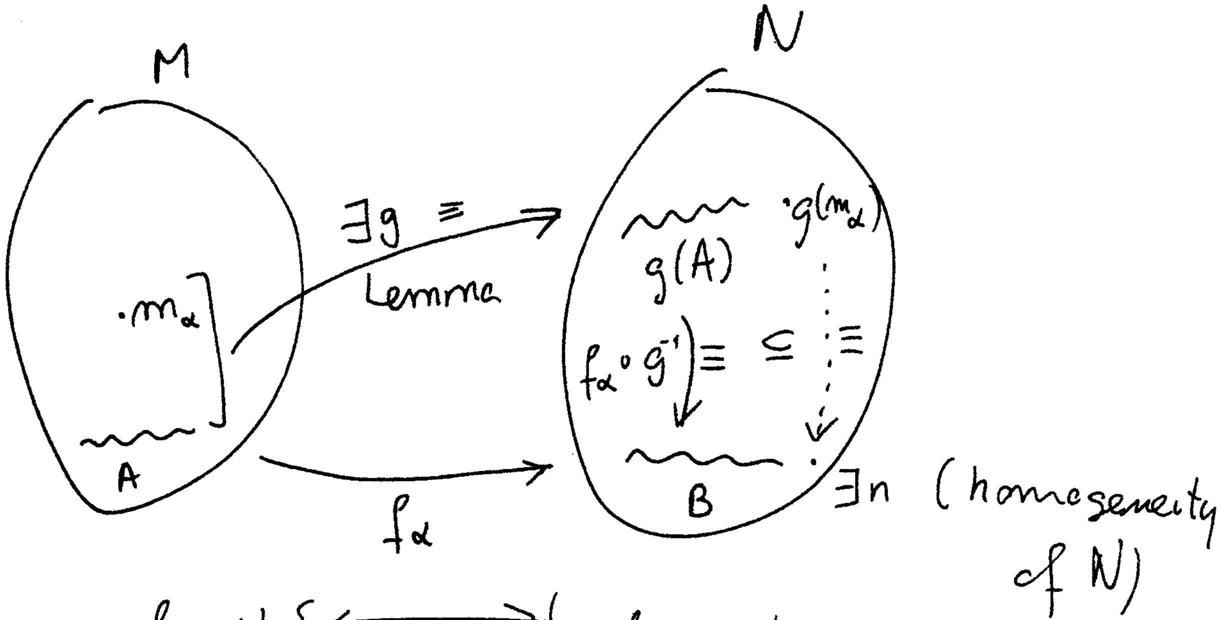
$f = \bigcup^{\uparrow}_{\alpha<\kappa} f_\alpha$, $f_\alpha : M \xrightarrow{\overset{\equiv}{\circ}} N$ (partial elementary), $\alpha < \kappa$

$|f_\alpha| \leq 2 \cdot |\alpha|$ + the same conditions as in the previous thm.

Inductive step $f_\alpha \longmapsto f_{\alpha+1}$

$A = \mathrm{Dom}\, f_\alpha \qquad B = \mathrm{Rng}\, f_\alpha$

"forth":



~~$f_\alpha \cup \{\langle m_\alpha, n\rangle\}$ elementary~~

$$h = (f_\alpha \circ g^{-1})\restriction_{g(A)} \cup \{\langle g(m_\alpha), n\rangle\} \text{ elementary}$$

$$\begin{array}{l} h \circ g : Am_\alpha \xrightarrow{\equiv} Bn \subseteq N \\ \;\cup| \\ f_\alpha \end{array}$$

"back": similarly.

Constructions of models:

• saturated $\Longrightarrow$ • (strongly) homogeneous $\not\Leftarrow$

Thm. $\underbrace{\varkappa = 2^{<\varkappa},\ \varkappa \in Reg}_{\substack{\Updownarrow \\ \varkappa^{<\varkappa} = \varkappa}},\ \varkappa > \aleph_0 \Rightarrow \exists M \models T$ saturated, of power $\varkappa$.

Proof

(*) $|S_1(A)| \leq 2^{|A| + \aleph_0}$, because: $|L_1(A)| = |A| + \aleph_0$

Here: $|A| < \varkappa \Rightarrow |S_1(A)| \leq \varkappa$.

Lemma $N \models T,\ \|N\| \leq \varkappa \Rightarrow \underset{\text{the set}}{\underset{\uparrow}{X_N}} := \bigcup \{S_1(A) : A \subseteq N \ \&\ |A| < \varkappa\}$ has power $\leq \varkappa$.

Pf • $|\{A \subset N : |A| \underset{\nwarrow}{<} \varkappa\}| \leq \varkappa^{<\varkappa} = \varkappa$.

• $|S_1(A)| \leq \varkappa$ for such A.

Proof of the thm.

$M_\alpha,\ \alpha < \varkappa$: elementary chain of models of T of power $\varkappa$.

- M_0 : whatever
- $M_\delta = \bigcup\limits_{\alpha<\delta}^{\uparrow} M_\alpha$, when $\delta < \varkappa$ limit.
- $M_{\alpha+1} \succ M_\alpha$ such that $\forall p \in X_{M_\alpha}\ p(M_{\alpha+1}) \neq \emptyset$:

$T' = Th(M_\alpha, m)_{m \in M_\alpha} \cup \bigcup\limits_{\beta<\varkappa} \{\varphi(\underset{\uparrow}{c_\beta}) : \varphi(x) \in p_\beta\}$,

where $X_{M_\alpha} = \{p_\beta : \beta < \varkappa\}$ new constant symbols,

and T' in language $L(M_\alpha) \cup \{\overset{\downarrow}{c_\beta} : \beta < \varkappa\}$.

T': consistent, has model of power $\varkappa$: $M_{\alpha+1}$
such that $M_\alpha \prec M_{\alpha+1}$.

$M = \bigcup\nolimits^{\uparrow}_{\alpha<\varkappa} M_\alpha$: of power $\varkappa$, saturated:

Let $A \subseteq M$, $|A| < \varkappa$, and $p \in S_1^M(A)$

$\varkappa \in Reg \Rightarrow A \subseteq M_\alpha$ for some $\alpha < \varkappa$.

proof: $A = \{a_\beta : \beta < \mu\}$ for some $\overset{\overset{\text{CN}}{\downarrow}}{\mu} < \varkappa$.

- $\forall \beta < \mu \quad \exists\, \alpha_\beta < \varkappa \quad a_\beta \in M_{\alpha_\beta}$
- $\{\alpha_\beta : \beta < \mu\} \subseteq \varkappa$, $\mu < cf(\varkappa) = \varkappa$

$\Rightarrow \exists\, \underset{\underset{A \subseteq M_\alpha.}{\uparrow}}{\alpha} < \varkappa \quad \forall \beta < \mu \quad \alpha_\beta < \alpha$

~~Let~~ $M_\alpha \prec M \Rightarrow p \in S_1^{M_\alpha}(A) = S_1^M(A)$

p realized in $M_{\alpha+1}$ by some $a \in M_{\alpha+1}$

$a \models p$ in $M_{\alpha+1} \underset{M_{\alpha+1} \prec M}{\Longrightarrow} a \models p$ in M.

Monster model:

Let $\bar{\varkappa}$: a large cardinal number.

"Ideal model" $\mathfrak{M} \models T$: saturated of power $\bar{\varkappa}$

because: $\forall M \models T \; (\|M\| < \bar{\varkappa} \Rightarrow \exists M' \prec \mathfrak{M} \quad M \cong M')$.

Advantages of saturated model $\mathcal{M}$:

(i) universality

(ii) strong homogeneity

~~More~~ Mae <u>Weakly</u> (a bit):

(1) $\bar{\kappa}$-universality

(2) strong $\bar{\kappa}$-homogeneity

$\mathrm{Aut}(M)$: the group of automorphisms of M

$\mathrm{Aut}(M/A) = \{ f \in \mathrm{Aut}(M) : f\restriction_A = id_A \}$: automorphisms of M <u>over A</u>.

$A \subseteq M$

<u>Lemma</u> Assume M is strongly κ-homogeneous, κ-saturated, $A \subseteq M$, $|A| < \kappa$. Then:

(1) For $a, b \in M$ ($tp(a/A) = tp(b/A) \iff a, b$ are in the same orbit of $\mathrm{Aut}(M/A)$ on M.

(2) $[\text{orbits } \mathrm{Aut}(M/A) \text{ on } M^n] \xleftrightarrow[\text{onto}]{1\text{-}1} S_n(A)$

<u>Proof</u> (1) $\Leftarrow$: $f \in \mathrm{Aut}(M/A)$, $f(a) = b$

$$\Downarrow$$

$$tp(a/A) = tp(b/A).$$

$\Rightarrow$: $tp(a/A) = tp(b/A) \implies f : Aa \xrightarrow{\equiv} Ab$

$f\restriction_A = id_A$, $f(a) = b$

strong κ-homogeneity $|A| < \kappa$ $\Rightarrow f \subseteq g \in \mathrm{Aut}(M)$, $g \in \mathrm{Aut}(M/A)$

$g(a) = b$: a, b in the same orbit of $\mathrm{Aut}(M/A)$

(2) $M^n \supseteq \mathcal{O} \xrightarrow[\varphi]{(1)} p_{\mathcal{O}} \in S_n(A)$

$\uparrow$ orbit of $\mathrm{Aut}(M/A)$

$p_{\mathcal{O}}$ $\|$ common type $tp(a/A)$ for $a \in \mathcal{O}$.

$$\begin{tikzcd} \Phi : \{\text{orbits of } \mathrm{Aut}(M/A) \text{ on } M^n\} \arrow[d, "\varphi"] \\ S_n(A) \end{tikzcd}$$

$\mathcal{O}_1 \neq \mathcal{O}_2 \overset{(1)}{\Rightarrow} p_{\mathcal{O}_1} \neq p_{\mathcal{O}_2}$ $\boxed{\text{so } \varphi : 1\text{-}1}$

[if $p_{\mathcal{O}_1} = p_{\mathcal{O}_2}$ then let $a \in \mathcal{O}_1$, $b \in \mathcal{O}_2 \Rightarrow \exists g \in \mathrm{Aut}(M/A)$ $g(a) = b$ ↯.]

M: κ-saturated $\Rightarrow$ φ: "onto".

<u>Def</u> Let $\bar{\kappa}$: a (large) cardinal number, $\mathcal{M} \models T$ <u>monster model</u> (w.r. to $\bar{\kappa}$), if $\mathcal{M}$: $\bar{\kappa}$-saturated, strongly $\bar{\kappa}$-homogeneous

Thm. Assume $\aleph_0 \le \kappa \in CN$. Then $\exists M$: κ-saturated ~~$\bar{\kappa}$~~ strongly $\bar{\kappa}$-saturated.

<u>Proof</u> $M = \bigcup^{\uparrow}_{\alpha < \kappa^+} M_\alpha$: union of elementary chain s.t.:

(1) $M_0 \models T$ any

(2) $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$ if $\delta \in Lim$,

(3) $M_{\alpha+1} \succ M_\alpha$ s.t.:

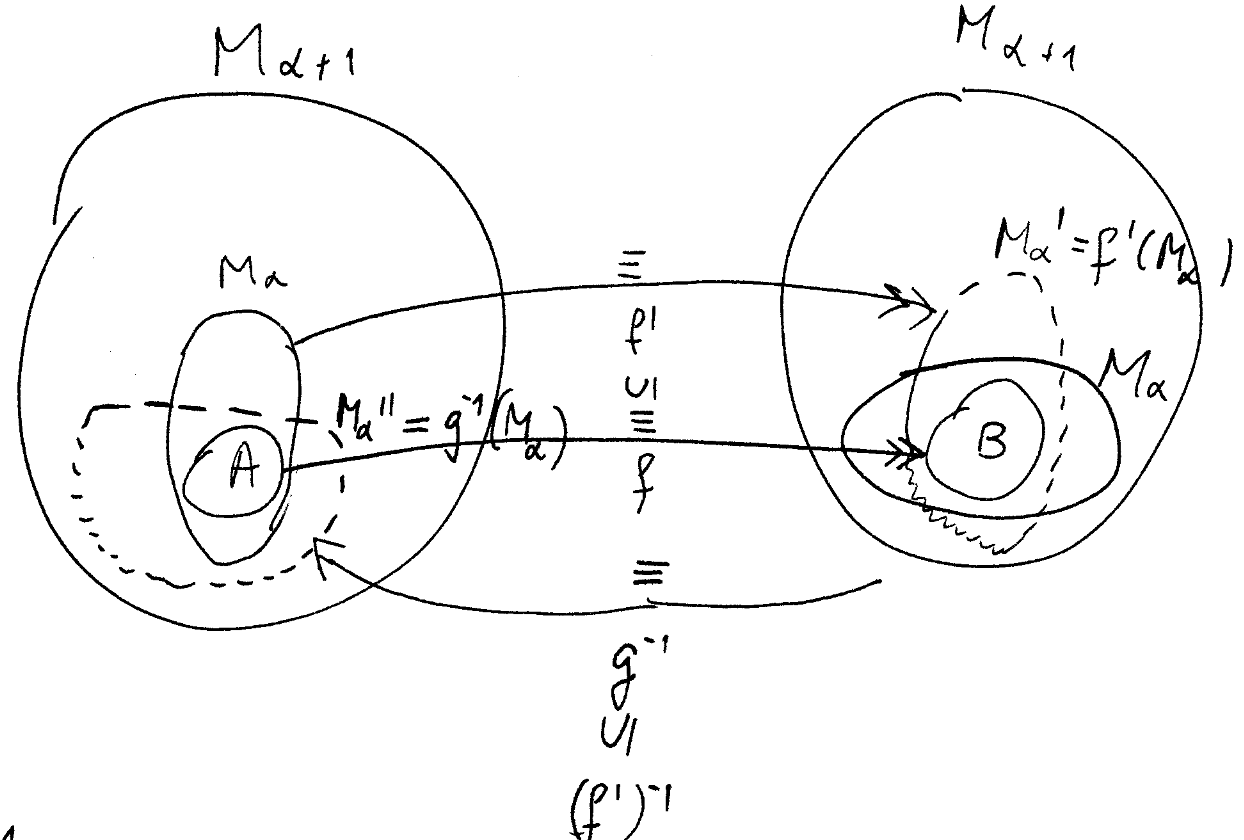
(a) $\forall p \in S_1(M_\alpha)$ p realized in $M_{\alpha+1}$

$$(b)\ (\forall f : \underset{M_\alpha}{\underset{\cap|}{A}} \xrightarrow{\equiv}\!\!\!\!\rightarrow \underset{M_\alpha}{\underset{\cap|}{B}})(\exists\, g \supseteq f)(g : \underset{M_\alpha}{\underset{\cup|}{A'}} \xrightarrow{\equiv}\!\!\!\!\rightarrow \underset{M_\alpha}{\underset{\cup|}{B'}} \text{ in } M_{\alpha+1})$$

It is enough that $M_{\alpha+1}$ is $\|M_\alpha\|^+$-saturated,

↑ To satisfy (a), (b). $M_{\alpha+1} \succ M_\alpha$

Proof of (b) for such $M_{\alpha+1}$:



I. M: $\varkappa$-saturated: <u>clear</u>

II M strongly $\varkappa$-homogeneous:

Assume $A \subset M$, $|A| < \varkappa$. Then $A \subseteq M_\alpha$, $B \subseteq M_\alpha$

$f : A \xrightarrow{\equiv} M$ for some $\alpha < \varkappa$.

$B = f[A]$

- we construct

a sequence f_β, $\alpha \le \beta < \varkappa^+$:.

- increasing $f_\beta : M \xrightarrow{\overset{\equiv}{0}} M$
  partial elementary
- $f \subseteq f_\alpha$

(*) $M_\beta \subseteq \operatorname{dom} f_\beta \cap \operatorname{rng} f_\beta$.

- f_α constructed according to 3)(b)

$$f_\alpha : M_{\alpha+1} \xrightarrow{\overset{\equiv}{0}} M_{\alpha+1}$$

- $f_\beta : M_{\beta+1} \xrightarrow{\overset{\equiv}{0}} M_{\beta+1}$, as in 3.(b)
  when β : successor.
- $f_\delta = \bigcup_{\beta<\delta} f_\beta$ when δ limit., still $f_\delta : M_\delta \xrightarrow{\overset{\equiv}{0}} M_\delta$.

$f_\infty = \bigcup^{\nearrow}_{\alpha\le\beta<\varkappa^+} f_\beta$, $f_\beta \in \operatorname{Aut}(M)$, $f \subseteq f_\beta$.

Assumptions Let $\bar{\varkappa}$: a cardinal number large enough so that:

(1) We consider only small models of T
$|||$
of power $< \bar{\varkappa}$, or even $\ll \bar{\varkappa}$

(2) We work within a monster model $\mathcal{M} \models T$ (w.r.to $\bar{\varkappa}$)

(3) We consider only small models $M \prec \mathcal{M}$
$|||$
of power $< \bar{\varkappa}$, or even $\ll \bar{\varkappa}$,

Consequences:

(1) For $M, N \prec \mathfrak{M}$, $\quad M \subseteq N \iff M \prec N$

(2) Convention: For $\bar{a} \subseteq \mathfrak{M}$

$\models \varphi(\bar{a})$ means $\mathfrak{M} \models \varphi(\bar{a})$

(3) For $A \subseteq M \prec \mathfrak{M}$:

$$S_n^{M}(A) = S_n^{\mathfrak{M}}(A) =: S_n(A)$$

Notation Assume $p(\bar{x})$, $q(\bar{x})$ types (small, over $\mathfrak{M}$)

- $p(\bar{x}) \vdash q(\bar{x}) \iff p(\mathfrak{M}) \subseteq q(\mathfrak{M})$
  "p implies q"
- $p(\bar{x}) \underset{\substack{\uparrow \\ \text{equivalent}}}{\equiv} q(\bar{x}) \iff p \vdash q \;\&\; q \vdash p$

Special case: $p(\bar{x}) = \{\varphi(\bar{x})\}$.

$\varphi(\bar{x}) \vdash q(\bar{x})$: "φ isolates q".

Remark: Syntactically:

$$\underset{\substack{\nwarrow \\ \text{types over } A}}{p(\bar{x})} \vdash \underset{\nearrow}{q(\bar{x})} \iff \forall \varphi(\bar{x}) \in q \;\; \exists \underset{\text{finite}}{p_0(\bar{x}) \subseteq p(\bar{x})}$$

$$T(A) \vdash \bigwedge p_0(x) \longrightarrow \varphi(x)$$

Remark (exercise)

$$p(\bar{x}) \vdash q(\bar{x}) \iff \forall \underset{|A|^+\text{-saturated}}{M \models T} \;\; p(M) \subseteq q(M).$$

Def. (reminder)

Let $p(\bar{x})$: a type over A.

p is isolated over $A \iff \exists \underset{\text{consistent (with T)}}{\varphi(\bar{x}) \in L(A)}$ $\varphi \vdash p$.

Thm (omitting types, Ehrenfeucht)

Assume $p_n(\bar{x}_n), n<\omega$: ~~non~~ a ~~fam~~ a family of non-isolated types over $\emptyset$. Then $\exists M \models T\ \forall n\ \underbrace{p_n(M)=\emptyset}_{M \text{ omits } p_n}$.

Lemma

Assume A is ctble, $p_n(\bar{x}_n), n<\omega$: a family of non-isolated types over A, $\varphi(x) \in L_1(A)$, $\underbrace{\varphi(\mathfrak{M}) \neq \emptyset}_{\text{i.e. } \varphi : \text{consistent}}$.

Then $\exists c \in \varphi(\mathfrak{M})\ \forall n$ p_n non-isolated over $A \cup \{c\}$.

Proof [Lemma $\Rightarrow$ Thm]

By the Lemma: $\exists \underbrace{\{a_n : n<\omega\}}_{\overset{\shortparallel}{A}} \subseteq \mathfrak{M}$ s.t.

(1) A satisfies the TV-test

(2) $p_n, n<\omega$: non-isolated over A.

Construction of $a_n, n<\omega$: recursion on n.

Let $\{\varphi_n(x,\bar{y}) : n<\omega\}$: all formulas of L of this form.

Suppose $n<\omega$ and $\{a_i : i<n\} = a_{<n}$ already constructed so that all $p_k, k<\omega$ still non-isolated over $a_{<n}$.

We consider a consistent formula $\varphi_n(x) \in L_1(a_{<n})$

By the Lemma we find $\underset{a_n}{\underset{\shortparallel}{c}} \in \varphi_n(\mathcal{M})$ so that all p_k, $k<\omega$, still non-isolated over $a_{\leq n}$.

- the formulas $\varphi_n(x)$, $n<\omega$ may be chosen so that after ω steps:

  $\underset{\text{consistent}}{\forall \varphi(x) \in L(A)}\ \exists n\ \varphi = \varphi_n$.

Then $A = \{a_n : n<\omega\}$ satisfies TV-test

$A = M \prec \mathcal{M}$, every p_k still non-isolated over A.

$p_k(M) = \emptyset$ [if not, some $\underset{M}{\underset{\cap}{\bar m}} \models p_k$. then ~~[illegible]~~ $(\bar x_k = \bar m) \vdash p_k(\bar x_k)$ ↯)

Proof of the Lemma.

Let $p(\bar x)$: one of the types $p_n(\bar x_n)$.

Let $h(\bar x, y, \bar a) \in L(A$

$$h(\bar x, c, \bar a) \vdash p(\bar x) \iff h(\mathcal{M}, c, \bar a) \subseteq p(\mathcal{M}).$$

$$\iff \forall \psi(\bar x) \in p(\bar x)\ \ h(\mathcal{M}, c, \bar a) \subseteq \psi(\mathcal{M})$$

$$\iff \forall \psi \in p\ \ \mathcal{M} \models \forall \bar x\,(h(\bar x, c, \bar a) \rightarrow \psi(\bar x))$$

$\Leftrightarrow \forall \psi \in p \quad \psi_h(y) \in tp(c/A)(y)$

where $\psi_h(y) = \forall \bar{x}\,(h(\bar{x}, y, \bar{a}) \to \psi(\bar{x}))$

<u>hence</u>:

$$tp(c/A) = tp(c'/A) \Rightarrow [\,h(\bar{x}, c, \bar{a}) \vdash p \Leftrightarrow h(\bar{x}, c', \bar{a}) \vdash p\,]$$

$$h(\bar{x}, c, \bar{a}) \text{ consistent} \Leftrightarrow (\exists \bar{x}\, h(\bar{x}, y, \bar{a})) \in tp(c/A)(y).$$

Let $\underset{\substack{\uparrow \\ \text{"bad types"}}}{X_{h,p}} = \{ q \in S_1(A) :$ For $c \models q$, $h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})\}$
and $h(\bar{x}, c, \bar{a})$ consistent.

Let $q \in S_1(A)$ then

$$q \in X_{h,p} \Leftrightarrow q(y) \in S_1(A) \cap \underset{\substack{\Updownarrow \\ \text{For } c \models q \;\; h(\bar{x},c,\bar{a}) \text{ consistent}}}{[\exists \bar{x}\, h(\bar{x}, y, \bar{a})]} \cap \underbrace{\bigcap_{\psi \in p} [\psi_h(y)]}_{\substack{\Updownarrow \\ \text{For } c \models q \;\; h(\bar{x},c,\bar{a}) \vdash p(\bar{x})}}$$

(*) $X_{h,p}$: nowhere dense in $S_1(A)$.

Proof of (*) : (a.a.)

Suppose $\theta(y) \in L_1(A)$ and $\emptyset \neq S_1(A) \cap [\theta] \subseteq X_{h,p}$.

Let $\underset{\substack{\cap \\ L(A)}}{\alpha(\bar{x})} = \exists y (h(\bar{x}, y, \bar{a}) \wedge \theta(y))$

- $\alpha(\bar{x})$: consistent :

Let $c \in \theta(\mathcal{M})$

$\Downarrow$ $[\theta] \subseteq X_{h,p}$)

$\mathcal{M} \models \exists \bar{x}\, h(\bar{x}, c, \bar{a})$.

Let $\bar{d} \subseteq \mathcal{M}$ s.t. $\mathcal{M} \models h(\bar{d}, c, \bar{a})$.

$\bar{d}$ satisfies in $\mathcal{M}$: $\underbrace{\exists y\, h(\bar{x}, y, \bar{a})}_{\overset{\shortparallel}{\alpha(\bar{x})}}$.

- $\alpha(\bar{x}) \vdash p(\bar{x})$, i.e. $\alpha(\mathcal{M}) \subseteq p(\mathcal{M})$.

Let $\bar{d} \in \alpha(\mathcal{M})$. So there is $c \in \mathcal{M}$ st

$\models h(\bar{d}, c, \bar{a}) \wedge \theta(c)$

$\Downarrow$ $[\theta] \subseteq X_{h,d}$

$\forall \psi \in p \quad \models \psi_h(c)$

$\Downarrow$

$$\forall \psi \in p \quad h(\bar{x}, c, \bar{a}) \vdash \psi(\bar{x}) \quad \Rightarrow \quad h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$$

$$\bar{d} \Rightarrow d \quad ↯$$

as : $p(\bar{x})$ non-isolated

Let $X = \bigcup_{h, p_n} X_{h,p_n} \subseteq S_1(A)$.

$\uparrow$ meager. Let $q \in S_1(A) \cap [\varphi] \setminus X$

$c \models q$ good.

(Pf) (a.a) Suppose $p = p_n$ isolated over $A \cup \{c\}$.

$$\exists\, h(\bar{x}, c, \bar{a}) \quad \underset{\text{consistent}}{h(\bar{x}, c, \bar{a})} \vdash p(x) \quad \Rightarrow \underset{\underset{tp(\bar{c}/A)}{\shortparallel}}{q} \in X_{h,p_n} \quad ↯.$$

14.03.2022

Def T is quantifier eliminable if $\forall \varphi \in L\ \exists \underset{\text{open}}{\psi} \in L$ $\quad$ open $\equiv$ q.f.

$T \vdash \varphi \leftrightarrow \psi$

Def. For $p(\bar{x}) \in S_n(\emptyset)$ let $p_0(\bar{x}) = \{\varphi(\bar{x}) \in p(\bar{x}) : \varphi \text{ open}\}$.

Remark T is q.e. $\Leftrightarrow \forall n\ \forall p \in S_n(\emptyset)\ \ p_0 \vdash p$

Proof „$\Rightarrow$" Obvious. „$\Leftarrow$" Let $\varphi(\bar{x}) \in L$.

- $\forall p \in [\varphi] \cap S_n(\emptyset)\ \exists \underset{\text{open}}{\psi \in p}\ \ p \in [\psi] \subseteq [\varphi]$

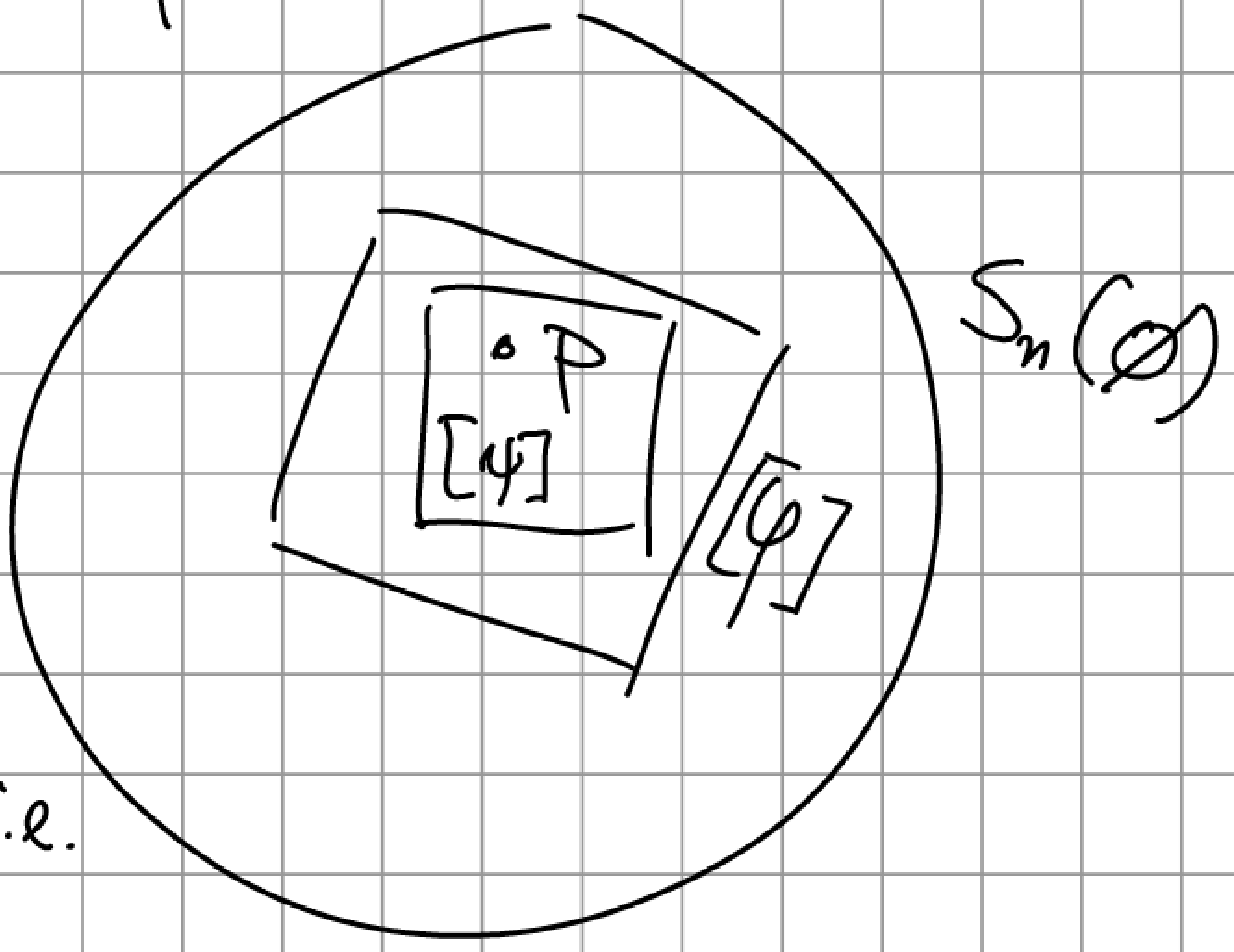
Why? $p_0 \vdash p$
$\qquad\qquad\ \ \ \rotatebox{90}{$\in$}$
$p_0 \vdash \varphi$

by compactness

$\exists$ finite $p_0' \subseteq p_0$ s.t.

$p_0' \vdash \varphi$, i.e.

$p_0'(\mathcal{M}) \subseteq \varphi(\mathcal{M})$



$$\psi(\mathcal{M}) = \Big(\bigwedge_{\psi' \in p_0'} \psi'\Big)(\mathcal{M}) \subseteq \varphi(\mathcal{M}) \rightsquigarrow \mathcal{M} \models \psi(\bar{x}) \to \varphi(\bar{x})$$

$\Updownarrow$

$$\psi \vdash \varphi \Rightarrow [\psi] \cap S_n(\emptyset) \subseteq [\varphi] \cap S_n(\emptyset)$$

∎

Application $L = \{+, \cdot, 0, 1\}$: the language of rings.

ACF_p: the theory of algebraically closed fields of char p, in L.

Axioms:

1) field axioms

2) char $= p \neq 0$: $\underbrace{1 + \ldots + 1}_{p} = 0$

2') $p = 0$: $\underbrace{1 + \ldots + 1}_{n} = 0$ for $n \geq 1$

3) Every polynomial of deg n has a root:
$0 < n$

$$\forall y_{n-1}, y_{n-2}, \ldots, y_0 \, \exists x \; x^n + y_{n-1} x^{n-1} + \ldots + y_0 = 0.$$

Fact ACF_p is complete.

Proof Let $M, N \models ACF_p$. Enough to show that $M \equiv N$. Let $\kappa > \|M\|, \|N\|$ and let $\underbrace{M' \succ M, N' \succ N}_{\text{power } \kappa}$.

M', N' : uncountable acl fields of the same power and char

$$\Downarrow \text{ algebra}$$

$$M' \cong N' \Rightarrow M' \equiv N'$$

$$\begin{array}{ccc} \rotatebox{90}{$\equiv$} & \Downarrow & \rotatebox{90}{$\equiv$} \\ M & \equiv & N \end{array}$$

Fact ACF_p is q.e. (Chevalley, Tarski)

Proof (in $\mathcal{M}$) We will show that $\forall p \in S_n(\emptyset)$ $p_0 \vdash p$ $(\Leftrightarrow p_0(\mathcal{M}) \subseteq p(\mathcal{M}))$. Let $\bar{a} \models p_0$, $\bar{b} \models p$, $\bar{a}, \bar{b} \subseteq \mathcal{M}$. It's enough to prove that $\exists f \in Aut(\mathcal{M})$ $f(\bar{a}) = \bar{b}$.

$$\boxed{\begin{array}{l} \bar{a} = (a_1, \dots, a_n) \\ \bar{b} = (b_1, \dots, b_n) \end{array}}$$

Let $\langle \bar{a} \rangle, \langle \bar{b} \rangle$: the subrings with 1 of $\mathcal{M}$ generated by $\bar{a}, \bar{b}$. $\bar{a}, \bar{b} \models p_0 \Rightarrow \langle \bar{a} \rangle \cong \langle \bar{b} \rangle$

$$a_i \overset{f}{\mapsto} b_i$$

Some algebraic magic.

$$\langle \bar{a} \rangle \cong \langle \bar{b} \rangle$$

$$\cap \quad \Downarrow \text{unique} \quad \cap$$

$$\mathcal{M} \supseteq \langle \bar{a} \rangle_0 \cong \langle \bar{b} \rangle_0 \subseteq \mathcal{M}$$

(fraction field) $\xrightarrow{\text{(not unique)}}$ $$\mathcal{M} \supseteq \underbrace{\langle \bar{a} \rangle_0^{alg}}_{F_a} \cong \underbrace{\langle \bar{b} \rangle_0^{alg}}_{F_b} \subseteq \mathcal{M}$$

$$\operatorname{trdeg}(\mathcal{M}/F_a) = \|\mathcal{M}\| = \operatorname{trdeg}(\mathcal{M}/F_b)$$

$$\Downarrow$$

$$\begin{tikzcd}
\mathcal{M} \arrow[r, phantom, "\cong"] \arrow[d, "f"', "\cong"] & F_a(X_\alpha, \alpha<\lambda)^{alg} \arrow[d, phantom, "\cong"] \arrow[ld, bend left] \\
\mathcal{M} \arrow[r, phantom, "\cong"] & F_b(x_\beta, \beta<\lambda)^{alg}
\end{tikzcd}$$

$$f \in \operatorname{Aut}(\mathcal{M}), \ f(\bar{a}) = \bar{b}.$$

∎

Types in $T = ACF_p$

Let $\mathcal{M} \models T$: a monster model.

subfield (small) $\overset{\cup\!|}{K}$, We will describe $S_n(K)$.

Let $\bar{a} \subseteq \mathcal{M}$, $|\bar{a}| = n$.

$K[\bar{x}] \rhd I(\bar{a}/K) = \{f \in K[\bar{x}] : f(\bar{a}) = 0\}$

<u>Remark</u> 1) $tp(\bar{a}/K) = tp(\bar{a}'/K) \Leftrightarrow I(\bar{a}/K) = I(\bar{a}'/K)$

2) $\forall \underset{\text{prime}}{I \lhd} K[\bar{x}] \ \exists \bar{a} \subseteq \mathcal{M} \ I(\bar{a}/K) = I.$

<u>Proof</u> 1) "$\Rightarrow$" $tp(\bar{a}/K) = tp(\bar{a}'/K) \Rightarrow \exists f \in \operatorname{Aut}(\mathcal{M}/K)$

$$I(\bar{a}/K) = I(\bar{a}'/K) \Leftarrow f(\bar{a}) = \bar{a}'$$

[alternatively: $f \in I(\bar{a}/K) \Leftrightarrow$ "$f(\bar{x}) = 0$" $\in tp(\bar{a}/K)$]

„$\Leftarrow$" Assume $I(\bar{a}/K) = I(\bar{a}'/K) = I$.

$$\Downarrow$$

$$K[\bar{a}] \underset{K}{\cong} K[\bar{x}]/I \cong K[\bar{a}']$$

$$\Downarrow$$

$$\exists f \in Aut(\mathcal{M}/K)\ f(\bar{a}) = \bar{a}'$$

$$\Downarrow$$

$$tp(\bar{a}/K) = tp(\bar{a}'/K).$$

2) $K \subseteq K[\bar{x}]/I = K[\bar{a}] \supseteq \bar{a} = \bar{x}/I$ and $I(\bar{a}/K) = I$.

$$\vdots$$

$\square$

$$S_1(K) = \{ tp(a/K) : a \in \mathcal{M} \} : \text{a top. space.}$$

$$\rotatebox{90}{$\in$}$$

$$p(x) = tp(a/K),\ I_p = I(a/K) \lhd K[x]$$

a) $I_p \neq \{0\}$, i.e. a is algebraic $/ K$, so

$\quad \rotatebox{90}{$\in$}$

$0 \neq f$ $\quad$ "$f(x) = 0$" $\in p(x)$. In fact,

irreducible over K $\nearrow$ "$f(x) = 0$" $\vdash p(x)$ (isolates)

Let $a' \in \mathcal{M}$ s.t. $f(a') = 0 \Rightarrow I(a'/K) \ni f \Rightarrow$

$$p = tp(a/K) = tp(a'/K) \Leftarrow I(a/K) = I(a'/K) \Leftarrow \text{f generates } I(a'/K)$$

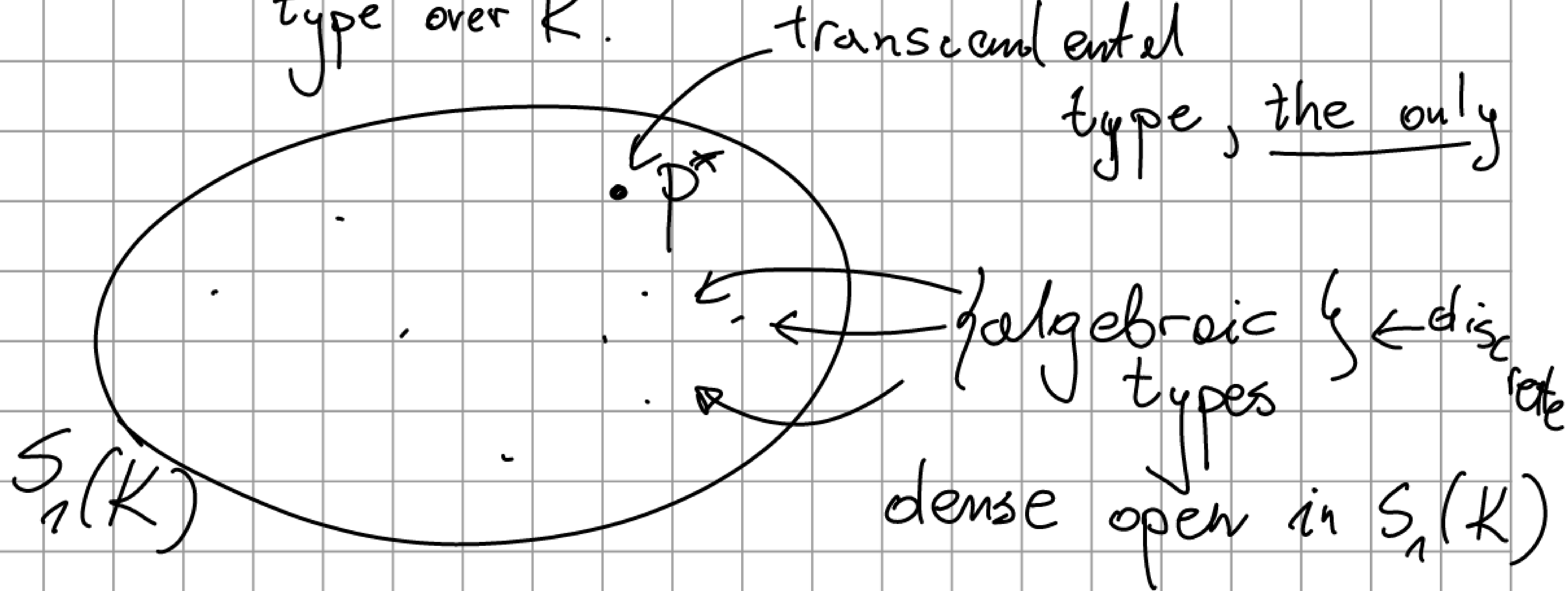
$p(x)$ here is called <u>algebraic</u>.

[More generally (T: arbitrary)

<u>Def</u> $\varphi(\bar{x}) \in L(A)$ is called <u>algebraic</u>, if $0 < |\varphi(\mathcal{M})| < \aleph_0$, similarly for q: a type.]

b) $I_p = \{0\}$: $p = tp(a/K)$ s.t. $a \in \mathcal{M}$ transcendental over K

the transcendental type over K.



If K ctble then $S_1(K)$ ctble

Corollary ACF_p is $\aleph_0$- stable [recall: T is κ-stable $\iff \forall A \subseteq \mathcal{M}, |A| \leq \kappa$ $|S_1(A)| \leq \kappa$]

Proof Let $\underset{\text{ctble}}{A} \subseteq \mathcal{M}$.

$$A \subseteq \underset{\substack{\uparrow \\ \text{ctble} \\ \text{subfield}}}{K} \subseteq \mathcal{M} \quad |S_1(A)| \leq |S_1(K)| = \aleph_0$$

Remark T is totally transcendental $\iff T$: $\aleph_0$-stable

Proof „$\Rightarrow$" from def, „$\Leftarrow$": (A.a.) Let $\kappa > \aleph_0$.

Suppose $|A| \leq \kappa < |S_1(A)|$ for some $A \subseteq \mathcal{M}$.

Shall find $\underset{\text{ctble}}{A_0} \subseteq A$ with $|S_1(A_0)| \geq 2^{\aleph_0}$.

(def) $\varphi(x) \in L(A)$ is <u>large</u> iff $|S_1(A) \cap [\varphi]| > \kappa$

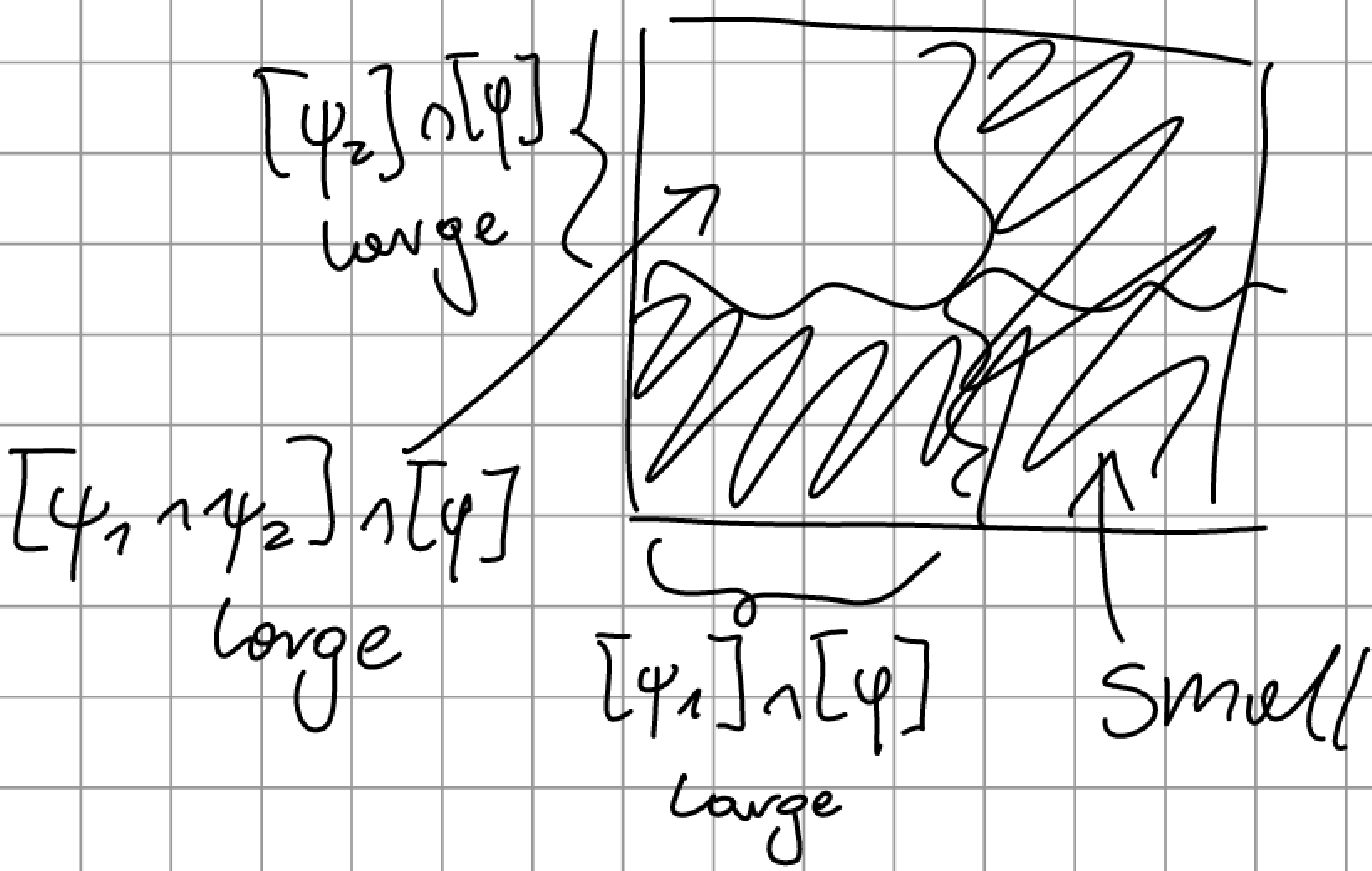
otherwise: $\varphi(x)$ small.

(a) "$x = x$" is large

(b) if $\varphi(x)$: large, then $\exists \psi_1, \psi_2 \in L(A)$ large s.t. $\varphi(\mathcal{M}) = \psi_1(\mathcal{M}) \,\dot\cup\, \psi_1(\mathcal{M})$

Pf. (b) <u>if not</u> then $p^* = \{\psi \in L_1(A) : \psi \wedge \varphi$ is large$\}$ is a complete type is $S_1(A) \cap [\varphi]$

1° p^*: consistent: $\psi_1, \psi_2 \in p^* \Longrightarrow \psi_1 \wedge \psi_2 \in p^*$



2° p^*: complete <u>OK</u>

$p^* \in S_1(A)\cap[\varphi]$: the only large type.

$$\underbrace{S_1(A)\cap[\varphi]}_{>\varkappa} = \underbrace{\{p^*\}\cup \bigcup_{\underbrace{\substack{\psi\in L_1(A)\\ \text{small}\\ \psi\vdash\varphi}}_{\leq\varkappa}} \underbrace{(S_1(A)\cap[\varphi])}_{\leq\varkappa}}_{\leq\varkappa}$$

↯

c) a tree of large formulas in $L_1(A)$ $\varphi_\eta(x)$,

$\eta \in 2^{<\omega}$ st. $\varphi_\eta(\mathcal{M}) = \varphi_{\eta 0}(\mathcal{M}) \odot \varphi_{\eta 1}(\mathcal{M})$

$\uparrow$ by (b)

Let $A_0 \subseteq A$: the set of all params of $\varphi_\eta, \eta \in 2^{<\omega}$

Then $|A_0| \leqslant \aleph_0$.

For $\eta = 2^\omega$: $p_\eta^0 = \{\varphi_{\eta|n}(x) : x < \omega\}$: a consistent 1-type over A_0

$$\rotatebox{-90}{$\subseteq$}$$

When $\nu \neq \eta$ then $p_\eta \neq p_\nu$ $\qquad p_\eta \in S_1(A_0)$

$$\Downarrow$$

$$|S_1(A_0)| \geqslant 2^{\aleph_0} > \aleph_0 \quad \lightning$$

21.03.2022

CONSTRUCTION OF SPECIAL MODELS: $N, M \models T$

Def. (1) M is atomic if $\underset{\text{finite}}{\forall \bar{a} \subseteq M}$ $tp(\bar{a}/\emptyset) = tp(\bar{a})$ is isolated.

(2) M is prime if $\forall N \models T\ \exists f : M \xrightarrow{\equiv} N$

Example $T = ACF_p$, F_p: prime field of char p

a) $\hat{F}_p$: atomic (exercise)

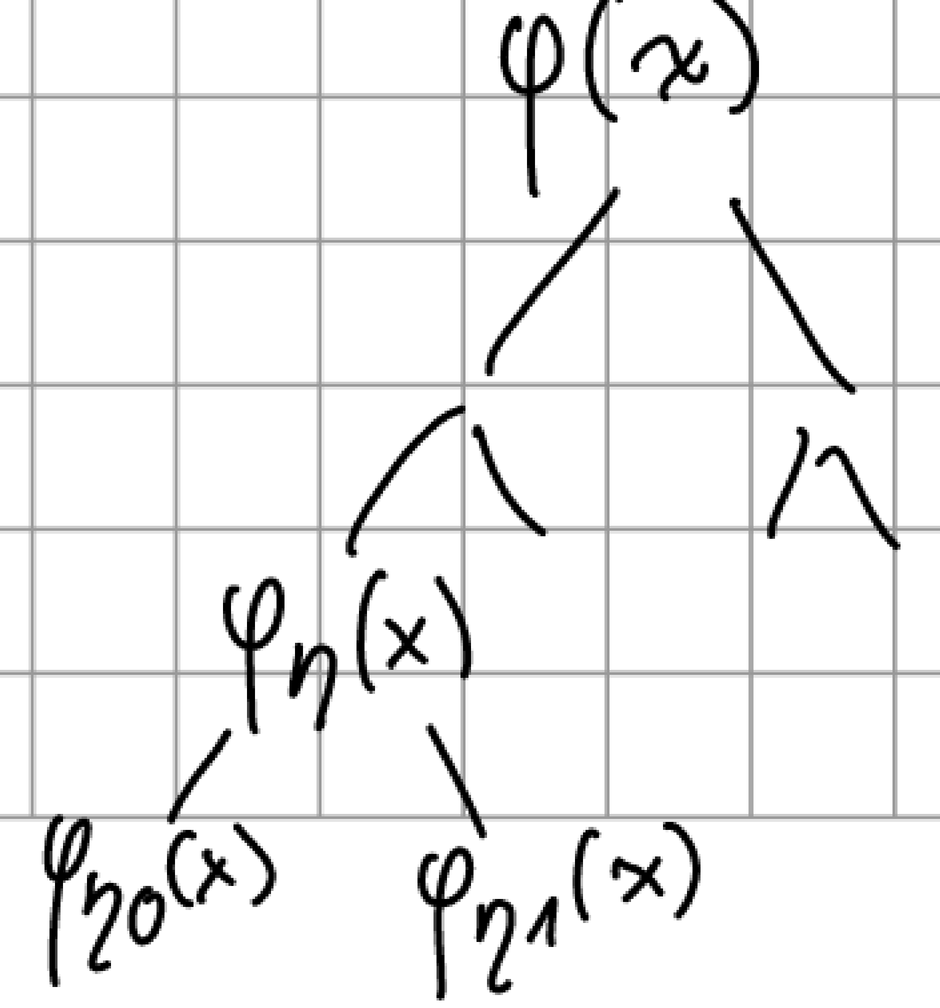
(b) $\hat{F}_p$: prime (exercise)

Thm. T : $\aleph_0$-stable $\Rightarrow$ T has a prime model.

Lemma 1 T : $\aleph_0$-stable $\Rightarrow$ $\forall A \subseteq \mathbb{M}$ $\{\text{isolated types}\} \underset{\text{dense}}{\subseteq} S_1(A)$

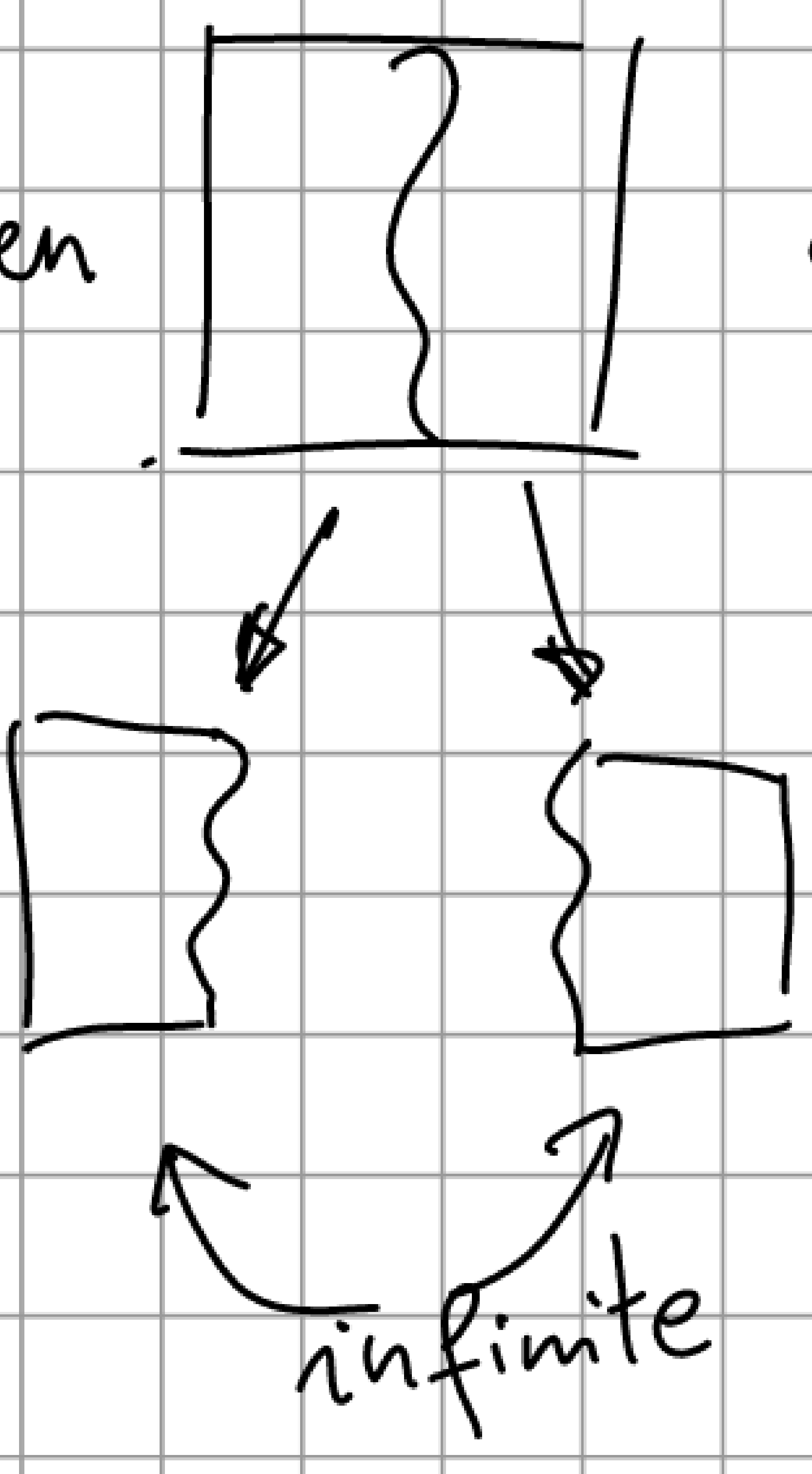
Pf. Suppose $\underset{\text{(consistent with } T)}{\varphi(x)} \in L_1(A)$ s.t. in $S_1(A) \cap [\varphi]$

there is no isolated types $\Rightarrow$ a tree of formulas $\varphi_\eta(x) \in L_1(A)$, $\eta \in 2^{<\omega}$



$\rightsquigarrow$ contradicts $\aleph_0$-stability of T

clopen $\subseteq S_1(A) \cap [\varphi]$



$\square$

<u>Lemma 2</u> $(a,b \in \mathcal{M})$ $tp(a)$ isolated and $tp(b/a)$ isolated
$\iff tp(ab)$ isolated

<u>Pf</u> „$\Rightarrow$": $\varphi(x) \vdash tp(a)$, $\psi(a,y) \vdash \underbrace{tp(b/a)}_{\substack{\shortparallel \\ p_a(y) \subseteq S_y(a)}}(y)$

Then: $\varphi(x) \wedge \psi(x,y) \vdash tp(ab)$

Let $a', b' \in \mathcal{M}$ satisfy $\varphi(x) \wedge \psi(x,y)$

$\models \varphi(a') \Rightarrow tp(a') = tp(a) \Rightarrow \psi(a',y) \vdash \underset{\substack{\cap \\ S_y(a')}}{p_{a'}}(y)$

$\models \psi(a',b') \Rightarrow \models p_{a'}(b')$

$$\Downarrow$$

$$ab \xrightarrow{\equiv} a'b'$$

and $tp(ab) = tp(a'b')$

"$\Leftarrow$": $\Theta(x,y) \vdash tp(a,b)$.

(a) "$\exists y\, \underset{\substack{\Uparrow \\ tp(a)}}{\Theta(x,y)}$" $\vdash tp(a)$.

<u>because</u> Let $a' \in \mathcal{M}$ satisfy "$\exists y\, \Theta(x,y)$".

So there is b' s.t. $\models \Theta(a',b')$

$\Rightarrow tp(ab) = tp(a',b') \Rightarrow tp(a) = tp(a')$.

(b) $\Theta(a,y) \vdash tp(b/a)(y)$

<u>because</u>: similar to (a) ■

<u>Proof of the thm.</u> Construction of a prime model of T:

$A = \{a_n : n < \omega\} \subseteq \mathcal{M}$ so that:

1) A satisfies TV-test

2) $\forall n$ $tp(a_n/a_{<n})$ is isolated

$[a_{<n} = \{a_k : k < n\}]$

At step n choose a_n:

Let $\varphi(x) \in L_1(a_{<n})$ consistent.

Let $a_n \in \varphi(\mathcal{M})$ s.t. $tp(a_n / a_{<n})$ is isolated
(lemma 1)

Suitable choice of φ's ensures ①.

$M \models T$ is prime. Let $N \models T$ arbitrary. We find $f(a_n) \in N$ for $n < \omega$ s.t. $\exists f : M \xrightarrow{\equiv} N$.

At step n $f[a_{<n}] \subseteq N$ with $f : a_{<n} \xrightarrow{\equiv} N$

Let $p(x) = tp(a_n / a_{<n})$ (isolated)

$$\Downarrow$$

$$\underset{\text{isolated}}{f^*(p)}(x) \in S(f(a_{<n}))$$

too, hence realised by $f(a_n)$

$$\overset{??}{\Downarrow}$$

$$f : a_{\leq n} \xrightarrow{\equiv} N$$

∎

Remark (1) A prime model $M \models T$ is atomic.

(2) If $M \models T$ ctble atomic, then M prime

Corrolary $\hat{F}_p$ is atomic.

Proof of remark

(1) Let $p(\bar{x}) \in S_n(\emptyset)$ non-isolated. Will show $p(M) = \emptyset$. Let $N \models T$ be ommiting p.
$\exists f : M \xrightarrow{\equiv} N \Rightarrow p(M) = \emptyset$.

(2) Let $\underset{\substack{\models \\ T}}{M} = \{a_n : n < \omega\}$ atomic.

$$
\begin{aligned}
&\text{Then } \forall n \ \ tp(a_{\leq n}) \text{ is isolated} \\
&\qquad \Big\Downarrow \text{ lemma 2} \\
&\forall n \ \ tp(a_n / a_{<n}) \text{ is isolated} \\
&\qquad \Big\Downarrow \text{ pf of thm} \\
&\qquad M \ \text{ prime.}
\end{aligned}
$$

▨

Corollary A prime model of T is unique (up to $\cong$)

Proof Let $M, N \models T$ both prime $\underset{\text{remark}}{\Rightarrow}$ M, N are ctble and atomic, so we have embeddings in both direction, using back-and-forth we get the iso.

Def $M \models T$ is minimal if $\neg \exists N \precneq M$

Example $\widehat{F}_p$ is minimal.

Fact T has a prime model $\iff \forall n \ \{\text{isolated types}\} \underset{\text{dense}}{\subseteq} S_n(\emptyset)$

Proof "$\Rightarrow$": Let $\underset{\text{prime}}{M \models T} \Rightarrow M$: atomic

what we need

$$\boxed{\underset{\text{any}}{N} \models T, \text{ then } \forall n \ \{p \in S_n(\emptyset) : p(N) \neq \emptyset\} \text{ is dense in } S_n(\emptyset)}$$

$\boxed{\text{exercise}}$

"$\Leftarrow$": **Claim** Assume $\underset{\text{finite}}{\bar{a} \subseteq \mathcal{M}}$ and $tp(\bar{a})$ is isolated.

Then $\{\text{isolated types}\} \underset{\text{dense}}{\subseteq} S_1(\bar{a})$.

Proof of claim Let $n = |a|$, $\varphi(\bar{x}) \vdash tp(\bar{a})$.

Let $\psi(\bar{x}, y) \in L_{n+1}(\emptyset)$ s.t. $\psi(\bar{a}, y)$ is consistent.

We seek $q(y) \in S_1(A) \cap [\psi(\bar{a}, y)]$ isolated.

Let $\chi(\bar{x}, y) = \varphi(\bar{x}) \wedge \psi(\bar{x}, y)$.

By assumptions of $\Leftarrow$ $\exists \underset{\text{isolated}}{p(\bar{x}, y)} \in S_{\bar{x},y}(\emptyset) \cap [\chi(\bar{x}, y)]$.

Let $\bar{a}', b' \models p(\bar{x}, y)$. Then $\bar{a}' \models \underbrace{p(x, y) \restriction_{\bar{x}}}_{\Uparrow \, \varphi(\bar{x})} = tp(\bar{a}$

Let $f \in Aut(\mathcal{M}) : f(\bar{a}') = \bar{a}$
$$b = f(b')$$

Then $\bar{a}'b' \overset{\equiv}{\underset{f}{\longmapsto}} \bar{a}b \Rightarrow \bar{a}b \models p(\bar{x}, y)$

so $\underset{\substack{\shortparallel \\ p}}{tp(\bar{a}b)}$ is isolated $\underset{\text{lemma 2}}{\Longrightarrow}$ $\underset{\substack{\psi \\ \psi(\bar{a}, y)}}{tp(b/\bar{a})}$ isolated

So $q(y) = tp(b/\bar{a})$

Claim $\leadsto$ we construct a model $M = \{a_n : n < \omega\} \prec \mathcal{M}$ s.t. $\forall n$ $tp(a_n / a_{<n})$ is isolated

$$\Downarrow \text{ lemma 2}$$

M atomic ctble $\Longrightarrow$ M prime.

<u>Corollary</u> If $\forall n$ $|S_n(\emptyset)| \leq \aleph_0$, then T has a prime model.

<u>Corollary</u> A prime model (of a ctble T) is homogeneous (exercise).

The number of cantable models of T: $I(T, \aleph_0)$, $n(T)$.

Remark $1 \leq n(T) \leq 2^{\aleph_0}$

$$\underset{\text{ctble}}{M} \models T \Rightarrow M \cong \underbrace{(\mathbb{N}, \ldots)}_{\leq 2^{\aleph_0}}$$ L-structures like that

Recall $n(T) = 1 \Leftrightarrow \forall n \ |S_n(\emptyset)| < \aleph_0$

(T: $\aleph_0$-categorical)

Vaught conjecture (1961)

$$n(T) > \aleph_0 \Rightarrow n(T) = 2^{\aleph_0}$$

Thm (M. Morley, 1971) $\aleph_0 < n(T) < 2^{\aleph_0} \Rightarrow n(T) = \aleph_1$

Thm (Vaught, 1961) $n(T) \neq 2$

Proof (A.a) suppose $n(T) = 2$.

$$n(T) < 2^{\aleph_0} \Rightarrow T \text{ small } (\text{i.e. } \forall n \ |S_n(\emptyset)| \leq \aleph_0)$$

⋮

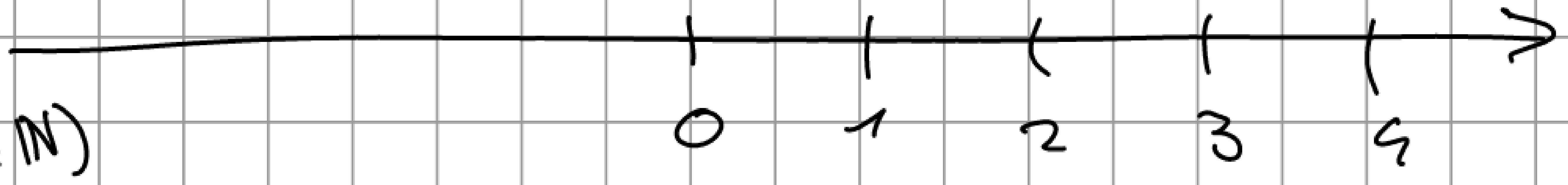
25.03.2022

Example (Andrzej Ehrenfeucht) Theory with exactly 3 ctble theories.

$T_0 = Th(\mathbb{Q}, \leq)$. Look at $T = T_0(\mathbb{N}) = Th(\mathbb{Q}, \leq, n)_{n \in \mathbb{N}}$.

T_0 is q.e. $\Rightarrow$ T is also q.e.

$S_1^T(\emptyset) = S_1^{T_0}(\mathbb{N})$. The types in $S_1^T(\mathbb{N})$:



realised in $(\mathbb{Q},\leq\mathbb{N})$ isolated

- $p_i(x) \equiv \{x = i\}$, $i \in \mathbb{N}$
- $r_i(x) \equiv \{i-1 \leq x \leq i\}$, $i \in \mathbb{N}$, $-1 \approx -\infty$
- $s(x) \equiv \{x > i : i \in \mathbb{N}\}$

ommited

We will point 3 ctble models $N \models T$.

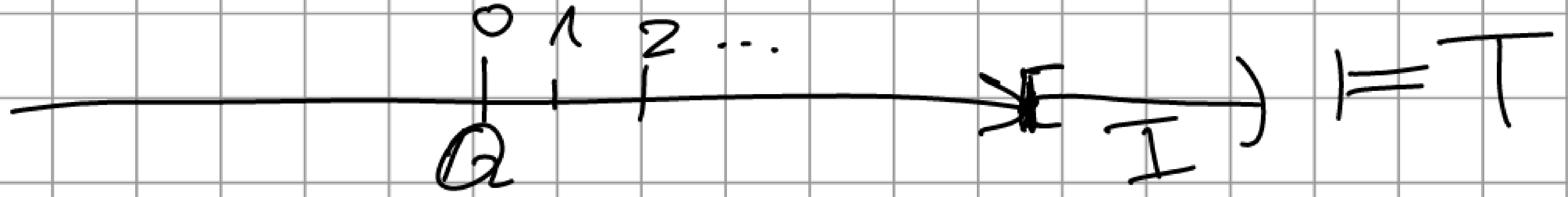
1° $N = M$: prime model of T.

2° $s(N)$ has the minimal element.

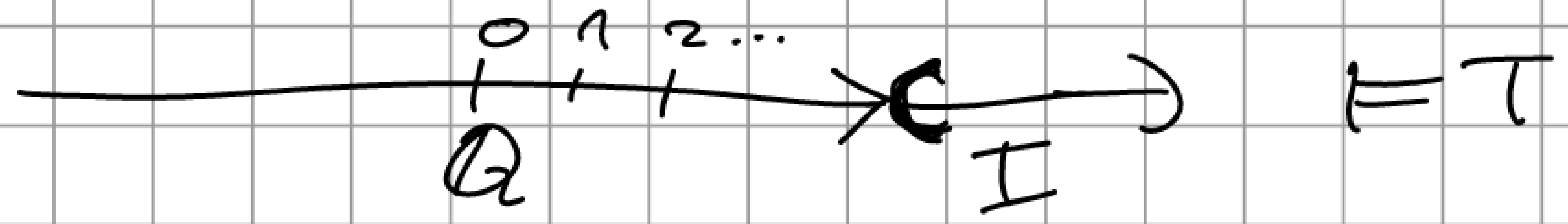
3° $s(N)$ hasn't minimal element.

$2°$: "$\mathbb{Q} \cup ([0,1) \cap \mathbb{Q})$" :

$\mathbb{N}$:



$3°$:



Variants: 3, 4, 5, 6, ...

<u>Problem</u> Does there exist a stable T with $1 < n(T) < \aleph_0$?

<u>Def</u> ($A \underset{\text{small}}{\subseteq} \mathfrak{M}$). $M \prec \mathfrak{M}$ is <u>prime over A</u> if:

(1) $A \subseteq M$

(2) $\forall \underset{\underset{A}{\cup\!|}}{N} \prec \mathfrak{M}\ \exists f : M \xrightarrow{\equiv} N : f\restriction_A = \mathrm{id}_A$

<u>Equivalently</u> M is a prime model of $T(A)$.

- If A is ctble $\longrightarrow$ full descripton of prime models of $T(A)$.
- If A is unctble $\longrightarrow$ in general not much can be set

Thm T $\aleph_0$-stable $\Longrightarrow \forall \underset{\mathfrak{M}}{\underset{\cap}{A}}\ \exists\, \underset{\text{prime}}{M \models T(A)}$

Proof $M = A \cup \{a_\alpha : \alpha < ?\}$

Construction of a_α's s.t.:

(i) $A \cup \{a_\alpha : \alpha < ?\}$ satisfies the TV-test

(ii) $\forall \alpha < ?$ $tp(a_\alpha / A a_{<\alpha})$ is isolated.

At some point it has to terminate (i.e. we cannot add more elements).

Claim Assume $\underset{A}{\underset{\cup\!\!\!\shortmid}{N}} \prec \mathfrak{M}$. Then $\forall \alpha\ \exists f : A \cup a_{<\alpha} \xrightarrow{\equiv} N$ s.t. $f\restriction_A = id_A$.

Proof We define $f(a_\beta)$ for all $\beta < \alpha$ by ind. on β so that $f\restriction_A = id_A$ and $f : A a_{\leq \beta} \xrightarrow{\equiv} N$.

Take $\beta < \alpha$ and suppose $\forall \beta' < \beta$ $f(a_{\beta'})\downarrow$ so that the condition holds.

$p(x) = tp(a_\beta / A a_{<\beta})$ is isolated.

$f : A a_{<\beta} \xrightarrow{\equiv} f[A a_{<\beta}] \subseteq N$ $\qquad f(p)$ is realised by c

$$\underset{\text{isolated}}{p(x)} \in S_1(A a_{<\beta}) \overset{f^*}{\rightsquigarrow} \underset{\text{isolated}}{f(p)} \in S_1(f[A a_{<\beta}])$$

Now we put $f(a_\beta) = c$.

Claim $\blacksquare$

By the claim after some time we cannot get any more elements.

Additional property of the construction:
At the step α we consider a consistent formula $\varphi(\alpha) \in L(A a_{<\alpha})$ with no realisation in $A a_{<\alpha}$, choose a_α s.t. $\models \varphi(a_\alpha)$.

Problems Is a prime model over A unique up to isomorphism over A?

Answer: not always. However the prime model M over A constructed by the previous construction is unique up to $\underset{A}{\cong}$ and it's called primary over A.

Thm M, N: primary over $A \Longrightarrow M \underset{A}{\cong} N$.

Proof $M = A \cup \{a_\alpha : \alpha < \gamma\}$: an „isolated construction" of M over A, i.e. $tp(a_\alpha / A a_{<\alpha})$ is isolated by a formula $\varphi_\alpha(x)$ over $A a_{C_\alpha}$ s.t. $\underset{\text{finite}}{C_\alpha} \subseteq \gamma$

Def. $X \subseteq \gamma$ is closed if $\forall \alpha \in X\ C_\alpha \subseteq X$

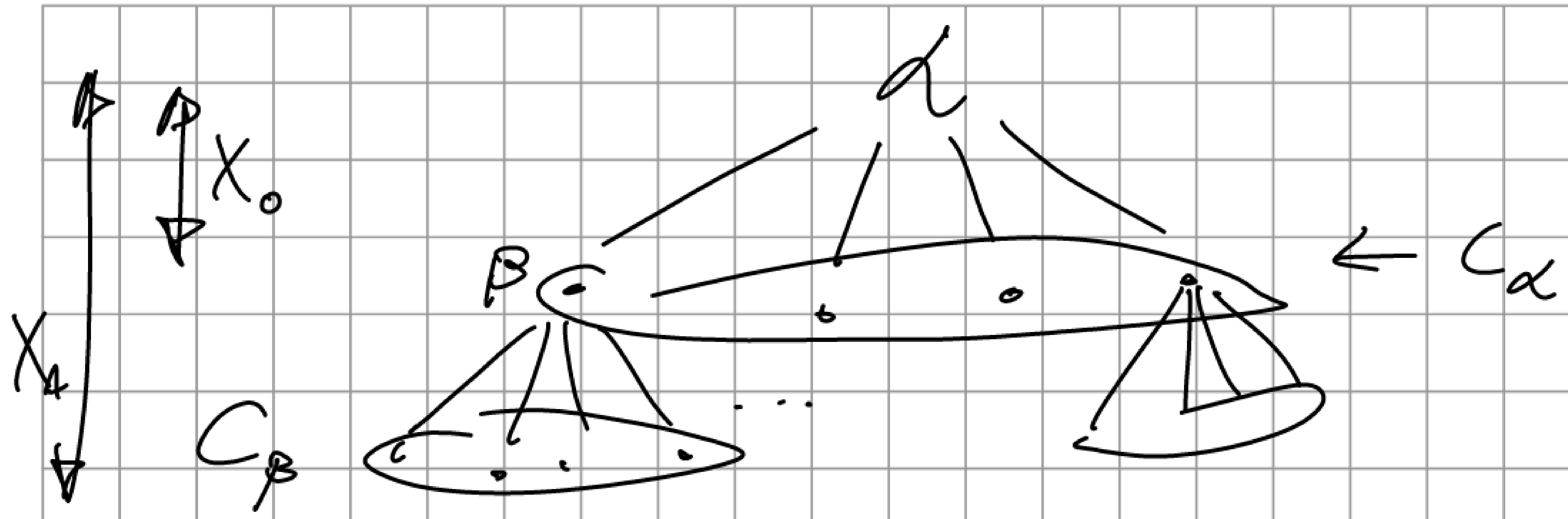
Remark (1) $\alpha \in \gamma \Rightarrow \exists$ minimal $\underset{\text{finite}}{X \subseteq \alpha}$ s.t. $\underbrace{X \cup \{\alpha\}}_{cl(\alpha)}$ is closed.

(2) A union of family of closed subsets of γ is closed.

Proof (remark) (2) is obvious.

(1): Take $X_0 = C_\alpha$, then $\underset{\substack{\uparrow \\ \text{finite} \\ \text{because } C_\beta \text{ finite}}}{X_n} = X_{n+1} \cup \bigcup_{\beta \in X_n} C_\beta$

Then $X = \bigcup_n^{\nearrow} X_n$. Then $X \cup \{\alpha\}$ is finite and closed!



This tree has no infinite branch (because there are no infinite decreasing sequence of ordinals).

remark$\square$

Remark Assume $\begin{array}{c}X\\ \cap\\ \gamma\end{array}$: closed. Then

$$A \cup \langle a_\alpha : \alpha \in X\rangle \underset{\substack{\uparrow\\ \text{concatenation}}}{\frown} \langle \alpha_\alpha : \alpha \in \gamma \setminus X\rangle$$

is an isolated construction over A.

Pf. (remark) (1) $A \cup \langle a_\alpha : \alpha \in X\rangle$ is an i-construction over A (by the fact that X is closed)

(2) Suppose that $\alpha < \gamma$ and $\alpha \notin X$. Will show that $tp(a_\alpha / A a_X a_{X^c \cap (<\alpha)})$ is isolated.

$\varphi_\alpha(a) \vdash tp(a_\alpha / Aa_{<\alpha}) \underset{\uparrow \text{will show}}{\vdash} tp(a_\alpha / Aa_X a_{X^c \cap \gamma < \alpha})$

Let $\underset{\text{finite}}{X_0} \subseteq X$ s.t. $X_0 \cap \alpha \neq \emptyset$.

Enough to show that $tp(a_\alpha / Aa_{<\alpha}) \vdash tp(a_\alpha / Aa_{<\alpha} a_{X_0})$

Wlog By the remark $X_0 \cup \alpha$ is closed.

So: $A \cup \langle a_\beta : \beta < \alpha \rangle \hat{} \langle a_\beta : \beta \in X_0 \rangle$ is

an i-construction over A, but for $\beta \in X_0$

$tp(a_\beta / Aa_{<\alpha} a_{(<\beta) \cap X_0}) \vdash tp(a_\beta / Aa_{<\beta})$

Because $\varphi_\beta(x) \vdash$ and $\varphi_\beta \in$.

So it implies also $tp(a_\beta / Aa_{\leq \alpha} a_{(<\beta) \cap X_0})$.

the proof of e1l3 $\Longrightarrow$ $tp(a_\alpha / Aa_{<\alpha} a_{(<\beta) \cap X_0}) \vdash tp(a_\alpha / Aa_{<\alpha} a_{(\leq \beta) \cap X})$

(we switch a_α with a_β)

We just continue with induction on β

(start with $\beta = \min X_0$).

roman

Claim M: primary/A $\Rightarrow$ atomic/A

Pf. Let $\bar{m} \subseteq M = A \cup \{a_\alpha : \alpha < \gamma\}$.

$\bar{m} \subseteq A \cup a_X$, $\underset{\text{finite closed}}{X \subseteq \gamma}$

$A \cup a_X$: a partial i-construction.

$\Downarrow$

$tp(a_X/A)$ is isolated

$\Downarrow$

$tp(\bar{m}/A)$ —— " ——

claim $\square$

Pf (of thm) $M = A \cup \{a_\alpha : \alpha < \gamma\}$,

$N = A \cup \{b_\alpha : \alpha < \delta\}$: i-constructions/A.

We construct $f : M \xrightarrow{\cong} N$, $f = \bigcup_\alpha f_\alpha$: elementary.

(i) $\mathrm{Dom}\, f_\alpha \supseteq A$, $\mathrm{Rng}\, f_\alpha \supseteq A$, $f_\alpha \restriction_A = id_A$

(ii) $|\mathrm{Dom}\, f_\alpha \setminus A|$, $|\mathrm{Rng}\, f_\alpha \setminus A| \leq |\alpha| \cdot \aleph_0$

(iii) $\beta \in \mathrm{Lim}$ $f_\beta = \bigcup_{\alpha<\beta} f_\alpha$

(iv) $a_\alpha \in \mathrm{Dom}\, f_{\alpha+1}$, $b_{\alpha+1} \in \mathrm{Rng}\, f_{\alpha+1}$.

(v) $\mathrm{Dom}\, f_\alpha \setminus A = a_X$, $\mathrm{Rng}\, f_\alpha \setminus A = b_Y$, where $X \subseteq \gamma$, $Y \subseteq \delta$ are closed.

The recursive step from f_α to $f_{\alpha+1}$.

Let $A' = A \cup \mathrm{Dom} f_\alpha$: an construction over A,

likewise $A'' = A \cup \mathrm{Rng} f_\alpha$: —— " ——

and M is primary over A' (by the remark)

and N is primary over A''.

$p(x) = tp(a_\alpha/A')$ is isolated, so $f_\alpha(p)$ is isolated too, therefore $\exists b \in N$ s.t. $f_\alpha(p) = tp(b/A'')$.