

# Preliminaries

MT1/1

$T$ : a complete consistent theory, in language  $L$   
with infinite models (countable)

that is,  $T = \text{Th}(M)$ ,  $M$ :  $L$ -structure.  
infinite

$L$  denotes also the set of formulas of language  $L$

$M = (|M|; \dots)$ , but  $M$  also denotes  $|M|$ .  
 $\emptyset \neq \uparrow$  universe of  $M$  (for brevity)

usually we omit  $| \cdot |$  in  $|M|$ .

$M \supseteq A$ : a set of parameters.

$L_n(A) = \{ \varphi(x_1, \dots, x_n, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in L, \bar{a} \subseteq A \}$

$L(A) = \bigcup_n L_n(A)$ , also  $L(A)$ : language  $L$   
extended by names for elements of  $A$ .

$L_n(A)$ : Lindenbaum algebra.

[formally: on  $L_n(A)$ :  $\varphi \sim \psi \Leftrightarrow T(A) \vdash \varphi \leftrightarrow \psi$   
 $\Leftrightarrow M \models \varphi \leftrightarrow \psi$

here:  $T(A) = \text{Th}(M, a)_{a \in A}$

a complete theory  
in language  $L(A)$ .



$L_m(A)/\sim$  : a Boolean algebra  
(Lindenbaum algebra)

$$[\varphi]_{\sim} \wedge [\psi]_{\sim} = [\varphi \wedge \psi]_{\sim} \text{ etc.}$$

shorthand:  $L_m(A)$  denotes also  $L_m(A)/\sim$ .

$S_m(A) = \{ \text{complete } n\text{-types over } A, \text{ in } \mathcal{M}_n \}$   
in variables  $x_1, \dots, x_n$

consistent  $n$ -type over  $A \mapsto$  proper filter in  $L_m(A)$

An  $n$ -type  $p(\bar{x})$  over  $A$  is complete if

$$L_m(A) \begin{cases} \cdot p(\bar{x}) : \text{consistent type} \\ \cdot \forall \varphi(\bar{x}) \in L_m(A) (\varphi(\bar{x}) \in p \text{ or } (\neg\varphi(\bar{x})) \in p) \end{cases}$$

$$S(A) := S_1(A)$$

(default)

$S_m(A)$  : topological space :

for  $\varphi(\bar{x}) \in L_m(A)$

$$[\varphi] = \{ p \in S_m(A) : \varphi \in p \}$$

basic open set [clopen]

closed and open

$S_m(A)$  : compact Hausdorff space, 0-dimensional  
(i.e. basis of clopen sets)

complete  $n$ -types /  $A \leftrightarrow$  ultrafilters in  $L_n(A)$  MT1/3

So  $S_n(A) = S(L_n(A))$ , the Stone space  
of ultrafilters in  $L_n(A)$

• the ~~top~~ topology  
on  $S_n(A)$  = the Stone space topology.

For  $p(\bar{x}) \in S_n(A)$

$$p(M) = \{ \bar{a} \in M^n : \bar{a} \text{ satisfies } p \}$$

$\bar{a} \models p$ , i.e.  $M \models \varphi(\bar{a})$  for  
every  $\varphi(\bar{x}) \in p(\bar{x})$

• The same notation for  
arbitrary type (also incomplete)

• A formula  $\varphi(\bar{x}) \in L(M)$ : a special case of a  
type  $\{ \varphi(\bar{x}) \}$ .

$$\varphi(M) = \dots$$

• When  $p \in S_n(A)$ ,  $\bar{a} \subseteq M$  and  $\bar{a} \models p$ , then

$$p = \text{tp}^M(\bar{a}/A) = \{ \varphi(\bar{x}) \in L_n(A) : M \models \varphi(\bar{a}) \}$$

Example Assume  $p(\bar{x})$ : a consistent type over  $M$ .

Then  $\exists N \supseteq M$   $p$  is realized in  $N$

i.e.  $p(N) \neq \emptyset$ .



From now on "a type" means "a consistent type". MT1/4

Def A type  $p(\bar{x})$  over  $A$  is isolated, if:

$$\exists \varphi(\bar{x}) \in L_n(A) \left\{ \begin{array}{l} \textcircled{1} \varphi(\bar{x}) \text{ is consistent (wrt } T), \text{ i.e.} \\ \varphi(M) \neq \emptyset \Leftrightarrow T(A) \vdash \exists \bar{x} \varphi(\bar{x}) \end{array} \right.$$

symbolically:  $\varphi(\bar{x}) \vdash p(\bar{x}) \rightarrow$

$$\left\{ \begin{array}{l} \textcircled{2} \varphi(\bar{x}) \\ \forall \psi(\bar{x}) \in p(\bar{x}) \quad \varphi(M) \subseteq \psi(M) \\ \updownarrow \\ T(A) \vdash \varphi(\bar{x}) \rightarrow \psi(\bar{x}) \end{array} \right.$$

• When  $p(\bar{x})$ : a complete type over  $A$ , then:

$p(\bar{x})$  is isolated  $\Leftrightarrow p$  is isolated in  $S_n(A)$   
in the topological sense  
(i.e.  $\{p\}$  is open)

### Tarski - Vaught test

Assume  $A \subseteq M$ . Then  $A = |N|$  for some  $N \prec M$  iff

$$\forall \varphi(x) \in L_1(A) \quad [\varphi(M) \neq \emptyset \Rightarrow \varphi(M) \cap A \neq \emptyset]$$

Construction of an elementary submodel of  $M$  containing  $A$ :

•  $A_n \subseteq M$ ,  $n < \omega$ , increasing chain of sets

recursive construction:

$$A_0 = A$$

$$A_n \subseteq A_{n+1} \subseteq M \text{ such that } \forall \psi(x) \in L_1(A_n)$$

$$[\psi(M) \neq \emptyset \Rightarrow \psi(M) \cap A_{n+1} \neq \emptyset]$$

$$A_\infty = \bigcup_{n < \omega} A_n \text{ satisfies TV-test.}$$

# Omitting types theorem

MT1/5

Assume  $p_n(\bar{x}_n)$ ,  $n < \omega$ : a family of non-isolated types in theory  $T$ , over  $\emptyset$ . Then:

$(\exists M \models T)$   $M$  omits every  $p_n$  [i.e.  $p_n(M) = \emptyset$ ]

Assume  $M, N \models T$   
 $\underset{A}{\cup}$

Def.  $f: A \rightarrow N$  is elementary ( $f: A \xrightarrow{\equiv} N$ ) if:

$$\forall \bar{a} \in A \forall \varphi(\bar{x}) \in L \quad (M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a})))$$

$$(\Leftrightarrow) \text{tp}^M(\bar{a}) = \text{tp}^N(f(\bar{a}))$$

Elementary diagram of  $A \subseteq M$ :

$$D_e(A) = T(A) = \text{Th}(M, a)_{a \in A}$$

Remark  $f: A \rightarrow N$  is elementary  $\Leftrightarrow (N, f(a))_{a \in A} \models T(A)$

Atomic diagram of  $A \subseteq M$ :

$$D_{\text{at}}(A) = \{ \varphi \in D_{\text{el}}(A) : \varphi \text{ is a quantifier free sentence} \}$$
$$= \{ \varphi(\bar{a}) \in L(A) : M \models \varphi(\bar{a}) \text{ and } \varphi(\bar{a}) : \text{q.f. sentence} \}$$

Remark  $f: M \rightarrow N$  is a monomorphism (i.e.:

$$f: M \xrightarrow{\cong} f(M) \subseteq N$$

↑ substructure

$$\Leftrightarrow (N, f(a))_{a \in M} \models D_{\text{at}}(M).$$



Here always  $f: M \rightarrow N$  denotes a monomorphism. MT 1/6

$M \subseteq N$  :  $M$  is a submodel (substructure) of  $N$

$M < N$  :  $M$  is an elementary submodel of  $N$ , i.e.:

$$M \subseteq N \text{ and } \text{id}_M: M \xrightarrow{\equiv} N$$

Remark Assume  $M < N$ ,  $A \subseteq M$ .

(1) Assume  $p(\bar{x}) \subseteq L_m(A)$ . Then

$p(\bar{x})$  is a consistent type in  $M \Leftrightarrow p(\bar{x})$  is a consistent type in  $N$

(2) Assume  $A \subseteq B \subseteq M$

• If  $p(\bar{x})$ : a type over  $B$ , then  $p \upharpoonright_A \stackrel{\text{def}}{=} p(\bar{x}) \cap L(A)$   
a type over  $A$

Let  $r: S_m(B) \rightarrow S_m(A)$ ,  $r(p) \stackrel{\text{def}}{=} p \upharpoonright_A$ .

Then  $r$ : continuous and "onto".

(3) If  $p(\bar{x})$ : a type over  $A$ , then  $\exists q(\bar{x}) \in S_m(A)$   $p(\bar{x}) \subseteq q(\bar{x})$

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Saturation, universality, (strong) homogeneity.

Let  $\kappa \in \mathbb{C}N$ ,  $\kappa \neq \aleph_0$ .

Def. (1)  $M$   $\kappa$ -saturated if  $\forall A \subseteq M$   $\forall p \in S_1(A)$   $p(M) \neq \emptyset$   
(nasyrony)  $|A| < \kappa$

$M$  is saturated if  $M$  is  $\|M\|$ -saturated

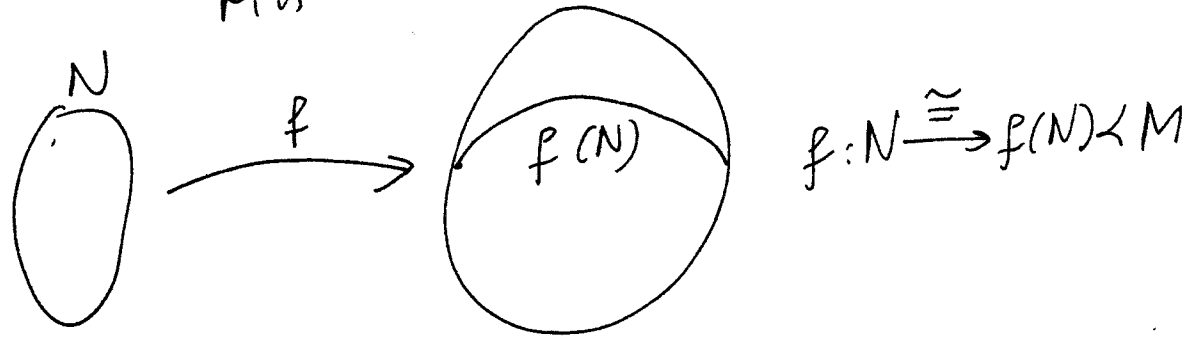
(2)  $M$  is  $\kappa$ -universal if  $\forall N \equiv M$  ( $\|N\| \leq \kappa \Rightarrow \exists f: N \xrightarrow{\equiv} M$ )

↑  
elementarily equivalent

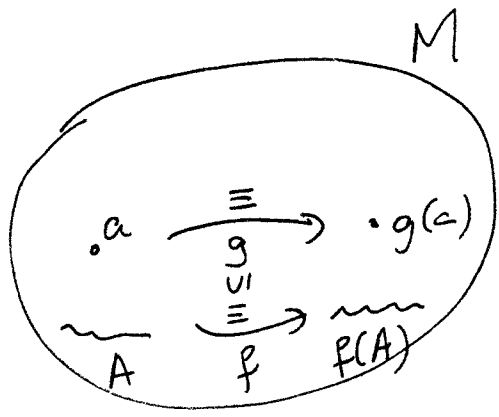
i.e.  $\text{Th}(N) = \text{Th}(M)$

$M$ : universal  $\Leftrightarrow$   $\|M\|$ -universal

MT1/7



(3)  $M$ :  $\kappa$ -homogeneous if  $\forall A \subseteq M \forall a \in M \forall f: A \xrightarrow{\cong} M$   
 $|A| < \kappa \quad \exists g: A \cup \{a\} \xrightarrow{\cong} M$   
 homogeneous =  $\|M\|$ -homogeneous.



4.  $M$  strongly  $\kappa$ -homogeneous if  $\forall A \subseteq M \forall f: A \xrightarrow{\cong} M$   
 $|A| < \kappa \quad \exists g: M \xrightarrow{\cong} M$

strongly homogeneous = strongly  $\|M\|$ -homogeneous.

5.  $M$  is  $\kappa$ -compact if  $(\forall 1$ -type  $p(x)$  over  $M)$   
 $(|p| < \kappa \Rightarrow p(M) \neq \emptyset)$

### Elementary chains of structures

Def  $\langle M_\alpha : \alpha < \mu \rangle, \mu \in \text{Ord}$ , : an elementary chain of structures if  $(\forall \alpha < \beta < \mu) M_\alpha \prec M_\beta$ .

Union of chain (when  $\mu \in \text{Lim}$ )

$$M_\mu = \bigcup_{\alpha < \mu} M_\alpha \quad ?$$

$$\cdot |M_\mu| := \bigcup_{\alpha < \mu} |M_\alpha|$$

$c \in L$  constant symbol

$$c^{M_\mu} = c^{M_\alpha} \text{ for } \alpha < \mu$$

$P$ : relation symbol

$$P^{M_\mu}(a_1, \dots, a_n) \Leftrightarrow M_\alpha \models P(a_1, \dots, a_n) \text{ for } \alpha < \mu$$

$\bigcap$   
 $|M_\mu|$

sufficiently large  
[so that  $\bar{a} \subseteq M_\alpha$ ]

$$\cdot f^{M_\mu}(\bar{a}) = b \Leftrightarrow M_\alpha \models f(\bar{a}) = b \text{ for } \alpha < \mu$$

sufficiently large

Fact (Tarski)  $M_\alpha < M_\mu$  for all  $\alpha < \mu$ .

Proof (1)  $M_\alpha \subseteq M_\mu$  (substructure): exercise

$$(2) \forall \varphi(\bar{x}) \in L \forall \alpha < \mu \forall \bar{a} \subseteq M_\alpha (M_\alpha \models \varphi(\bar{a}) \Leftrightarrow M_\mu \models \varphi(\bar{a}))$$

$$(a) \varphi \text{ atomic: } M_\alpha \subseteq M_\mu \checkmark$$

$$(b) \varphi = \psi_1 \wedge \psi_2, \varphi = \neg \psi : \text{easy}$$

$$(c) \varphi(\bar{x}) = \exists y \psi(\bar{x}, y)$$

$$M_\alpha \models \varphi(\bar{a}) \Rightarrow M_\alpha \models \psi(\bar{a}, b) \text{ for some } b \in M_\alpha$$

$\Downarrow$  ind. assumption for  $\psi$

$$M_\mu \models \psi(\bar{a}, b)$$

$$\Downarrow$$

$$M_\mu \models \varphi(\bar{a})$$



$$M_\mu \models \varphi(\bar{a}) \Rightarrow M_\mu \models \psi(\bar{a}, b) \text{ for some } b \in M_\mu$$

$$\exists y \psi(\bar{a}, y)$$

$\Downarrow$  ind. assumption

$$b \in M_\beta \text{ for some } \alpha \leq \beta < \mu$$

$$M_\beta \models \psi(\bar{a}, b)$$

$\Downarrow$

$$M_\beta \models \varphi(\bar{a})$$

$$\Downarrow M_\alpha < M_\beta$$

$$M_\alpha \models \varphi(\bar{a})$$

Elementary directed systems of structures:

Let  $(I, \leq)$ : a directed set, i.e.:

(1)  $\leq$ : partial order on  $I$

(2)  $(\forall a, b \in I)(\exists c \in I)(a \leq c \wedge b \leq c)$

Example  $J$ : a set  $\mapsto ([J]^{<\omega}, \leq)$ : directed set.

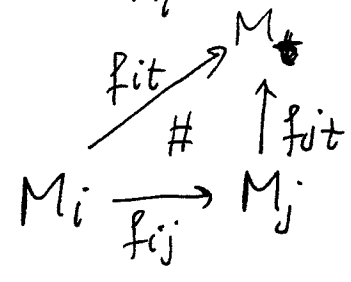
Directed system of structures:

$$\mathcal{M} = (M_i, f_{ij})_{i \leq j \in I}$$

connecting functions  $f_{ij}: M_i \rightarrow M_j$ ,  $f_{ii} = id_{M_i}$ . such that

$$(\forall i \leq j \leq t \in I) f_{it} = f_{jt} \circ f_{ij}$$

(compatibility)



System  $\mathcal{M}$  is elementary if all  $f_{ij}$  are elementary.

# Example Elementary chain $(M_\alpha)_{\alpha < \mu}$

MT1/10

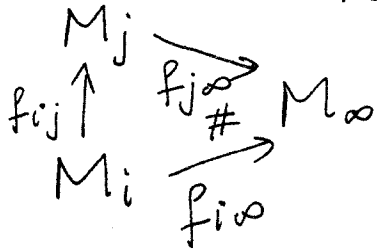
$\mathcal{M} = (M_\alpha, f_{\alpha\beta})_{\alpha \leq \beta < \mu}$   $f_{\alpha\beta} = id_{M_\alpha} : M_\alpha \xrightarrow{\cong} M_\beta$   
elementary directed system of structures

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Direct limit of a directed system  $\mathcal{M} : M_\infty = \varinjlim \mathcal{M}$

$(M_\infty, f_{i\infty})_{i \in I}$ , where  $f_{i\infty} : M_i \rightarrow M_\infty$  such that

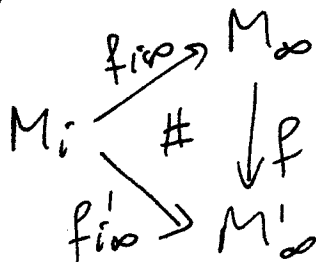
(1)  $\forall i \leq j \in I$   $f_{i\infty} = f_{j\infty} \circ f_{ij}$  [compatible with connecting functions]



(2)  $(\forall (M'_\infty, f'_{i\infty})_{i \in I})$  satisfying (1)  $\exists ! f : M_\infty \rightarrow M'_\infty$

(universality)

$(\forall i \in I) f'_{i\infty} = f \circ f_{i\infty}$



Fact  $M_\infty$  exists (and is unique up to  $\cong$ ).

If  $\mathcal{M}$  is elementary, then  $f_{i,\infty} : M_i \xrightarrow{\cong} M_\infty$ .

Proof 1. Construction of  $M_\infty$ :

$S := \dot{\bigcup}_{i \in I} |M_i|$  : formally disjoint union.

$\sim$  on  $S$  : an equivalence relation

$$M_i \quad M_j \quad \text{MTI/II}$$

$$\downarrow \quad \downarrow \quad \text{def}$$

$$x \sim y \Leftrightarrow f_{it}(x) = f_{jt}(y) \text{ for some } (= \text{every})$$

$$t \geq i, j$$

exercise:  $\sim$  is transitive.

$$|M_\infty| := S/\sim$$

- $\sim \upharpoonright |M_i|$ : the equality (because  $f_{ij}$  is 1-1 (monomorphism))
- $f_{i\infty}(x) = x/\sim$ ,  $f_{i\infty}: |M_i| \xrightarrow{1-1} |M_\infty|$ .

L-structure on  $|M_\infty|$ :

- $c^{M_\infty} = c^{M_i}/\sim$
- $P^{M_\infty}(a_{i_1}/\sim, \dots, a_{i_m}/\sim) \Leftrightarrow M_t \models P(f_{i_1 t}(a_{i_1}), \dots, f_{i_m t}(a_{i_m}))$   
 $a_{ij} \in M_{ij}$  for  $t \geq i_1, \dots, i_m$
- $f^{M_\infty}$ : similarly

the rest is an exercise.

How to extend elementary mappings?

MT2/1

~~Def.~~  $\mathbf{BA}_{lg}$ : Category of Boolean algebras

$\mathbf{Comp}_0$ :  $\mathcal{T}$  of compact Hausdorff 0-dimensional spaces

$$F: \mathbf{BA}_{lg} \rightarrow \mathbf{Comp}_0$$

$$G: \mathbf{Comp}_0 \rightarrow \mathbf{BA}_{lg}$$

$$F(A) = S(A)$$

$$G(X) = C(\text{open}(X))$$

$F, G$ : contravariant functors "inverse" to each other

~~$(F, G)$  is a duality of categories. (look it up)~~  
Categories  $\mathbf{BA}_{lg}$  and  $\mathbf{Comp}_0$  are dually equivalent.

$A, B$ : Boolean algebras

$$f: A \rightarrow B \text{ homomorphism} \Rightarrow F(f): S(B) \rightarrow S(A)$$

$$F(f)(p) = f^{-1}[p]$$

continuous.

$$\text{Assume } f: A \xrightarrow{\cong} B$$

$$\begin{matrix} \cap & & \cap \\ T \neq M & , & N \neq T \end{matrix}$$

$$\text{Then } \hat{f}: L_n(A) \rightarrow L_n(B)$$

$$\hat{f}(\varphi(\bar{x}, \bar{a})) = \varphi(\bar{x}, f(\bar{a}))$$

homomorphism

of Boolean algebras.

even: monomorphism.

We skip  $\hat{\phantom{f}}$  in  $\hat{f}$ , so:

$$f: L_n(A) \rightarrow L_n(B) \text{ monomorphism}$$

$$f^*: S_n(B) \rightarrow S_n(A) \text{ epimorphism in } \mathbf{Comp}_0$$

i.e. continuous onto

Lemma (on extensions of elementary mappings) MT2/2

Assume  $M, N \models T$ ,  $A \subseteq M$ ,  $B \subseteq N$ ,  $f: A \xrightarrow{\equiv} B$  "onto".

Assume  $\overset{\psi}{\underset{a}{\#}}, \overset{\psi}{\underset{b}{\#}}$ ,  $p = \text{tp}(a/A)$ ,  $q = \text{tp}(b/B)$ .

Then  $f \cup \{Ka, b\}$  is elementary  $\Leftrightarrow f^*(q) = p$ .

[here  $f^*: S(B) \xrightarrow{\cong} S(A)$   
homeomorphism]

Proof exercise.

Def.  $M$  is  $(< \kappa_0)$ -universal  $\Leftrightarrow \forall n \forall p \in S_n(\emptyset) p(M) \neq \emptyset$ .

Remark  $M: \kappa$ -universal  $\Rightarrow M: (< \kappa_0)$ -universal.

Proof Let  $p \in S_n(\emptyset)$ .

Choose a countable  $N \models T$  with  $p(N) \neq \emptyset$ .

$M: \kappa$ -universal  $\Rightarrow \exists f: N \xrightarrow{\equiv} M$   
 $\overset{\psi}{\underset{a}{\#}} \neq p \mapsto \overset{\psi}{\underset{f(a)}{\#}} \neq p$ .

Thm. (1)  $M: \kappa$ -saturated  $\Rightarrow M: \kappa$ -homogeneous  
and  $\kappa$ -universal.

(2)  $M: \kappa$ -~~universal~~<sup>homogeneous</sup> and  $(< \kappa_0)$ -universal  $\Rightarrow$   
 $M: \kappa$ -saturated.

Proof. (1)  $\kappa$ -homogeneity of  $M$ :

Assume  $f: A \xrightarrow{\equiv} M$ ,  $A \subseteq M$ ,  $|A| < \kappa$ ,  $a \in M$ .

We seek  $b \in M$  s.t.  $g = f \cup \{ \langle a, b \rangle \}$  elementary

MT2/3

$\Updownarrow$  Lemma

$$f^*(tp(b/B)) = tp(a/A).$$

⊗ Let  $p = tp(a/A)$ ,  $q = (f^*)^{-1}(p) \in S_1(B)$

$\uparrow$   
 $S_1(A)$

Let  $b \in M$  (exists by  $\kappa$ -saturation)  
 $\uparrow$   
good. of  $M$

•  $\kappa$ -universality of  $M$ :

Assume  $N \equiv M$ ,  $\|N\| \leq \kappa$ .

We seek  $f: N \xrightarrow{\equiv} M$ .

Let  $\{a_\alpha : \alpha < \mu\}$ : an enumeration of  $N$ ,  $\mu = \|N\|$ .

We define  $f(a_\alpha)$  by induction on  $\alpha < \mu$ :

• Suppose  $f(a_\beta)$  defined for all  $\beta < \alpha$  so that

$$f: \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} M$$

Want to find  $f(a_\alpha)$  so that

$$f: \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} M.$$

~~By the Lemma it is enough that~~

Let  $p = tp(a_\alpha / \{a_\beta : \beta < \alpha\})$ .

By the lemma it is enough to find  $f(a_\alpha) \in M$

so that  $f^*(tp(f(a_\alpha) / \{f(a_\beta) : \beta < \alpha\})) = p$ .

So let  $q_f = (f^*)^{-1}(p) \in S_{\kappa}(\underbrace{\{f(a_\beta) : \beta < \alpha\}}_{\text{power} < \kappa})$

(MT2/4)

$M$   $\kappa$ -saturated  $\Rightarrow q_f$  realized in  $M$ .

Let  $f(a_\alpha) \in M$  s.t.  $f(a_\alpha) \neq q_f$ .

(2) Assume  $M$  is  $\kappa$ -homogeneous &  $(< \aleph_0)$ -~~saturated~~ <sup>universal</sup>.

Want:  $M$ :  $\kappa$ -saturated.

So: Let  $A \subseteq M$ ,  $|A| < \kappa$ ,  $p \in S_{\kappa}(A)$ . Show:  $p(M) \neq \emptyset$ .

Induction on  $|A|$ .

Case (a):  $|A| < \aleph_0$ .

$N$

$\exists N \supseteq M$   $p(N) \neq \emptyset$ . So let  $b \in p$ .

Let  $A^* = A \cup \{b\}$   
 $\quad \quad \quad \cup \{a_1, \dots, a_k\}$

Let  $q_f = t_p^N(a_1, \dots, a_k, b) \in S_{\kappa+1}(\emptyset)$

$q_f$  is realized in  $M$  ( $(< \aleph_0)$ -universality),

by  $\langle \underbrace{a'_1, \dots, a'_k}_{A'}, b' \rangle$

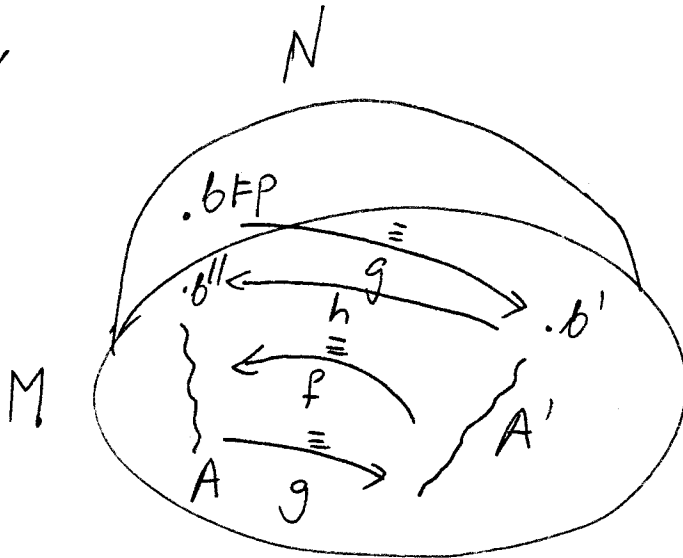
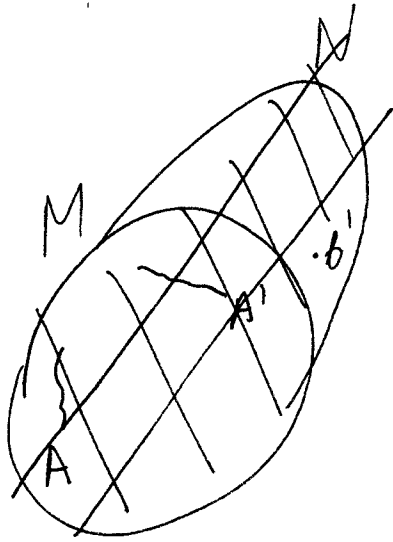
Let  $g: A \cup \{b\} \rightarrow A' \cup \{b'\}$ ,  $g(a_i) = a'_i$ ,  $g(b) = b'$ .

$g$ : elementary.

$$\Rightarrow g \uparrow_A : A \xrightarrow{\cong} A'$$

$$\Downarrow f := (g \uparrow_A)^{-1} : A' \xrightarrow{\cong} A$$

M:  $\kappa$ -homogeneous  $\Rightarrow \exists h : A' \cup \{b''\} \xrightarrow{\cong} A \cup \{b''\}$   
for some  $b'' \in M$ .



$$\begin{array}{ccc} A \cup b & \xrightarrow{\cong} & A \cup b'' \\ g \downarrow \cong & & \cong \uparrow h \\ A' \cup b' & & \end{array}$$

Let  $s = h \circ g$

$$s \uparrow_A = \underbrace{(h \uparrow_{A'})}_{(g \uparrow_A)^{-1}} \circ (g \uparrow_A) = id_A$$

$$s^*(\cancel{tp(b''/A)}) = \cancel{tp(b''/A)}$$

$$s \uparrow_A = id_A \Rightarrow s^* : S(A) \xrightarrow{\cong} S(A)$$

$\cong$   
 $id_{S(A)}$

$$\text{hence: } p = tp(b/A) \underset{\uparrow \text{Lemma}}{=} s^*(tp(b''/A)) \underset{\uparrow}{=} tp(b''/A)$$

and  $b'' \neq p$   $s^* = id_{S(A)}$

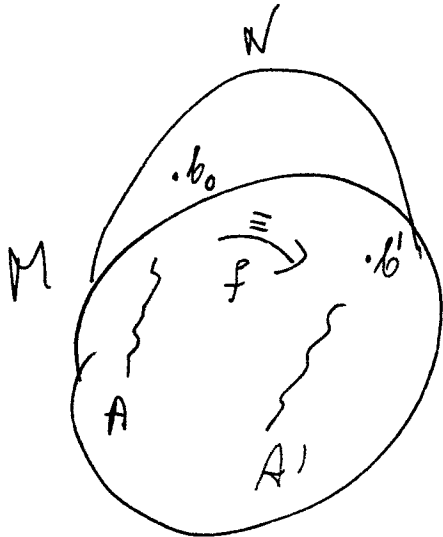


Case (b)  $|A| = \mu$ ,  $x_0 \leq \mu < \kappa$ .

$A = \{a_\alpha : \alpha < \mu\}$ ,  $p \in S_1(A)$ .

$p \upharpoonright \emptyset \in S_1(\emptyset) \implies \exists b' \in M \ b' \neq p \upharpoonright \emptyset$   
 $M: \langle x_0 \rangle$ -universal

$\exists N \supset M \ \exists b_0 \in N$   
 $\begin{matrix} \pi \\ \downarrow \\ p \end{matrix}$



Will find  $A' = \{a'_\alpha : \alpha < \mu\} \subseteq M$

s.t.  $f: A \cup b_0 \rightarrow A' \cup b'$   
 given by  $f(a_\alpha) = a'_\alpha$   
 $f(b_0) = b'$

is elementary!

We find  $a'_\alpha$ ,  $\alpha < \mu$  by induction on  $\alpha < \mu$ .

So suppose  $\alpha < \mu$  and  $a'_\beta$  already defined for all  $\beta < \alpha$

so that  $\overline{p \upharpoonright \{a_\beta : \beta < \alpha \cup b_0\}} = \{a_\beta : \beta < \alpha \cup b_0\} \equiv \{a'_\beta : \beta < \alpha \cup b'\}$   
 $\equiv: f_0$

We look for  $a'_\alpha$ .

Let  $q = \text{tp}(a_\alpha / \{a_\beta : \beta < \alpha \cup \{b_0\}\})$

then  $(f_0^*)^{-1}(q) \in S(\underbrace{\{a'_\beta : \beta < \alpha \cup \{b'\}\}}_{\text{power} < \mu \leq |A|})$

power  $< \mu \leq |A|$

By the lemma it is enough that  $a'_\alpha \neq (f_0^*)^{-1}(q)$ .

But  $M: \kappa$ -

$M$ .

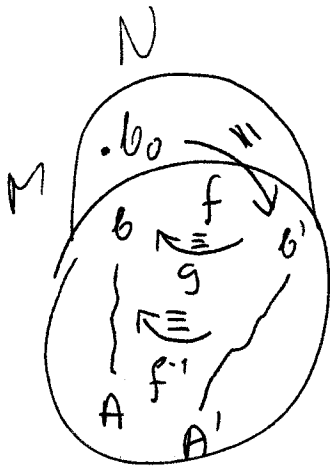
~~Let~~ By the inductive assumption on  $A$ :

MT2/7

$(f_0^*)^{-1}(q)$  is realized in  $M$ , so we are done with constructing  $A'$ .

Now:  $f^{-1}: A' \xrightarrow{\cong} A$  in  $M \leftarrow \kappa$ -homogeneous, so

$\exists \tilde{g}: A'b' \xrightarrow{\cong} Ab$  for some  $b \in M$ .



~~Let~~ Let  $s = g \circ f$

$$s: Ab_0 \xrightarrow{\cong} Ab$$

$$s \uparrow_A = (g \uparrow_{A'}) \circ (f \uparrow_A) = id_A,$$

$$\parallel$$

$$(f \uparrow_A)^{-1}$$

$$p = \text{tp}(b_0/A) = \text{tp}(b/A)$$

and  $p$  is realized in  $M$ .

### Corollary

$M$  is  $\kappa$ -saturated  $\Leftrightarrow M$  is  $\kappa$ -homogeneous and  $\kappa$ -universal and  $\kappa$ -universal.

Proof  $\Rightarrow$  by Thm (1).

$\Leftarrow$   $\kappa$ -homogeneous +  $\kappa$ -universal  $\Rightarrow$

$\kappa$ -homogeneous +  $(\langle \cdot \rangle_0)$ -universal  $\Rightarrow$   $\kappa$ -saturated  
Thm (2)

Properties of saturated models.

Thm. Assume  $M, N \models T$  saturated models of the same power. Then  $M \cong N$ .

Proof  $M = \{m_\alpha : \alpha < \kappa\}$ ,  $N = \{n_\alpha : \alpha < \kappa\}$ ,  
 $\kappa = \|M\| = \|N\|$ . We find  $f: M \xrightarrow{\cong} N$

back-and-forth method:

$$f = \bigcup_{\alpha < \kappa} f_\alpha \quad f_\alpha: M \xrightarrow{\cong} N \quad \text{s.t.}$$

partial, elementary

(1)  $m_\alpha \in \text{Dom } f_{\alpha+1}$

$n_\alpha \in \text{Rng } f_{\alpha+1}$  ,  $|f_\alpha| \leq 2 \cdot |\alpha|$

(2)  $f_0 = \emptyset$

(3) For  $\delta \in \text{Lim}$ ,  $f_\delta = \bigcup_{\alpha < \delta} f_\alpha$ .

(4)  $f_{\alpha+1} = f_\alpha \cup \{ \langle \underset{\substack{\uparrow \\ N}}{m_\alpha}, m \rangle, \langle m, \underset{\substack{\uparrow \\ M}}{n_\alpha} \rangle \}$

Inductive step:

Suppose we have  $f_\alpha$ . Want:  $f_{\alpha+1}$ .

Let  $A_\alpha = \text{Dom } f_\alpha \subseteq M$ ,  $B_\alpha = \text{Rng } f_\alpha \subseteq N$ .

$$f_\alpha: A_\alpha \xrightarrow{\cong} B_\alpha$$

$$\downarrow$$

$$f_\alpha^*: S(B_\alpha) \xrightarrow{\cong} S(A_\alpha).$$

"forth": Find  $n \in N$  st.  $f_\alpha \cup \{ \langle m_\alpha, n \rangle \}$  elementary MT2/9

$$\begin{array}{c} \Downarrow \\ (f_\alpha^*)^{-1}(tp(m_\alpha/A_\alpha)) = tp(n/B_\alpha). \end{array}$$

Let  $p = tp(m_\alpha/A_\alpha)$ .

So  $(f_\alpha^*)^{-1}(p) \in S(B_\alpha)$  is realized in  $N$  by some  $n$ .

"~~back~~": similarly.  
back

Thm Assume  $M, N \models T$  are homogeneous, of the same power and  $\forall n < \omega \forall p \in S_n(\emptyset) (p(M) \neq \emptyset \Leftrightarrow p(N) \neq \emptyset)$ .  
Then  $M \cong N$ .

Lemma Under the assumptions of the Thm,

$$\forall A \subseteq M \exists f: A \xrightarrow{\cong} N.$$

Proof. Induction on  $|A|$ .

Case (a)  $|A| < \aleph_0$ .  $A = \{a_1, \dots, a_n\}$ .

Let  $p = tp(\langle a_1, \dots, a_n \rangle) \in S_n(\emptyset)$ . realized in  $M$   
 $\Downarrow$   
 realized in  $N$

by some  $\langle b_1, \dots, b_n \rangle \in N$ .  
 $f(a_i) = b_i$  is good.

Case (b)  $|A| = \mu \geq \aleph_0$ ,  $A = \{a_\alpha : \alpha < \mu\}$

We find  $f(a_\alpha)$  by induction on  $\alpha < \mu$ .

Inductive step.

Suppose  $\alpha < \mu$  and for every  $\beta < \alpha$  we have  $f|_{a_\beta}$

$$\text{s.t. } f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} N.$$

We shall find  $f(a_\alpha) \in N$  s.t.  $f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} N$ .

Let  $a_{<\alpha} := \{a_\beta : \beta < \alpha\}$ . Likewise  $a_{\leq \alpha}$ .

$$|a_{\leq \alpha}| < \mu = |A|$$

By inductive assumption:  $\exists g : a_{\leq \alpha} \xrightarrow{\equiv} N$ .

$$\text{Then } f \circ g^{-1} : \underbrace{g(a_{<\alpha})}_N \xrightarrow{\equiv} \underbrace{f(a_{<\alpha})}_N$$

By homogeneity of  $N$ :  $\exists f(a_\alpha) \in N$  s.t.

$$f \circ g^{-1} : \underbrace{g(a_{<\alpha}) \cup g(a_\alpha)}_{g(a_{\leq \alpha})} \xrightarrow{\equiv} f(a_{<\alpha}) \cup f(a_\alpha) = f(a_{\leq \alpha})$$

$$\text{Then } f = (f \circ g^{-1}) \circ g : a_{\leq \alpha} \xrightarrow{\equiv} N.$$

Proof of the theorem

$$\kappa := \|M\| = \|N\|$$

$f : M \xrightarrow{\cong} N$  constructed by back-and-forth method

$$f = \bigcup_{\alpha < \kappa} f_\alpha, \quad f_\alpha : M \xrightarrow{\cong} N \text{ (partial elementary), } \alpha < \kappa$$

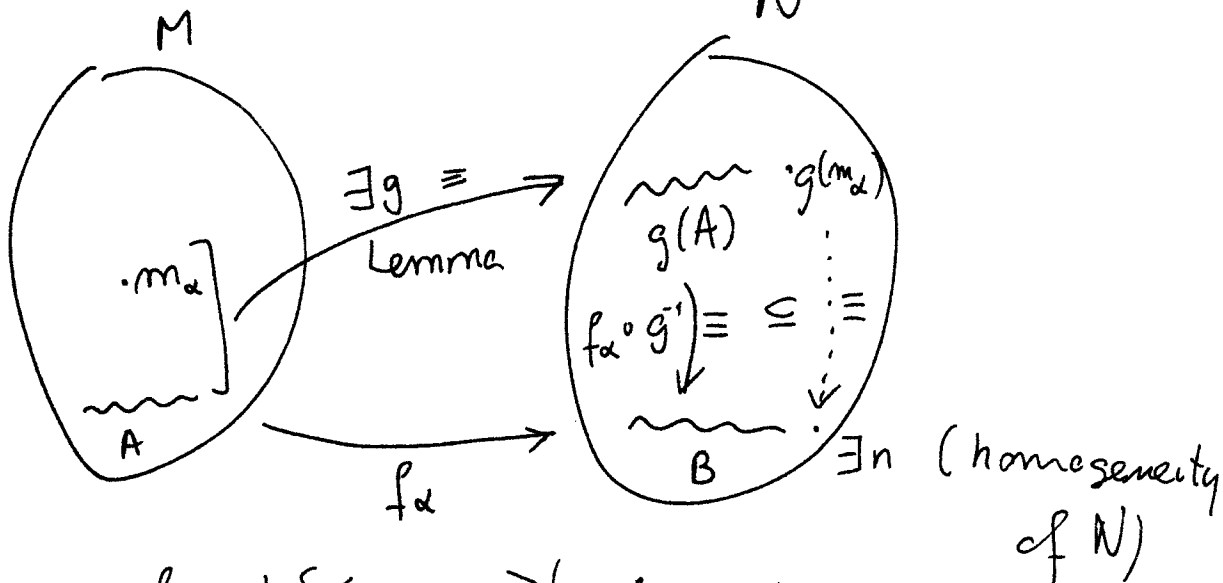
$|f_\alpha| \leq 2 \cdot |\alpha|$  + the same conditions as in the previous thm.

inductive step  $f_\alpha \mapsto f_{\alpha+1}$

$A = \text{Dom } f_\alpha$

$B = \text{Rng } f_\alpha$

"forth"



~~$f_\alpha \cup \{ \langle m_\alpha, n \rangle \}$  elementary~~

$h = (f_\alpha \circ g^{-1}) \upharpoonright_{g(A)} \cup \{ \langle g(m_\alpha), n \rangle \}$  elementary

$h \circ g : A \cup m_\alpha \xrightarrow{\equiv} B \cup n \subseteq N$

$\cup$   
 $f_\alpha$

"back": similarly.

Constructions of models:

MT3/1

- Saturated  $\implies$  • (strongly) homogeneous  $\bar{\equiv}$

Thm.  $\kappa = 2^{<\kappa}$ ,  $\kappa \in \text{Reg}$ ,  $\kappa > \aleph_0 \implies \exists M \models T$   
 saturated, of power  $\kappa$ .  
 $\parallel \kappa^{<\kappa} = \kappa$

Proof

(\*)  $|S_i(A)| \leq 2^{|A| + \aleph_0}$ , because:  $|L_i(A)| = |A| + \aleph_0$

Here:  $|A| < \kappa \implies |S_1(A)| \leq \kappa$ .

Lemma  $N \models T$ ,  $\|N\| \leq \kappa \implies X_N := \bigcup \{S_i(A) : A \subseteq N \text{ \& } |A| < \kappa\}$   
 the set has power  $\leq \kappa$ .

Pf  $|\{A \subseteq N : |A| < \kappa\}| \leq \kappa^{<\kappa} = \kappa$ .

- $|S_i(A)| \leq \kappa$  for such  $A$ .

Proof of the thm.

$M_\alpha$ ,  $\alpha < \kappa$ : elementary chain of models of  $T$  of power  $\kappa$ .

- $M_0$ : whatever

•  $M_\delta = \bigcup_{\alpha < \delta} M_\alpha$ , when  $\delta < \kappa$  limit.

•  $M_{\alpha+1} \supset M_\alpha$  such that  $\forall p \in X_{M_\alpha} p(M_{\alpha+1}) \neq \emptyset$ :

$$T' = \text{Th}(M_\alpha, m)_{m \in M_\alpha} \cup \bigcup_{\beta < \kappa} \{ \varphi(c_\beta) : \varphi(x) \in p_\beta \},$$

where  $X_{M_\alpha} = \{ p_\beta : \beta < \kappa \}$

↑  
new constant symbols,

and  $T'$  in language  $L(M_\alpha) \cup \{ c_\beta : \beta < \kappa \}$ .

$T^1$ : consistent, has model of power  $\kappa$ :  $M_{\alpha+1}$   
such that  $M_\alpha < M_{\alpha+1}$ .

$M = \bigcup_{\alpha < \kappa} M_\alpha$ : of power  $\kappa$ , saturated:

Let  $A \subseteq M$ ,  $|A| < \kappa$ , and  $p \in S_1^M(A)$   
 $\kappa \in \text{Reg} \Rightarrow A \subseteq M_\alpha$  for some  $\alpha < \kappa$ . CW  
↓

proof:  $A = \{a_\beta : \beta < \mu\}$  for some  $\mu < \kappa$ .

$\forall \beta < \mu \exists \alpha_\beta < \kappa \ a_\beta \in M_{\alpha_\beta}$

$\{\alpha_\beta : \beta < \mu\} \subseteq \kappa$ ,  $\mu < \text{cf}(\kappa) = \kappa$

$\Rightarrow \exists \alpha < \kappa \ \forall \beta < \mu \ \alpha_\beta < \alpha$   
↑  
 $A \subseteq M_\alpha$ .

~~Let~~  $M_\alpha < M \Rightarrow p \in S_1^{M_\alpha}(A) = S_1^M(A)$

$p$  realized in  $M_{\alpha+1}$  by some  $a \in M_{\alpha+1}$

$a \models p$  in  $M_{\alpha+1} \Rightarrow a \models p$  in  $M$ .

$M_{\alpha+1} < M$

Monster model:

Let  $\bar{\kappa}$ : a large cardinal number.

"Ideal model"  $M \models T$ : saturated of power  $\bar{\kappa}$

because:  $\forall M \models T$  ( $\|M\| < \bar{\kappa} \Rightarrow \exists M' < M \ M \cong M'$ ).



# Advantages of saturated model $M$ :

MT 3/3

(i) universality

(ii) strong homogeneity

~~More~~ <sup>More</sup> Weakly (a bit):

(1)  $\bar{\kappa}$ -universality

(2) strong  $\bar{\kappa}$ -homogeneity

$\text{Aut}(M)$ : the group of automorphisms of  $M$

$\text{Aut}(M/A) = \{ f \in \text{Aut}(M) : f|_A = \text{id}_A \}$ : automorphisms of  $M$  over  $A$   
 $A \subseteq M$

Lemma Assume  $M$  is strongly  $\kappa$ -homogeneous,  $\kappa$ -saturated,  $A \subseteq M$ ,  $|A| < \kappa$ . Then:

(1) For  $a, b \in M$  ( $\text{tp}(a/A) \stackrel{!}{=} \text{tp}(b/A) \Leftrightarrow a, b$  are in the same orbit of  $\text{Aut}(M/A)$  on  $M$ ).

(2) [orbits  $\text{Aut}(M/A)$  on  $M^n$ ]  $\xleftrightarrow[\text{onto}]{1:1}$   $S_n(A)$

Proof (1)  $\Leftarrow$ :  $f \in \text{Aut}(M/A)$ ,  $f(a) = b$

$$\Downarrow \text{tp}(a/A) = \text{tp}(b/A)$$

$\Rightarrow$ :  $\text{tp}(a/A) = \text{tp}(b/A) \Rightarrow f: Aa \xrightarrow{\equiv} Ab$

strong  $\kappa$ -homogeneity  $f|_A = \text{id}_A, f(a) = b$

$|A| < \kappa \Rightarrow f \in g \in \text{Aut}(M), g \in \text{Aut}(M/A)$   
 $g(a) = b : a, b$  in the same orbit of  $\text{Aut}(M/A)$

$$(2) M^n \supseteq \mathcal{O} \xrightarrow[\varphi]{(1)} p_{\mathcal{O}} \in S_n(A)$$

$\uparrow$   
 orbit of  
 $\text{Aut}(M/A)$

$\parallel$   
 common  
 type  $tp(a/A)$   
 for  $a \in \mathcal{O}$ .

$\mathcal{O} : \{ \text{orbits of } \text{Aut}(M/A) \text{ on } M^n \}$

$$\downarrow \varphi$$

$$S_n(A)$$

$$\mathcal{O}_1 \neq \mathcal{O}_2 \xrightarrow{(1)} p_{\mathcal{O}_1} \neq p_{\mathcal{O}_2} \quad \boxed{\text{so } \varphi: 1-1}$$

[if  $p_{\mathcal{O}_1} = p_{\mathcal{O}_2}$  then let  $a \in \mathcal{O}_1, b \in \mathcal{O}_2 \Rightarrow \exists g \in \text{Aut}(M/A)$

$M: \kappa$ -saturated  $\Rightarrow \varphi: \text{"onto"}$

$$g(a) = b \quad \checkmark.]$$

Def Let  $\bar{\kappa} : \text{a (large) cardinal number,}$

$M \models T$  monster model, if  $M: \bar{\kappa}$ -saturated,  
 (w.r. to  $\bar{\kappa}$ ) strongly  $\bar{\kappa}$ -homogeneous

Thm. Assume  $\aleph_0 \leq \kappa \in \mathcal{C}N$ . Then

$\exists M: \kappa$ -saturated ~~is~~ strongly  $\bar{\kappa}$ -saturated.

Proof  $M = \bigcup_{\alpha < \kappa^+} M_\alpha$  : union of elementary chain  
 s.t.:

(1)  $M_0 \models T$  any

(2)  $M_\delta = \bigcup_{\alpha < \delta} M_\alpha$  if  $\delta \in \text{Lim}$ ,

(3)  $M_{\alpha+1} \supset M_\alpha$  s.t.:

(a)  $\forall p \in S_1(M_\alpha)$   $p$  realized in  $M_{\alpha+1}$

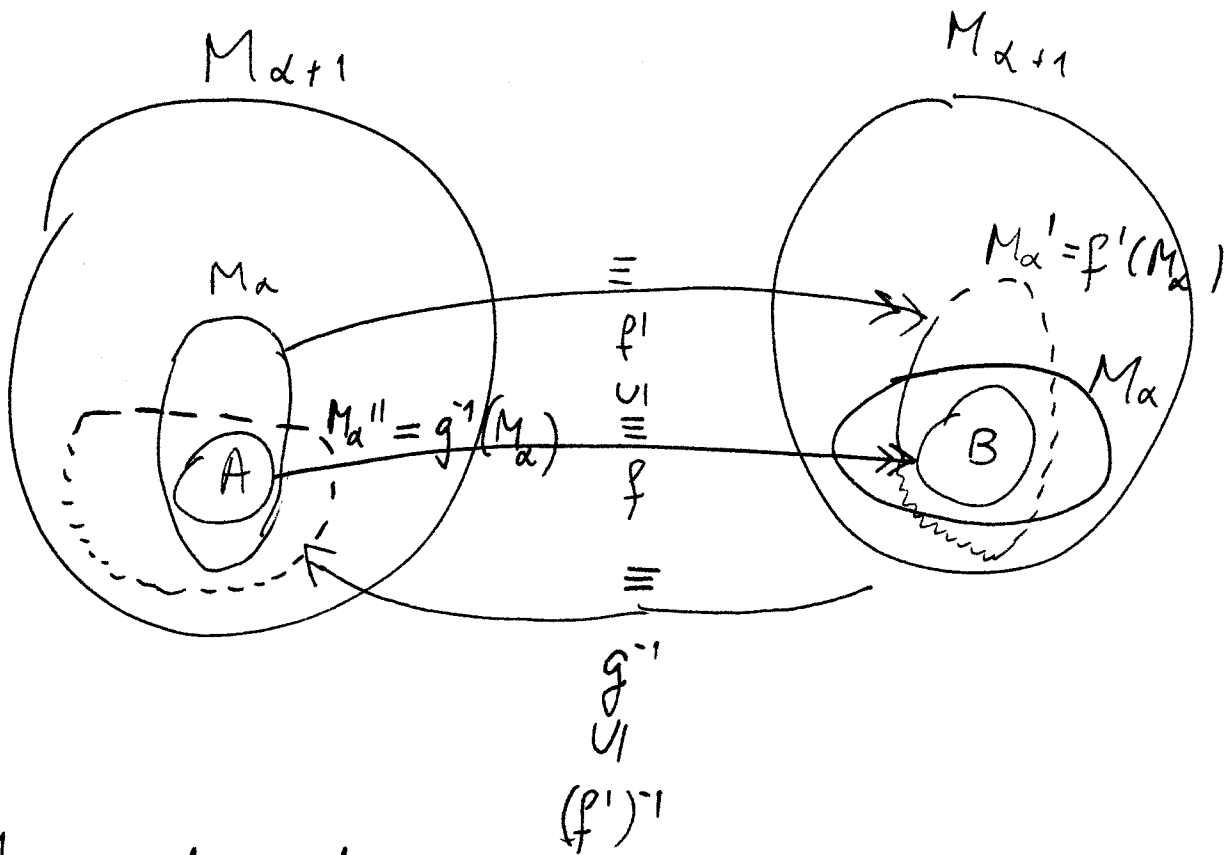
$$(b) \left( \forall f: A \xrightarrow{\equiv} B \right) \left( \exists g \geq f \right) \left( g: A' \xrightarrow{\equiv} B' \text{ in } M_{\alpha+1} \right)$$

$\begin{matrix} \cap & & \cap \\ M_\alpha & & M_\alpha \end{matrix} \qquad \begin{matrix} \cup \\ M_\alpha \end{matrix} \qquad \begin{matrix} \cup \\ M_\alpha \end{matrix}$

MT3/5.

It is enough ~~to~~ that  $M_{\alpha+1}$  is  $\|M_\alpha\|^+$ -saturated,  
 To satisfy (a), (b).  $M_{\alpha+1} \succ M_\alpha$

Proof of (b) for such  $M_{\alpha+1}$ :



I.  $M$ :  $\kappa$ -saturated: clear

II.  $M$  strongly  $\kappa$ -homogeneous:

Assume  $A \subset M$ ,  $|A| < \kappa$ . Then  $A \subseteq M_\alpha$ ,  $B \subseteq M_\alpha$   
 $f: A \xrightarrow{\equiv} M$  for some  $\alpha < \kappa$ .  
 $B = f[A]$

• we construct

a sequence  $f_\beta$ ,  $\alpha \leq \beta < \kappa^+$  :-

• increasing  $f_\beta : M \xrightarrow{\equiv} M$

•  ~~$f_\alpha$~~   $f_\alpha \subseteq f_{\alpha+1}$  partial elementary

(\*)  $M_\beta \subseteq \text{dom } f_\beta \cap \text{rng } f_\beta$ .

•  ~~$f_\alpha$~~   $f_\alpha$  constructed according to 3)(b)

$$f_\alpha : M_{\alpha+1} \xrightarrow{\equiv} M_{\alpha+1}$$

•  $f_\beta : M_{\beta+1} \xrightarrow{\equiv} M_{\beta+1}$ , as in 3.(b)

when  $\beta$  : successor.

•  $f_\delta = \bigcup_{\beta < \delta} f_\beta$  when  $\delta$  limit, still  $f_\delta : M_\delta \xrightarrow{\equiv} M_\delta$ .

$$f_\infty = \bigcup_{\alpha \leq \beta < \kappa^+} f_\beta, \quad f_\beta \in \text{Aut}(M), \quad f \subseteq f_\beta.$$

Assumptions let  $\bar{\kappa}$  : a cardinal number large enough  
so that:

(1) We consider only small models of  $T$

||  
of power  $< \bar{\kappa}$ , or even  $\ll \bar{\kappa}$

(2) We work within a monster model  $\mathcal{M} \models T$  (w.r. to  $\bar{\kappa}$ )

(3) We consider only small models  $M \prec \mathcal{M}$

||  
of power  $< \bar{\kappa}$ , or even  $\ll \bar{\kappa}$ ,

Consequences:

(1) For  $M, N < \mathcal{M}$ ,  $M \subseteq N \Leftrightarrow M < N$

(2) Convention: For  $\bar{a} \in \mathcal{M}$   
 $\vDash \varphi(\bar{a})$  means  $\mathcal{M} \vDash \varphi(\bar{a})$

(3) For  $A \subseteq M < \mathcal{M}$ :

$$S_m^M(A) = S_m^{\mathcal{M}}(A) =: S_m(A)$$

Notation Assume  $p(\bar{x}), q(\bar{x})$  types (small, over  $\mathcal{M}$ )

•  $p(\bar{x}) \vdash q(\bar{x}) \Leftrightarrow p(\mathcal{M}) \subseteq q(\mathcal{M})$   
 "p implies q"

•  $p(\bar{x}) \equiv q(\bar{x}) \Leftrightarrow p \vdash q \ \& \ q \vdash p$   
 ↑  
 equivalent

Special case:  $p(\bar{x}) = \{ \varphi(\bar{x}) \}$ .

$\varphi(\bar{x}) \vdash q(\bar{x})$ : "φ isolates q".

Remark: Syntactically:

$$p(\bar{x}) \vdash q(\bar{x}) \Leftrightarrow \forall \varphi(\bar{x}) \in q \ \exists p_0(\bar{x}) \subseteq p(\bar{x}) \text{ finite} \\
 \begin{matrix} \uparrow & \uparrow \\ \text{types over } A & \end{matrix} \quad T(A) \vdash \bigwedge p_0(x) \rightarrow \varphi(x)$$

Remark (exercise)

$$p(\bar{x}) \vdash q(\bar{x}) \Leftrightarrow \forall M \models T \text{ IA-saturated } p(M) \subseteq q(M).$$

Def. (reminder)

Let  $p(\bar{x})$ : a type over  $A$ .

$p$  is isolated over  $A \iff \exists \varphi(\bar{x}) \in L(A)$   $\varphi \vdash p$ .  
consistent (with  $T$ )

Thm (omitting types, Ehrenfeucht)

Assume  $p_n(\bar{x}_n), n < \omega$ : ~~non~~ a family of non-isolated types over  $\emptyset$ . Then  $\exists M \models T \forall n \underbrace{p_n(M) = \emptyset}$ ,  
 $M$  omits  $p_n$ .

Lemma

Assume  $A$  is stable,  $p_n(\bar{x}_n), n < \omega$ : a family of non-isolated types over  $A$ ,  $\varphi(\bar{x}) \in L_1(A)$ ,  $\varphi(M) \neq \emptyset$ .

Then  $\exists c \in \varphi(M) \forall n$   $p_n$  non-isolated over  $A \cup \{c\}$ .  
i.e.  $\varphi$ : consistent

Proof [Lemma  $\Rightarrow$  Thm]

By the lemma:  $\exists \underbrace{\{a_n : n < \omega\}}_A \subseteq M$  s.t.

(1)  $A$  satisfies the TV-test

(2)  $p_n, n < \omega$ : non-isolated over  $A$ .

Construction of  $a_n, n < \omega$ : recursion on  $n$ .

Let  $\{\varphi_n(x, \bar{y}) : n < \omega\}$ : all formulas of  $L$  of this form.

Suppose  $n < \omega$  and  $\{a_i : i < n\} = a_{<n}$  already  
 so that all  $p_k, k < \omega$  still non-isolated over  $a_{<n}$ .

MT3/9

We consider a consistent formula  $\varphi_n(x) \in L_1(a_{<n})$

By the Lemma we find  $c \in \mathcal{M}$  so that  
 $\downarrow$   
 $a_n$

all  $p_k, k < \omega$ , still non-isolated over  $a_{\leq n}$ .

• the formulas  $\varphi_n(x), n < \omega$  may be chosen so that  
 after  $\omega$  steps:

$$\forall \varphi(x) \in L(A) \exists n \varphi = \varphi_n.$$

consistent

Then  $A = \{a_n : n < \omega\}$  satisfies TV-test

$A = M \prec \mathcal{M}$ , every  $p_k$  still non-isolated  
 over  $A$ .

$$p_k(M) = \emptyset \text{ [if not,}$$

some  $\bar{m} \models p_k$ . then  $\bar{m} \in \bar{a}_0$

$$(\bar{x}_k = \bar{m}) \vdash p_k(\bar{x}_k) \downarrow)$$

### Proof of the Lemma.

Let  $p(\bar{x})$ : one of the types  $p_n(\bar{x}_n)$ .

Let  $h(\bar{x}, y, \bar{a}) \in L(A)$

$$h(\bar{x}, c, \bar{a}) \vdash p(\bar{x}) \Leftrightarrow h(\mathcal{M}, c, \bar{a}) \subseteq p(\mathcal{M}).$$

$$\Leftrightarrow \forall \psi(\bar{x}) \in p(\bar{x}) \quad h(\mathcal{M}, c, \bar{a}) \subseteq \psi(\mathcal{M})$$

$$\Leftrightarrow \forall \psi \in p \quad \mathcal{M} \models \forall \bar{x} (h(\bar{x}, c, \bar{a}) \rightarrow \psi(\bar{x}))$$

$$\Leftrightarrow \forall \psi \in p \quad \psi_h(y) \in t_p(c/A)(y)$$

$$\text{where } \psi_h(y) = \forall \bar{x} (h(\bar{x}, y, \bar{a}) \rightarrow \psi(\bar{x}))$$

hence:

$$t_p(c/A) = t_p(c'/A) \Rightarrow [h(\bar{x}, c, \bar{a}) \vdash p \Leftrightarrow h(\bar{x}, c', \bar{a}) \vdash p]$$

$$h(\bar{x}, c, \bar{a}) \text{ consistent} \Leftrightarrow (\exists \bar{x} h(\bar{x}, y, \bar{a})) \in t_p(c/A)(y).$$

$$\text{Let } X_{h,p} = \{q \in S_1(A) : \text{For } c \models q, h(\bar{x}, c, \bar{a}) \vdash p(\bar{x}) \text{ and } h(\bar{x}, c, \bar{a}) \text{ consistent.}\}$$

"bad types"

$$\text{Let } q \in S_1(A) \text{ then} \quad \text{For } c \models q, h(\bar{x}, c, \bar{a}) \text{ consistent}$$

$$q \in X_{h,p} \Leftrightarrow q(y) \in S_1(A) \cap [\exists \bar{x} h(\bar{x}, y, \bar{a})] \cap \bigcap_{\psi \in p} [\psi_h(y)]$$

$$\text{For } c \models q, h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$$

(\*)  $X_{h,p}$ : nowhere dense in  $S_1(A)$ .

Proof of (\*): (a.a.)

Suppose  $\theta(y) \in L_1(A)$  and  $\emptyset \neq S_1(A) \cap [\theta] \subseteq X_{h,p}$ .

$$\text{Let } \alpha(\bar{x}) = \exists y (h(\bar{x}, y, \bar{a}) \wedge \theta(y))$$

$\uparrow$   
 $L(A)$



•  $\alpha(\bar{x})$  : consistent :

MT3 / M

Let  $c \in \Theta(\mathcal{M})$

$\Downarrow [\theta] \subseteq X_{n,p}$

$\mathcal{M} \models \exists \bar{x} h(\bar{x}, c, \bar{a})$ .

Let  $\bar{d} \in \mathcal{M}$  s.t.  $\mathcal{M} \models h(\bar{d}, c, \bar{a})$ .

$\bar{d}$  satisfies in  $\mathcal{M}$  :  $\underbrace{\exists y h(\bar{x}, y, \bar{a})}_{\alpha(\bar{x})}$

•  $\alpha(\bar{x}) \vdash p(\bar{x})$ , i.e.  $\alpha(\mathcal{M}) \subseteq p(\mathcal{M})$ .

Let  $\bar{d} \in \alpha(\mathcal{M})$ . So there is  $c \in \mathcal{M}$  s.t.

$\models h(\bar{d}, c, \bar{a}) \wedge \theta(c)$

$\Downarrow [\theta] \subseteq X_{n,d}$

$\forall \psi \in p \models \psi_n(c)$

$\forall \psi \in p \models h(\bar{x}, c, \bar{a}) \vdash \psi(\bar{x}) \Rightarrow h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$   
 $\neq \bar{d} \Rightarrow \frac{\perp}{\bar{d}}$  (y)

as:  $p(\bar{x})$  non-isolated

Let  $X = \bigcup_{h,p_n} X_{h,p_n} \subseteq S_1(A)$ .

meager. Let  $q_i \in S_1(A) \cap [\varphi] \setminus X$

$c \models q_i$  good.

(pf) (a.e) Suppose  $p = p_n$  isolated over  $A \cup \Sigma c \mathcal{L}$ .

$\exists h(\bar{x}, c, \bar{a}) \models p(\bar{x}) \Rightarrow q_i \in X_{h,p_n} \Downarrow$   
 consistent  $\uparrow$   $\uparrow p(\bar{c}/A)$

14.03.2022

Def  $T$  is quantifier eliminable if  $\forall \varphi \in L \exists \psi \in L$

$$T \vdash \varphi \leftrightarrow \psi$$

open  
 $\equiv$  q.f.

Def. For  $p(\bar{x}) \in S_n(\emptyset)$  let  $p_o(\bar{x}) = \exists \varphi(\bar{x}) \in p(\bar{x})$ .

Remark  $T$  is q.e.  $\Leftrightarrow \forall n \forall p \in S_n(\emptyset) p_o \vdash p$   $\varphi$  open.

Proof " $\Rightarrow$ " Obvious. " $\Leftarrow$ " Let  $\varphi(\bar{x}) \in L$ .

$$\bullet \forall p \in [\varphi] \cap S_n(\emptyset) \exists \psi \in p \quad p \in [\psi] \subseteq [\varphi]$$

Why?  $p_o \vdash p$   
 $p_o \vdash \varphi$

by compactness

$$\exists \text{ finite } p'_o \subseteq p_o \text{ s.t. } p'_o \vdash \varphi, \text{ i.e.}$$

$$p'_o(\mathcal{M}) \subseteq \varphi(\mathcal{M})$$

$\Downarrow$

$$\varphi(\mathcal{M}) = \left( \bigwedge_{\psi' \in p'_o} \psi' \right) (\mathcal{M}) \subseteq \varphi(\mathcal{M}) \rightsquigarrow \mathcal{M} \models \psi(\bar{x}) \rightarrow \varphi(\bar{x})$$

$$\varphi \vdash \varphi \Rightarrow [\varphi] \cap S_n(\emptyset) \subseteq [\varphi] \cap S_n(\emptyset)$$



Application  $L = \{+, \cdot, 0, 1\}$ : the language of rings.

$ACF_p$ : the theory of algebraically closed fields of char  $p$ , in  $L$ .

Axioms:

1) field axioms

2) char  $= p \neq 0$ :  $\underbrace{1 + \dots + 1}_p = 0$

2')  $p = 0$ :  $\underbrace{1 + \dots + 1}_n = 0$  for  $n \geq 1$

3) Every polynomial of deg  $n$  has a root:  
 $0 < n$   
 $\forall y_{n-1}, y_{n-2}, \dots, y_0 \exists x \quad x^n + y_{n-1}x^{n-1} + \dots + y_0 = 0.$

Fact  $ACF_p$  is complete.

Proof Let  $M, N \models ACF_p$ . Enough to show that  $M \equiv N$ . Let  $\varepsilon > \|M\|, \|N\|$  and

let  $M' \succ M, N' \succ N$ .

power  $\varepsilon$ .

$M', N'$ : uncountable acl fields of the same power and char

$$\begin{array}{c}
 \Downarrow \text{algebra} \\
 M' \cong N' \Rightarrow M' \equiv N' \\
 \quad \quad \quad \Downarrow \\
 \quad \quad \quad M \equiv N
 \end{array}$$

Fact  $ACF_p$  is q.e. (Chevalley, Tarski)

Proof (in  $\mathcal{M}$ ) We will show that  $\forall p \in S_n(\emptyset)$

$p_0 + p \Leftrightarrow p_0(\mathcal{M}) \subseteq p(\mathcal{M})$ . Let  $\bar{a} \models p_0$ ,

$\bar{b} \models p$ ,  $\bar{a}, \bar{b} \in \mathcal{M}$ . It's enough to prove

that  $\exists f \in \text{Aut}(\mathcal{M}) f(\bar{a}) = \bar{b}$ .

$$\begin{array}{l}
 \bar{a} = (a_1, \dots, a_n) \\
 \bar{b} = (b_1, \dots, b_n)
 \end{array}$$

Let  $\langle \bar{a} \rangle, \langle \bar{b} \rangle$ : the subfields with  $\mathbb{A}$

of  $\mathcal{M}$  generated by  $\bar{a}, \bar{b}$ .  $\bar{a}, \bar{b} \models p_0 \Rightarrow \langle \bar{a} \rangle \cong \langle \bar{b} \rangle$

$$\langle \bar{a} \rangle \cong \langle \bar{b} \rangle$$

$\Downarrow$  unique  $\Downarrow$

$$\mathcal{M} \cong \langle \bar{a} \rangle_0 \cong \langle \bar{b} \rangle_0 \subseteq \mathcal{M}$$

(fraction field)

$$\mathcal{M} \cong \underbrace{\langle \bar{a} \rangle_0^{\text{alg}}}_{F_a} \cong \underbrace{\langle \bar{b} \rangle_0^{\text{alg}}}_{F_b} \subseteq \mathcal{M}$$

Some algebraic magic.

$$\text{trdeg}(\mathcal{M}/\mathbb{F}_a) = \|\mathcal{M}\| = \text{trdeg}(\mathcal{M}/\mathbb{F}_b)$$

$$\begin{array}{ccc} \mathcal{M} \cong \mathbb{F}_a(X_\alpha, \alpha < \lambda)^{\text{alg}} & \Downarrow & \\ f \cong \downarrow & \curvearrowright & \cong \\ \mathcal{M} \cong \mathbb{F}_b(X_\beta, \beta < \lambda)^{\text{alg}} & & \end{array}$$

$$f \in \text{Aut}(\mathcal{M}), f(\bar{a}) = \bar{b}.$$



### Types in $T = \text{ACF}_p$

Let  $\mathcal{M} \models T$ : a monster model.

subfield  $K$ , <sup>UI</sup> We will describe  $S_n(K)$ .  
(small)

Let  $\bar{a} \subseteq \mathcal{M}$ ,  $|\bar{a}| = n$ .

$$K[\bar{x}] \triangleright I(\bar{a}/K) = \{ f \in K[\bar{x}] : f(\bar{a}) = 0 \}$$

Remark 1)  $\text{tp}(\bar{a}/K) = \text{tp}(\bar{a}'/K) \Leftrightarrow I(\bar{a}/K) = I(\bar{a}'/K)$

2)  $\forall I \triangleleft K[\bar{x}]$  <sub>prime</sub>  $\exists \bar{a} \subseteq \mathcal{M} \ I(\bar{a}/K) = I$ .

Proof 1) " $\Rightarrow$ "  $\text{tp}(\bar{a}/K) = \text{tp}(\bar{a}'/K) \Rightarrow \exists f \in \text{Aut}(\mathcal{M}/K)$   
 $I(\bar{a}/K) = I(\bar{a}'/K) \stackrel{f}{=} f(\bar{a}) = \bar{a}'$

[alternatively:  $f \in I(\bar{a}/K) \Leftrightarrow "f(\bar{x}) = 0" \in \text{tp}(\bar{a}/K)$ ]

" $\Leftarrow$ " Assume  $I(\bar{a}/K) = I(\bar{a}'/K) = I$ .

$$K[\bar{a}] \cong_K K[\bar{x}] / I \cong K[\bar{a}']$$

$$\Downarrow$$

$$\exists f \in \text{Aut}(K[\bar{x}]/K) \quad f(\bar{a}) = \bar{a}'$$

$$\Downarrow$$

$$t_p(\bar{a}/K) = t_p(\bar{a}'/K).$$

2)  $K \subseteq K[\bar{x}] / I = K[\bar{a}] \cong \bar{a} = \bar{x} / I$  and  $I(\bar{a}/K) = I$ .

•  
•  
•

□

$$S_1(K) = \{ t_p(a/K) : a \in \mathcal{A} \} : a \text{ top. space.}$$

$$\Downarrow$$

$$p(x) = t_p(a/K), \quad I_p = I(a/K) \triangleleft K[x]$$

a)  $I_p \neq \{0\}$ , i.e.  $a$  is algebraic /  $K$ , so

$0 \neq f \in I_p$  "  $f(x) = 0$  "  $\in p(x)$ . In fact,

irreducible over  $K$  "  $f(x) = 0$  "  $\in p(x)$  (isolates)

Let  $a' \in \mathcal{M}$  s.t.  $f(a') = 0 \Rightarrow I(a'/K) \ni f \ni$

$$p = \text{tp}(a/K) = \text{tp}(a'/K) \Leftarrow I(a/K) = I(a'/K) \Leftarrow \begin{matrix} f \text{ generates} \\ I(a'/K) \end{matrix}$$

$p(x)$  here is called algebraic.

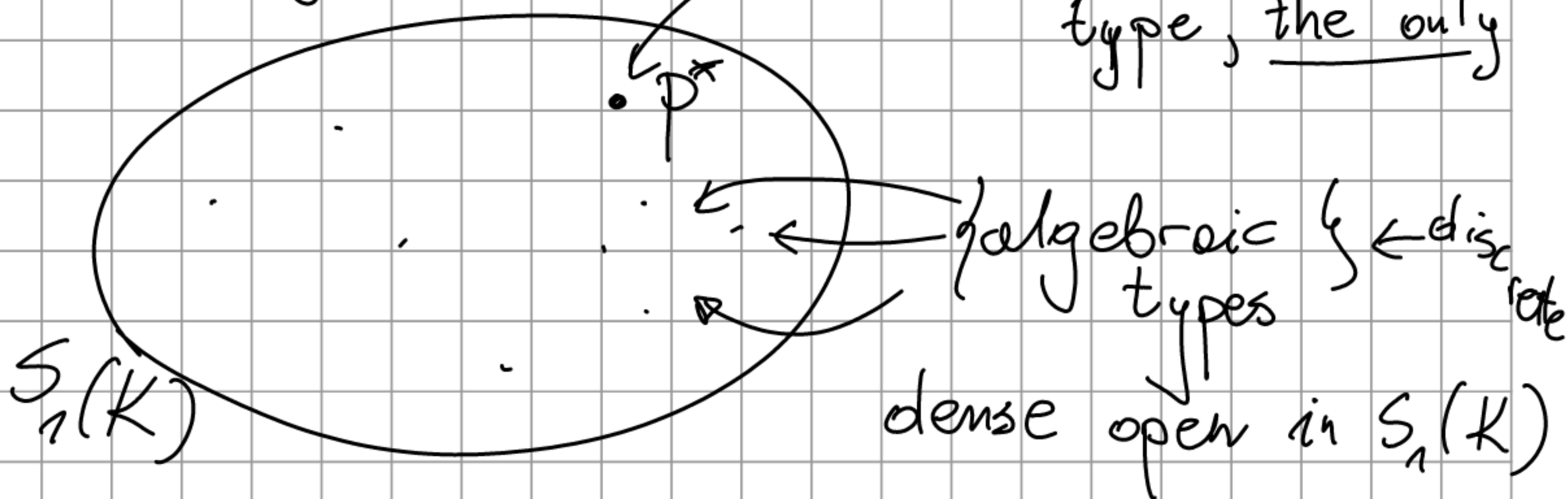
More generally ( $T$ : arbitrary)

Def  $\varphi(\bar{x}) \in L(A)$  is called algebraic, if  $0 < |\varphi(\mathcal{M})| < \aleph_0$ , similarly for  $q$ : a type.

b)  $I_p = \{0\}$ :  $p = \text{tp}(a/K)$  s.t.  $a \in \mathcal{M}$  transcendental over  $K$

↑  
the transcendental type over  $K$ .

transcendental type, the only



If  $K$  cble then  $S_n(K)$  cble

Corollary  $ACF_p$  is  $\aleph_0$ -stable [recall:  $T$  is  $\kappa$ -stable  $\Leftrightarrow \forall A \subseteq \mathcal{M}, |A| \leq \kappa$   
 $|S_n(A)| \leq \kappa$ ]

Proof Let  $A \subseteq \mathcal{M}$ .  
 $A$  is abelian

$$A \subseteq K \subseteq \mathcal{M} \quad |S_n(A)| \leq |S_n(K)| = \aleph_0$$

$\uparrow$   
 abelian subfield

Remark  $T$  is totally transcendental  $\Leftrightarrow T: \aleph_0$ -stable

Proof " $\Rightarrow$ " from def, " $\Leftarrow$ ": (A.a.) Let  $\kappa > \aleph_0$ .

Suppose  $|A| \leq \kappa < |S_n(A)|$  for some  $A \subseteq \mathcal{M}$ .

Shall find  $A_0 \subseteq A$  with  $|S_n(A_0)| \geq 2^{\aleph_0}$ .  
 $A_0$  is abelian

(def)  $\varphi(x) \in L(A)$  is large iff  $|S_n(A) \cap [\varphi]| > \kappa$   
 otherwise:  $\varphi(x)$  small.

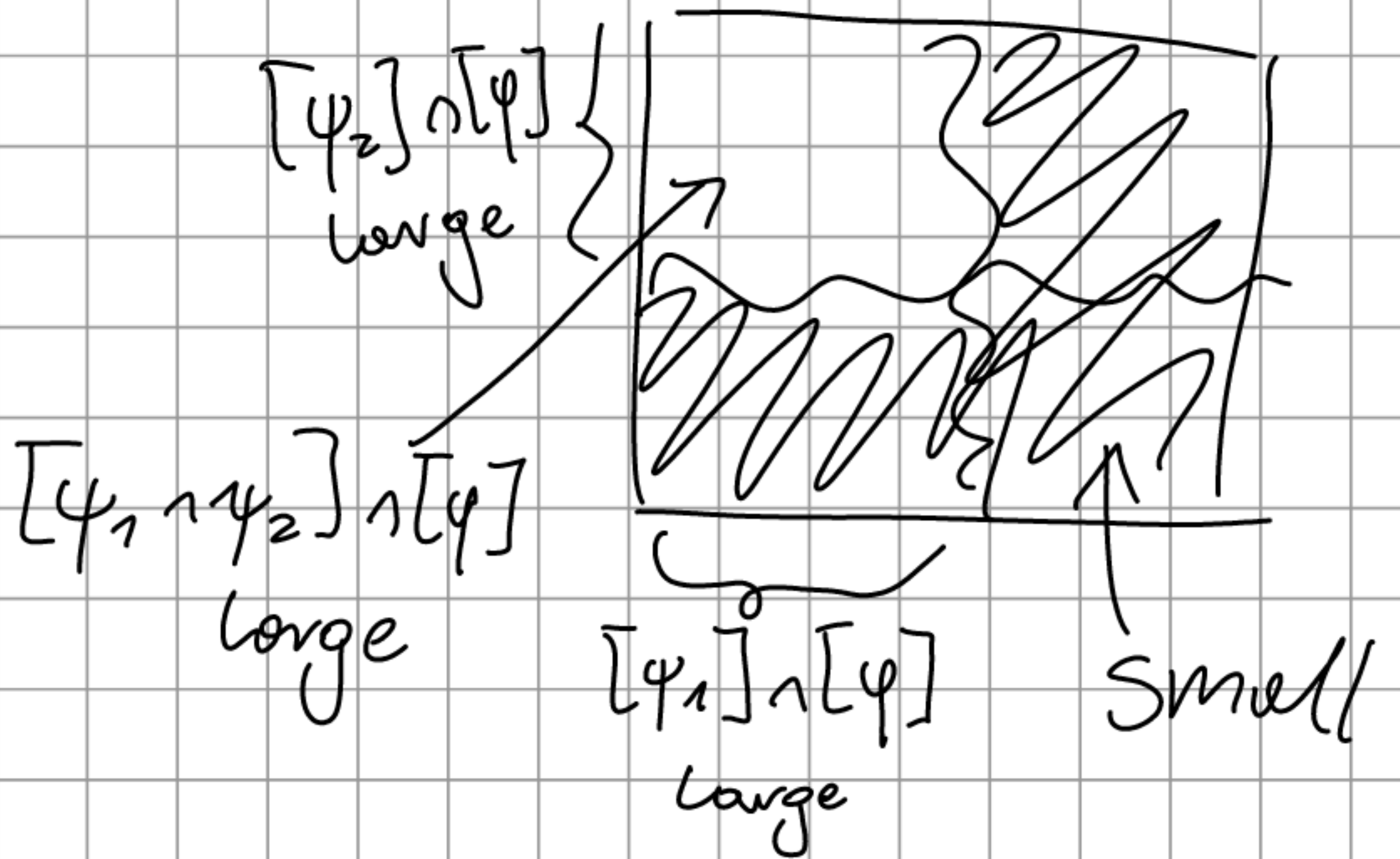
(a) " $x=x$ " is large

(b) if  $\varphi(x)$  is large, then  $\exists \psi_1, \psi_2 \in L(A)$  large  
 s.t.  $\varphi(\mathcal{M}) = \psi_1(\mathcal{M}) \cup \psi_2(\mathcal{M})$

Pf. (b) if not then  $\exists \psi \in L_n(A)$ :  $\psi \wedge \varphi$  is large  
 is a complete type is  $S_n(A) \cap [\psi]$



1°  $\mathcal{P}^*$ : consistent:  $\psi_1, \psi_2 \in \mathcal{P}^* \Rightarrow \psi_1 \wedge \psi_2 \in \mathcal{P}^*$



2°  $\mathcal{P}^*$ : complete OK

$\mathcal{P}^* \in S_n(A) \cap [\phi]$ : the only large type.

$$S_n(A) \cap [\phi] = \underbrace{\mathcal{P}^*}_{\text{large}} \cup \underbrace{\bigcup_{\substack{\psi \in L_n(A) \\ \psi \vdash \phi}}}_{\text{small}} (S_n(A) \cap [\phi])$$

$\leq \aleph$

$> \aleph$

$\leq \aleph$

↘

c) a tree of large formulas in  $L_1(A)$   $\varphi_\eta(x)$ ,  
 $\eta \in 2^{<\omega}$  st.  $\varphi_\eta(\mathcal{M}) = \varphi_{\eta_0}(\mathcal{M}) \cup \varphi_{\eta_1}(\mathcal{M})$   
↑  
by (b)

Let  $A_0 \subseteq A$ : the set of all params of  $\varphi_\eta, \eta \in 2^{<\omega}$

Then  $|A_0| \leq \aleph_0^{\aleph_0}$ .

For  $\eta = 2^{<\omega}$ :  $\mathcal{P}_\eta^0 = \{ \varphi_{\eta|n}(x) : x < \omega \}$ : a consistent  
 $\mathcal{L}$ -type over  $A_0$

When  $\nu \neq \eta$   
then  $\mathcal{P}_\eta \neq \mathcal{P}_\nu$

$\mathcal{P}_\eta \in S_\eta(A_0)$

$$\Downarrow \quad |S_\eta(A_0)| \geq 2^{\aleph_0} > \aleph_0 \quad \Downarrow$$

21.03.2022

## CONSTRUCTION OF SPECIAL MODELS: N, M ⊨ T

Def.  $M$  is atomic if  $\forall \bar{a} \in M$   $\text{tp}(\bar{a}/\emptyset) = \text{tp}(\bar{a})$  is isolated.

(2)  $M$  is prime if  $\forall N \models T \exists f: M \xrightarrow{\cong} N$

Example  $T = \text{ACF}_p$ ,  $F_p$ : prime field of char  $p$

a)  $F_p$ : atomic (exercise)

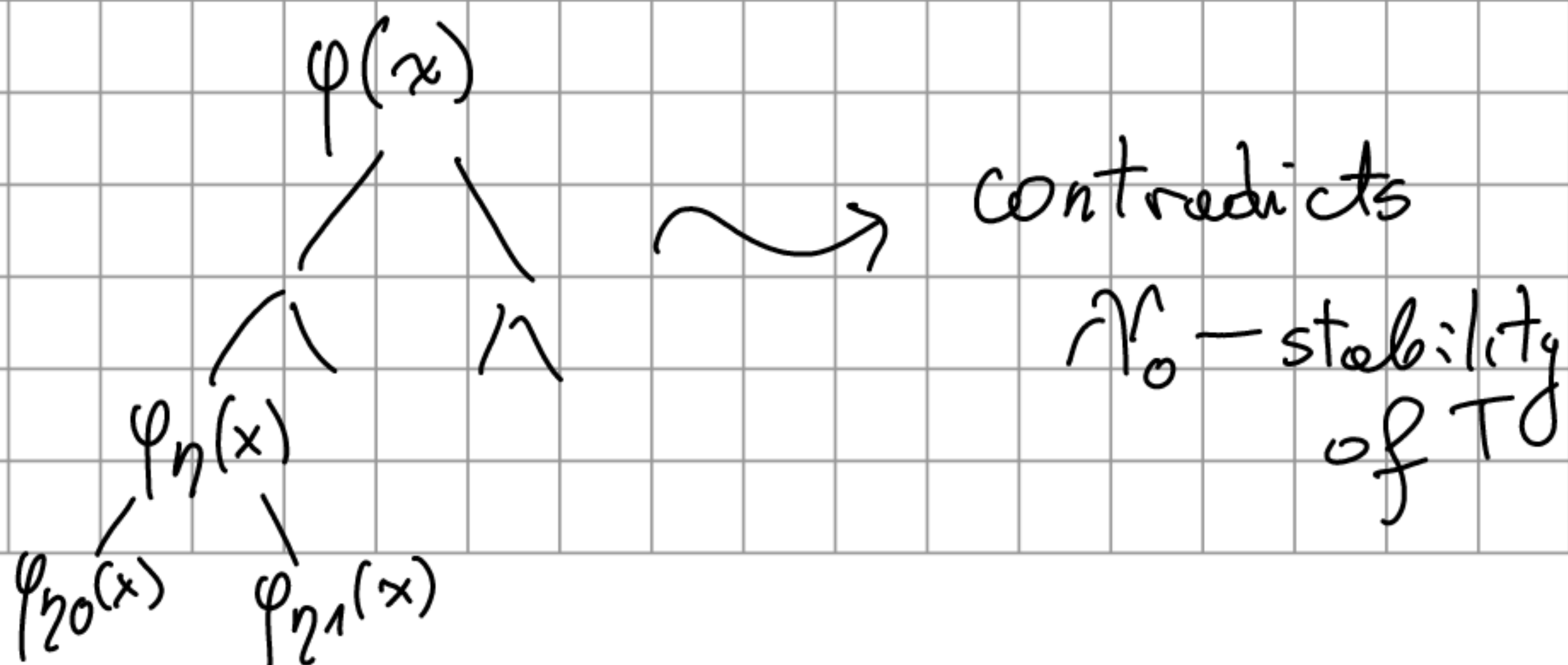
b)  $F_p$ : prime (exercise)

Thm.  $T: \aleph_0$ -stable  $\Rightarrow T$  has a prime model.

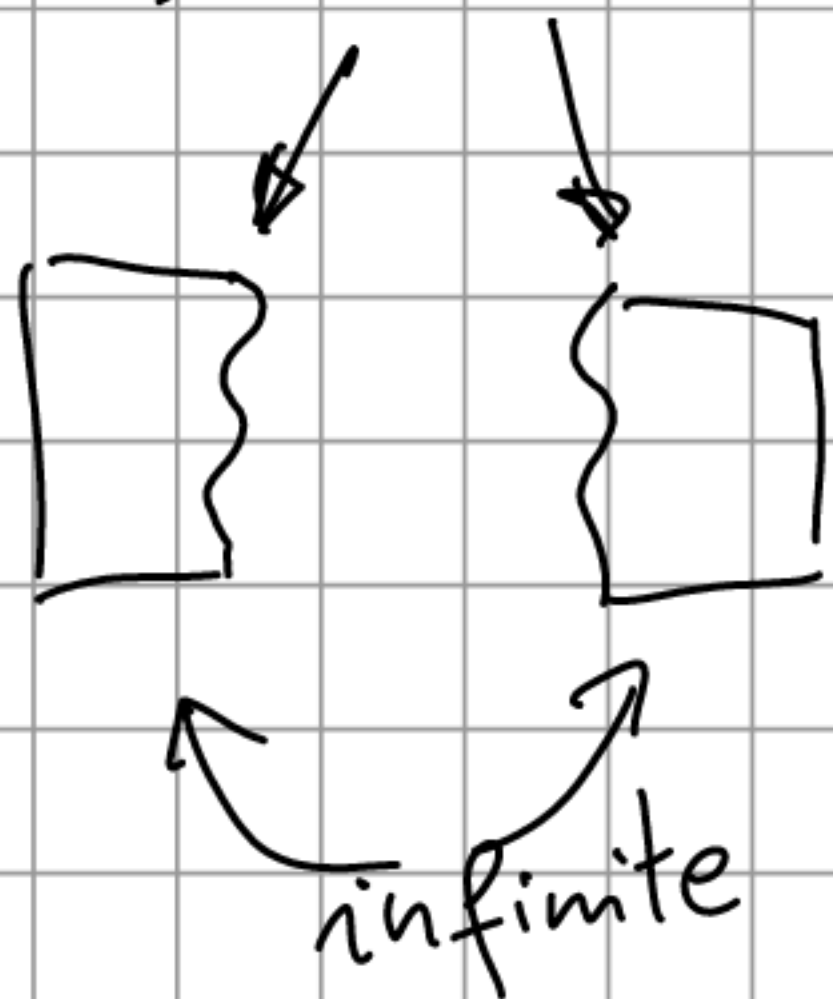
Lemma 1  $T: \aleph_0$ -stable  $\Rightarrow \forall A \subseteq \mathcal{M}$   $\{\text{isolated types}\} \subseteq S_n(A)$  dense

Pf. Suppose  $\varphi(x) \in L_n(A)$  s.t. in  $S_n(A) \cap [\varphi]$  (consistent with  $T$ )

there is no isolated types  $\Rightarrow$  a tree of formulas  $\varphi_\eta(x) \in L_n(A)$ ,  $\eta \in 2^{<\omega}$



clopen  $\{ \} \subseteq S_1(A) \cap [\varphi]$



□

Lemma 2  $(a, b \in \mathcal{M})$   $tp(a)$  isolated and  $tp(b/a)$  isolated

$\Leftrightarrow tp(ab)$  isolated

Pf " $\Rightarrow$ ":  $\varphi(x) \vdash tp(a), \psi(a, y) \vdash tp(b/a)(y)$

$p_a(y) \subseteq S_y(a)$

Then:  $\varphi(x) \wedge \psi(x, y) \vdash tp(ab)$

Let  $a', b' \in \mathcal{M}$  satisfy  $\varphi(x) \wedge \psi(x, y)$

$\models \varphi(a') \Rightarrow tp(a') = tp(a) \Rightarrow \psi(a', y) \vdash p_{a'}(y)$

$\supseteq S_y(a')$

$\models \psi(a', b') \Rightarrow \models p_{a'}(b')$

$\Downarrow$

$ab \equiv a'b'$

and  $tp(ab) = tp(a'b')$

" $\Leftarrow$ ":  $\Theta(x, y) \vdash \text{tp}(a, b)$ .

(a) " $\exists y \Theta(x, y)$ "  $\vdash \text{tp}(a)$ .

because Let  $a' \in \mathcal{M}$  satisfy " $\exists y \Theta(x, y)$ ".

So there is  $b'$  s.t.  $\models \Theta(a', b')$

$\Rightarrow \text{tp}(ab) = \text{tp}(a', b') \Rightarrow \text{tp}(a) = \text{tp}(a')$ .

(b)  $\Theta(a, y) \vdash \text{tp}(b/a)(y)$

because: similar to (a) ▀

Proof of the thm. Construction of a prime model of  $T$ :

$A = \{a_n : n < \omega\} \subseteq \mathcal{M}$  so that:

1)  $A$  satisfies TV-test

2)  $\forall n$   $\text{tp}(a_n/a_{<n})$  is isolated

At step  $n$  choose  $a_n$ :  $[a_{<n} = \{a_k : k < n\}]$

Let  $\varphi(x) \in L_n(a_{<n})$  consistent.

Let  $a_n \in \varphi(\mathcal{M})$  s.t.  $\text{tp}(a_n/a_{<n})$  is isolated (lemma 1)

Suitable choice of  $\varphi$ 's ensures (1).

$M \models T$  is prime. Let  $N \models T$ . We find  
 $f(a_n) \in N$  for  $n < \omega$  s.t.  $\exists$  <sup>arbitrary</sup>  $f: M \equiv \rightarrow N$ .

At step  $n$   $f[a_{<n}] \subseteq N$  with  $f: a_{<n} \equiv \rightarrow N$

Let  $p(x) = \text{tp}(a_n/a_{<n})$  (isolated)

$$\Downarrow \\ f^*(p)(x) \in S(f(a_{<n}))$$

isolated

too, hence realised by  $f(a_n)$

$$\Downarrow \\ f: a_{\leq n} \equiv \rightarrow N$$



Remark (1) A prime model  $M \models T$  is atomic

(2) If  $M \models T$  is atomic, then  $M$  prime

Corollary  $\hat{F}_p$  is atomic.

## Proof of remark

(1) Let  $p(\bar{x}) \in S_n(\emptyset)$  non-isolated. Will show

$p(M) = \emptyset$ . Let  $N \models T$  be omitting  $p$ .

$\exists f: M \xrightarrow{\cong} N \Rightarrow p(M) = \emptyset$ .

(2) Let  $M = \prod_T \{a_n : n < \omega\}$  atomic.

Then  $\forall n$   $tp(a_n)$  is isolated

$\Downarrow$  lemma 2  
 $\forall n$   $tp(a_n/a_{<n})$  is isolated

$\Downarrow$  pf of thm

$M$  prime. □

Corollary A prime model of  $T$  is unique (up to  $\cong$ )

Proof Let  $M, N \models T$  both prime  $\stackrel{\text{remark}}{\Rightarrow} M, N$

are stable and atomic, so we have embeddings

in both directions, using back-and-forth

we get the iso.

Def  $M \models T$  is minimal if  $\neg \exists N \cong M$

Example  $\hat{\mathbb{F}}_p$  is minimal.

Fact  $T$  has a prime model  $\Leftrightarrow \forall n$   $\{ \text{isolated types} \} \subseteq S_n(\emptyset)$   
dense

Proof " $\Rightarrow$ ": Let  $M \models T \Rightarrow M$ : atomic  
prime

what  
we need

$N \models T$ , then  $\forall n \exists p \in S_n(\emptyset)$ :  
any  $p(N) \neq \emptyset$   
exercise | is dense in  $S_n(\emptyset)$

" $\Leftarrow$ ": Claim Assume  $\bar{a} \subseteq M$  and  $\text{tp}(\bar{a})$  is isolated.  
finite

Then  $\{ \text{isolated types} \} \subseteq S_n(\bar{a})$ .  
dense

Proof of claim Let  $n = |\bar{a}|$ ,  $\varphi(\bar{x}) \vdash \text{tp}(\bar{a})$ .

Let  $\psi(\bar{x}, y) \in L_{n+1}(\emptyset)$  s.t.  $\psi(\bar{a}, y)$  is consistent.

We seek  $q(y) \in S_1(A) \cap [\psi(\bar{a}, y)]$  isolated.

Let  $\chi(\bar{x}, y) = \varphi(\bar{x}) \wedge \psi(\bar{x}, y)$ .

By assumptions of  $\Leftarrow \exists p(\bar{x}, y) \in S_{\bar{x}, y}(\emptyset) \cap [\chi(\bar{x}, y)]$ .  
isolated

Let  $\bar{a}', b' \models p(\bar{x}, y)$ . Then  $\bar{a}' \models p(\bar{x}, y) \upharpoonright_{\bar{x}} = \text{tp}(\bar{a})$   
 $\wedge$   
 $\varphi(\bar{x})$



Let  $f \in \text{Aut}(\mathcal{M}) : f(\bar{a}') = \bar{a}$   
 $b = f(b')$

Then  $\bar{a}'b' \stackrel{\equiv}{\underset{f}{\rightarrow}} \bar{a}b \Rightarrow \bar{a}b \models p(\bar{x}, y)$

so  $\text{tp}(\bar{a}b)$  is isolated  $\stackrel{\text{lemma 2}}{\Rightarrow}$   $\text{tp}(b/\bar{a})$  isolated  
 $\Downarrow$   
 $\psi(\bar{a}, y)$

So  $q(y) = \text{tp}(b/\bar{a})$

Given  $n \mapsto$  we construct a model  $M = \{a_n : n < \omega\}$  s.t.

$\forall n \text{ tp}(a_n/a_{<n})$  is isolated

$\Downarrow$  lemma 2

$M$  atomic dble  $\Rightarrow M$  prime.

Corollary If  $\forall n |S_n(\emptyset)| \leq \aleph_0$ , then  $T$  has a prime model.

Corollary A prime model (of a dble  $T$ ) is homogeneous (exercise).

The number of countable models of  $T: I(T, \aleph_0)$ ,  
 $n(T)$ .

Remark  $1 \leq n(T) \leq 2^{\aleph_0}$

$$M \models T \Rightarrow M \cong \underbrace{(N, \dots)}_{\leq 2^{\aleph_0} \text{ L-structures like that}}$$

Recall  $n(T) = 1 \Leftrightarrow \forall n \ |S_n(\emptyset)| < \aleph_0$

( $T: \aleph_0$ -categorical)

Vaught conjecture (1961)

$$n(T) > \aleph_0 \Rightarrow n(T) = 2^{\aleph_0}$$

Thm (M. Morley, 1971)  $\aleph_0 < n(T) < 2^{\aleph_0} \Rightarrow n(T) = \aleph_1$

Thm (Vaught, 1961)  $n(T) \neq 2$

Proof (A.a) suppose  $n(T) = 2$ .

$$n(T) < 2^{\aleph_0} \Rightarrow T \text{ small (i.e. } \forall n \ |S_n(\emptyset)| \leq \aleph_0)$$

⋮

25.03.2022

Example (Andrzej Ehrenfeucht) Theory with exactly 3 stable theories.

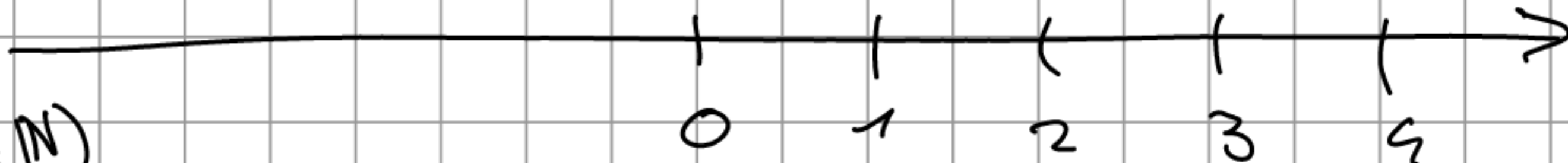
$T_0 = \text{Th}(\mathbb{Q}, \leq)$ . Look at  $T = T_0(N) = \text{Th}(\mathbb{Q}, \leq, n)_{n \in \mathbb{N}}$ .

$T_0$  is q.e.  $\Rightarrow T$  is also q.e.

$S_1^T(\emptyset) = S_1^{T_0}(N)$ . The types in  $S_1^T(N)$ :

realised in  $(\mathbb{Q}, \leq, N)$

isolated



- $p_i(x) \equiv \{x = i\}$ ,  $i \in \mathbb{N}$
- $r_i(x) \equiv \{i-1 \leq x \leq i\}$ ,  $i \in \mathbb{N}$ ,  $-1 \approx -\infty$
- $s(x) \equiv \{x > i : i \in \mathbb{N}\}$

omitted

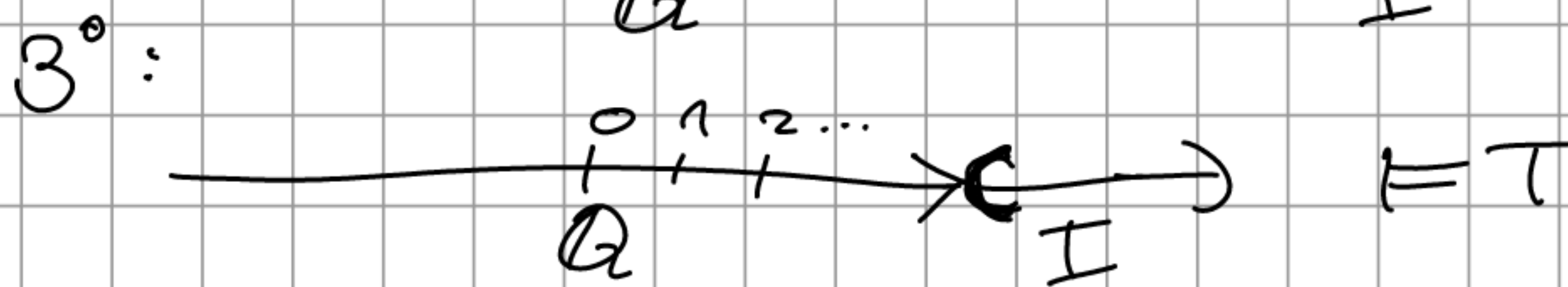
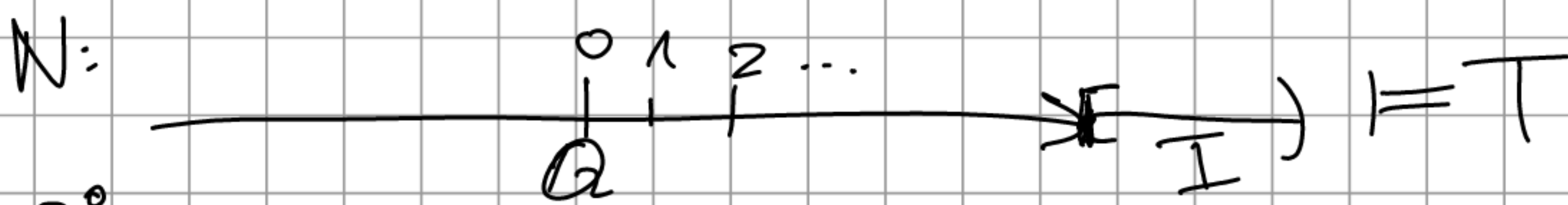
We will point 3 stable models  $N \models T$ .

1°  $N = M$ : prime model of  $T$ .

2°  $s(N)$  has the minimal element.

3°  $s(N)$  hasn't minimal element.

2°:  $\mathbb{Q} \cup ([0,1) \cap \mathbb{Q})$ :



Variants: 3, 4, 5, 6, ...

Problem Does there exist a stable  $T$  with  
 $1 < n(T) < \aleph_0$ ?

Def ( $A \subseteq \mathcal{U}$ ),  $M \prec \mathcal{U}$  is prime over  $A$  if:

(1)  $A \subseteq M$

(2)  $\forall N \prec \mathcal{U} \exists f: M \xrightarrow{\cong} N: f|_A = \text{id}_A$

Equivalently  $M$  is a prime model of  $T(A)$ .

- If  $A$  is stable  $\rightarrow$  full description of prime models of  $T(A)$ .
- If  $A$  is unstable  $\rightarrow$  in general not much can be set

Thm  $\aleph_0$ -stable  $\implies \forall A \exists M \neq T(A)$   
 $\mathcal{M}$  prime

Proof  $M = A \cup \{a_\alpha : \alpha < ?\}$

Construction of  $a_\alpha$ 's s.t.:

(i)  $A \cup \{a_\alpha : \alpha < ?\}$  satisfies the TV-test

(ii)  $\forall \alpha < ?$   $\text{tp}(a_\alpha / A a_{<\alpha})$  is isolated.

At some point it has to terminate

(i.e. we cannot add more elements).

Claim Assume  $N \not\prec \mathcal{M}$ . Then  $\forall \alpha \exists f: A \cup a_{<\alpha} \xrightarrow{\cong} N$   
 $\uparrow_A$   
 s.t.  $f|_A = \text{id}_A$ .

Proof We define  $f(a_\beta)$  for all  $\beta < \alpha$   
 by ind. on  $\beta$  so that  $f|_A = \text{id}_A$  and  $f: A \cup a_{\leq \beta} \xrightarrow{\cong} N$ .

Take  $\beta < \alpha$  and suppose  $\forall \beta' < \beta$   $f(a_{\beta'}) \downarrow$   
 so that the condition holds.

$p(x) = \text{tp}(a_\beta / A a_{<\beta})$  is isolated.

$f: A \cup a_{<\beta} \xrightarrow{\cong} f[A \cup a_{<\beta}] \subseteq N$

$f(p)$  is realised by  $c$

$p(x) \in S_1(A \cup a_{<\beta})$   $\xrightarrow{f^*}$   $f(p) \in S_1(f[A \cup a_{<\beta}])$   
 isolated  $\uparrow$  isolated

Now we put  $f(a_\beta) = c$ .

Claim ~~1~~

By the claim after some time we cannot get any more elements.

Additional property of the construction:

At the step  $\alpha$  we consider a formula  $\varphi(x) \in L(A_{\alpha \setminus \alpha})$  with no consistent realisation in  $A_{\alpha \setminus \alpha}$ , choose  $a_\alpha$  s.t.  $\models \varphi(a_\alpha)$ .

Problems Is a prime model over  $A$  unique up to isomorphism over  $A$ ?

Answer: not always. However the prime model  $M$  over  $A$  constructed by the previous construction is unique up to  $\cong_A$  and it's called primary over  $A$ .

Thm  $M, N$ : primary over  $A \Rightarrow M \cong_A N$ .

Proof  $M = A \cup \{a_\alpha : \alpha < \gamma\}$  : an "isolated construction" of  $M$  over  $A$ , i.e.  $\text{tp}(a_\alpha / A a_{\beta < \alpha})$  is isolated by a formula  $\varphi_\alpha(x)$  over  $A a_{\beta < \alpha}$  s.t.  $C_\alpha \subseteq \gamma$

Def.  $X \subseteq \gamma$  is closed if  $\forall \alpha \in X \ C_\alpha \subseteq X$

Remark (1)  $\alpha \in \gamma \Rightarrow \exists$  minimal  $X \subseteq \alpha$  s.t.  $X$  is finite

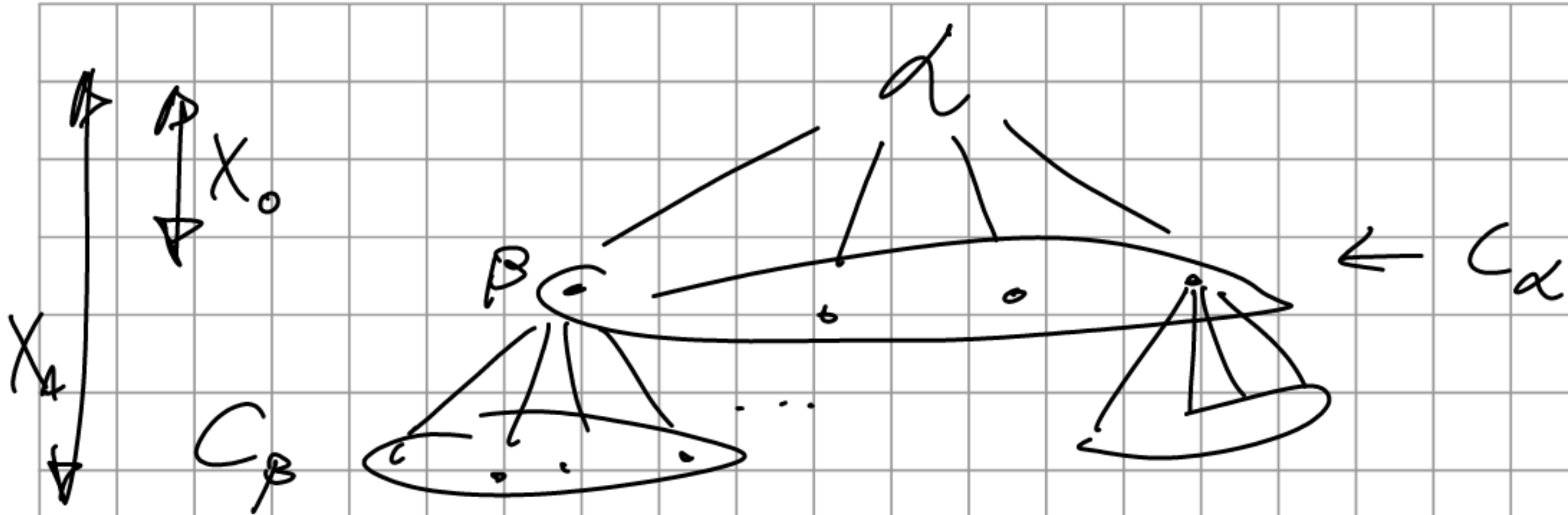
$\underbrace{X \cup \alpha}_\alpha$  is closed.

(2) A union of family of closed subsets of  $\gamma$  is closed.

Proof (remark) (2) is obvious.

(1): Take  $X_0 = C_\alpha$ , then  $X_n = X_{n+1} \cup \bigcup_{\beta \in X_n} C_\beta$   
 $\uparrow$   
 finite because  $C_\beta$  finite

Then  $X = \bigcup_n X_n$ . Then  $X \cup \alpha$  is finite and closed!



This tree has no infinite branch  
 (because there are no infinite  
 decreasing sequence of ordinals).

Remark II

Remark Assume  $X$ : closed. Then

$$A \cup \langle a_\alpha : \alpha \in X \rangle \overset{\text{concatenation}}{\uparrow} \langle a_\alpha : \alpha \in \gamma \setminus X \rangle$$

is an isolated construction over  $A$ .

Pf. (remark) as  $A \cup \langle a_\alpha : \alpha \in X \rangle$  is an  $i$ -construction  
 over  $A$   
 (by the fact that  $X$  is closed)

(2) Suppose that  $\alpha < \gamma$  and  $\alpha \notin X$ . Will  
 show that  $\text{tp}(a_\alpha / A a_\chi a_{\chi \cap (\kappa)})$  is isolated.



$$\varphi_\alpha(A) \vdash \text{tp}(a_\alpha / A a_{<\alpha}) \vdash \text{tp}(a_\alpha / A a_{\alpha \cap X^c \cap \alpha})$$

↑ will show

Let  $X_0 \subseteq X$  s.t.  $X_0 \cap \alpha \neq \emptyset$ .  
finite

Enough to show that  $\text{tp}(a_\alpha / A a_{<\alpha}) \vdash \text{tp}(a_\alpha / A a_{<\alpha} a_{X_0})$

Wlog By the remark  $X_0 \cup \alpha$  is closed.

So:  $A \cup \langle a_\beta : \beta < \alpha \rangle \wedge \langle a_\beta : \beta \in X_0 \rangle$  is

an  $i$ -construction over  $A$ , but for  $\beta \in X_0$

$$\text{tp}(a_\beta / A a_{<\alpha} a_{(\beta) \cap X_0}) \vdash \text{tp}(a_\beta / A a_{<\beta})$$

Because  $\varphi_\beta(x) \vee$  and  $\varphi_\beta \in$ .

So it implies also  $\text{tp}(a_\beta / A a_{<\alpha} a_{(\beta) \cap X_0})$ .

$$\Rightarrow \text{tp}(a_\alpha / A a_{<\alpha} a_{(\beta) \cap X_0}) \vdash \text{tp}(a_\alpha / A a_{<\alpha} a_{(\beta) \cap X})$$

the  
proof  
of e123

(we switch  $a_\alpha$  with  $a_\beta$ )

We just continue with induction on  $\beta$

(start with  $\beta = \min X_0$ ).

□

Claim M: Primary / A  $\Rightarrow$  atomic / A

Pf. Let  $\bar{m} \subseteq M = A \cup \{a_\alpha : \alpha < \gamma\}$ .

$\bar{m} \subseteq A \cup a_\chi, \chi \in \gamma$   
finite closed

$A \cup a_\chi$ : a partial  $i$ -construction.

$\Downarrow$   
 $\text{tp}(a_\chi / A)$  is isolated

$\Downarrow$   
 $\text{tp}(\bar{m} / A) \text{ --- } \parallel \text{ ---}$

claim  $\square$

Pf (of thm)  $M = A \cup \{a_\alpha : \alpha < \gamma\}$ ,

$N = A \cup \{b_\alpha : \alpha < \delta\}$ :  $i$ -constructions / A.

We construct  $f: M \xrightarrow{\cong} N$ ,  $f = \bigcup_{\alpha} f_{\alpha}$ : elementary.

(i)  $\text{Dom } f_{\alpha} \supseteq A$ ,  $\text{Rng } f_{\alpha} \supseteq A$ ,  $f_{\alpha} \upharpoonright_A = \text{id}_A$

(ii)  $|\text{Dom } f_{\alpha} \setminus A|, |\text{Rng } f_{\alpha} \setminus A| \leq |\alpha| \cdot \aleph_0$

(iii)  $\beta \in \text{Lim } f_{\beta} = \bigcup_{\alpha < \beta} f_{\alpha}$

(iv)  $a_{\alpha} \in \text{Dom } f_{\alpha+1}$ ,  $b_{\alpha+1} \in \text{Rng } f_{\alpha+1}$ .

(v)  $\text{Dom } f_{\alpha} \setminus A = a_{\chi}$ ,  $\text{Rng } f_{\alpha} \setminus A = b_{\chi}$ ,

where  $X \subseteq \mathcal{I}$ ,  $Y \subseteq \mathcal{J}$  are closed.

The recursive step from  $f_\alpha$  to  $f_{\alpha+1}$ .

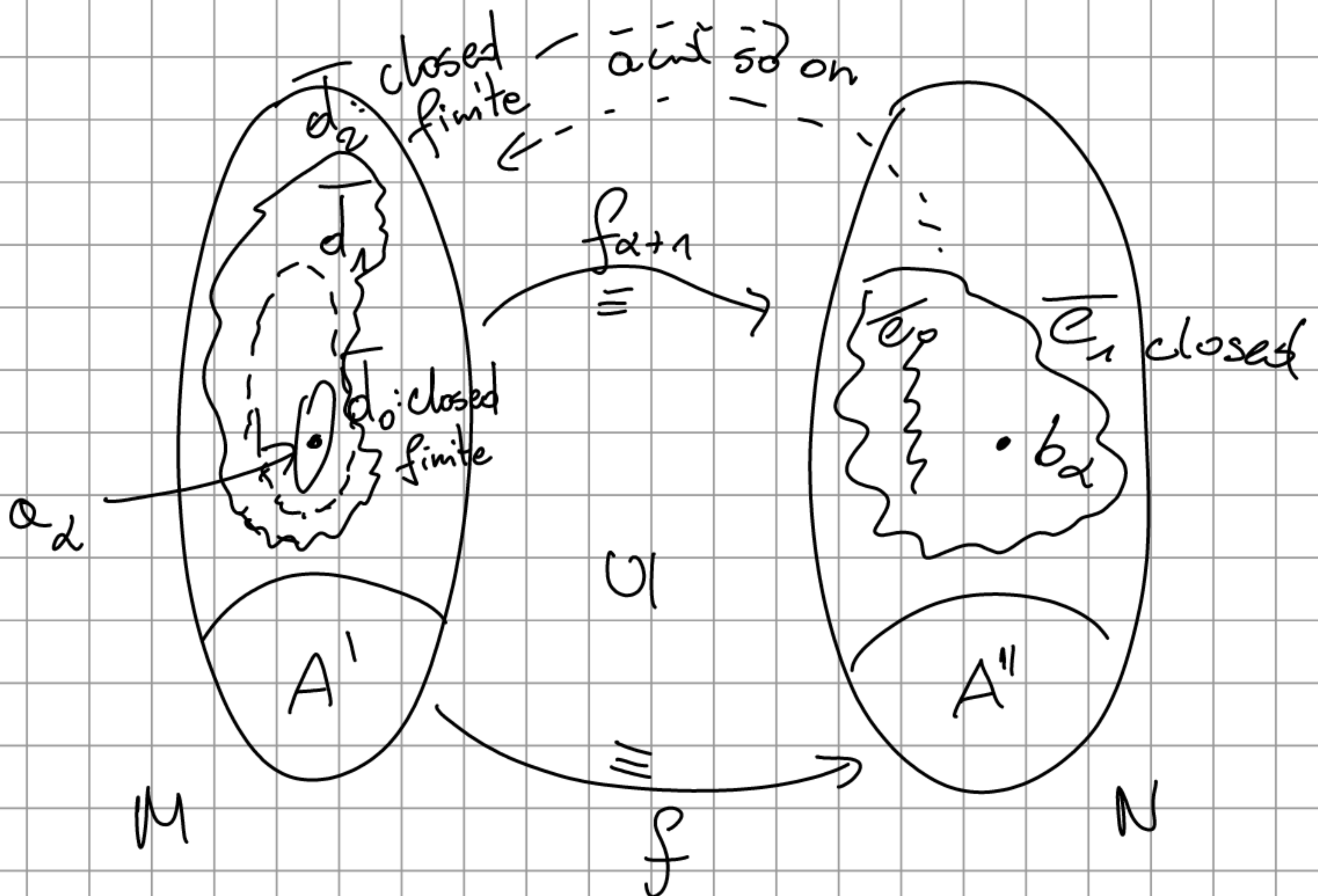
Let  $A' = A \cup \text{Dom } f_\alpha$ : an  $\mathcal{I}$ -construction over  $A$ ,

likewise  $A'' = A \cup \text{Rng } f_\alpha$ :  $\text{---} \cup \text{---}$

and  $M$  is primary over  $A'$  (by the remark)

and  $N$  is primary over  $A''$ .

$\varphi(x) = \text{tp}(a_\alpha / A')$  is isolated, so  $f_\alpha(p)$  is isolated too, therefore  $\exists b \in N$  st.  $f_\alpha(p) = \text{tp}(b / A'')$ .



4.04.22

Def. (1)  $\varphi(\bar{x}, \bar{a}) \in L_n(\bar{a})$  is algebraic if

$$0 < |\varphi(\mathcal{M})| < \aleph_0$$

(2) a type  $p(\bar{x})$  (over  $\mathcal{M}$ ) is algebraic if

$$0 < |p(\mathcal{M})| < \aleph_0$$

(3)  $a \in \text{acl}(A)$  if  $\text{tp}(a/A)$  is algebraic  
algebraic closure

(4)  $a \in \text{dcl}(A)$  if  $a \in \mathcal{M}$  is the only  
definable closure  
realisation of  $\text{tp}(a/A)$

Remark (1)  $p(\bar{x})$ : an algebraic type  $\Leftrightarrow p(\bar{x}) \vdash \varphi(\bar{x})$

for some algebraic formula  $\varphi(\bar{x})$

$$(2) |p(\mathcal{M})| = 1 \Leftrightarrow \exists \varphi (p \vdash \varphi \text{ and } |\varphi(\mathcal{M})| = 1)$$

Proof. 1 ( $\Leftarrow$ )  $p(\mathcal{M}) \subseteq \varphi(\mathcal{M})$

( $\Rightarrow$ ) (a.a.) let  $n \in \mathbb{N}$  arbitrary. Will show:  $|p(\mathcal{M})| \geq n$ .

Let  $\bar{x}_1, \dots, \bar{x}_n$ : disjoint tuples of variables,

$$|\bar{x}_i| = |\bar{x}|.$$

$\{ \varphi(\bar{x}_i) : \varphi(\bar{x}) \in p, i=1, \dots, n \} \cup \{ \bar{x}_i \neq \bar{x}_j : 1 \leq i < j \leq n \}$ :

a consistent type.

$\mathcal{J}$  is realised in  $\mathcal{M}$  so it has  $\geq n$  realisations.  $\Downarrow$

Fact  $\text{acl}(A) = \bigcup \{ \varphi(\mathcal{M}) : \varphi(x) \in L_n(A) \text{ algebraic} \}$

$\text{dcl}(A) = \bigcup \{ \varphi(\mathcal{M}) : \varphi(x) \in L_n(A) \wedge |\varphi(\mathcal{M})| = 1 \}$

Remark Let  $\varphi(\bar{x}) \in L_n(\mathcal{M})$ . Then  $\varphi(\bar{x})$  algebraic  $\Leftrightarrow 0 < |\varphi(\mathcal{M})| < \aleph_0$

Proof  $\mathcal{M} \models \mathcal{M}, |\varphi(\mathcal{M})| = k \Leftrightarrow \mathcal{M} \models (\exists!^k \bar{x}) \varphi(\bar{x})$   
 $\Leftrightarrow \mathcal{M} \models (\exists!^k \bar{x}) \varphi(\bar{x}) \Leftrightarrow |\varphi(\mathcal{M})| = k$

Remark Let  $A \subseteq \mathcal{M}$ , then:  $\text{tp}(ab/A)$  is algebraic  
 $a, b \in \mathcal{M}$

$\Leftrightarrow \text{tp}(a/A)$  is algebraic and  $\text{tp}(b/Aa)$  is algebraic

Pf. ( $\Rightarrow$ )  $p(x, y) = \text{tp}(ab/A)$ . Let  $q(x) = p \upharpoonright_x$   
 $= \text{tp}(a/A)$ . Let  $f: \mathcal{M}^2 \rightarrow \mathcal{M}$  projection

to the first coord. Then  $f: p(\mathcal{M}) \rightarrow q(\mathcal{M})$

Why?

Take  $a' \in q$ , choose  $g \in \text{Aut}(\mathcal{M}/A), g(a) = a'$ ,

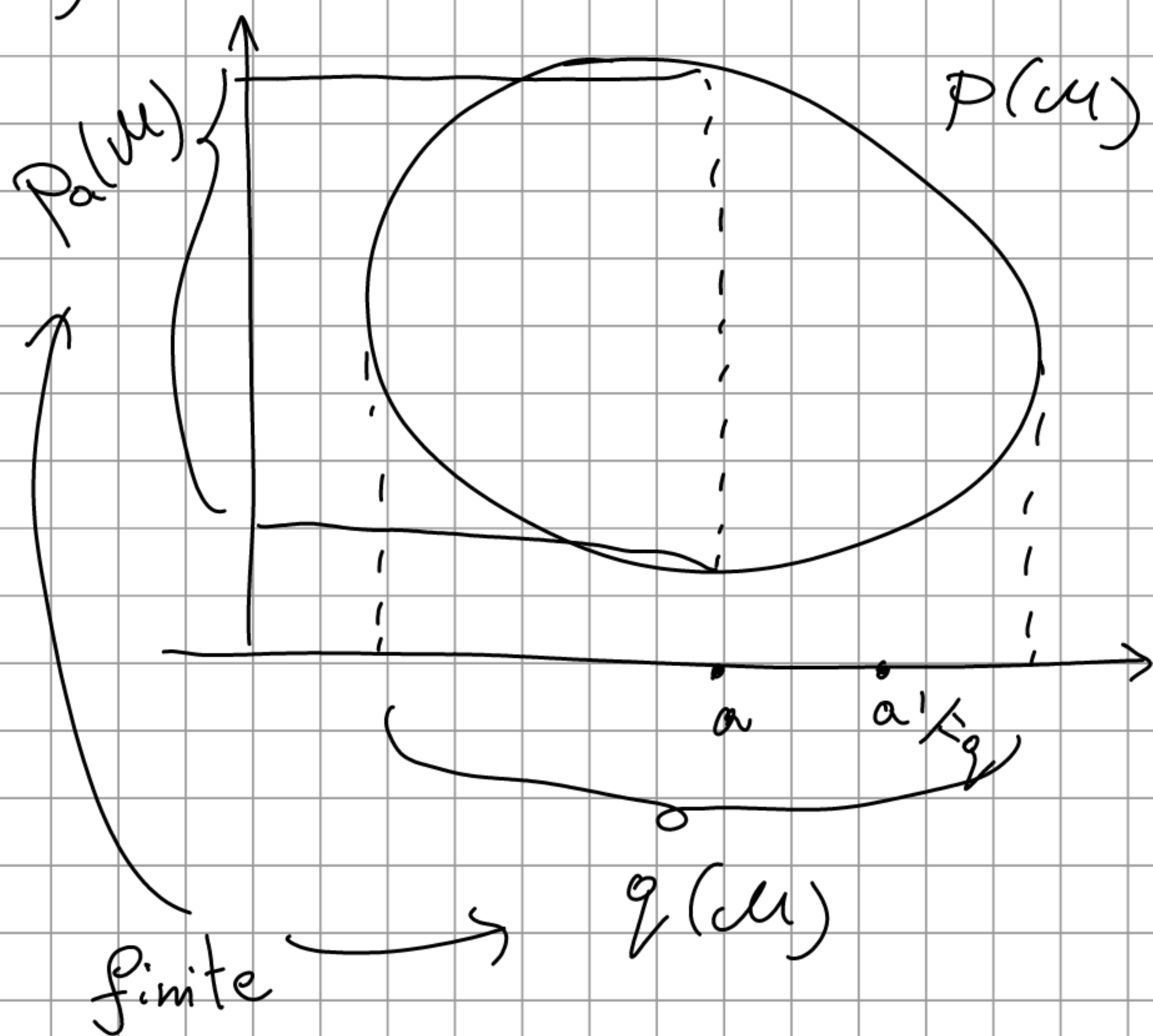
then  $b' = g(b) \Rightarrow \text{tp}(ab/A) = \text{tp}(a'b'/A)$ .

Then  $|p(\mathcal{U})| < \aleph_0 \Rightarrow |p_q(\mathcal{U})| < \aleph_0$  and  $(p_a(\mathcal{U}))$

Let  $p_a(y) = \text{tp}(b/Aa) = \{ \varphi(a, y) : \varphi(x, y) \in P \}$

Then  $p_a(\mathcal{U}) = p(\mathcal{U})_a \leftarrow \text{"verticalization"}$   
 $\uparrow$   
 this is finite

( $\Leftarrow$ )



$p(\mathcal{U})_{a'} = g(p(\mathcal{U})_a)$  for any  $g \in \text{Aut}(\mathcal{U}/A)$   
 with  $g(a) = a'$ .

$p(\mathcal{U}) = \bigcup_{a' \in G \cdot a} (\{a'\} \times p_{a'}(\mathcal{U})) : \text{finite} \quad \square$

Remark (1) <sup>(i)</sup>  $A \subseteq \text{acl}(A)$ , <sup>(ii)</sup>  $\text{acl}(\text{acl}(A)) = \text{acl}(A)$ ,  
<sup>(iii)</sup>  $A \subseteq B \Rightarrow$  <sup>(iv)</sup>  $\text{acl}(A) \subseteq \text{acl}(B)$  (a closure operator)  
 $\text{acl}(A) = \bigcup_{\substack{A_0 \subseteq A \\ \text{finite}}} \text{acl}(A_0)$  (finite character)

(2) The same for  $\text{dcl}$

Pf. 1 •  $a \in \text{acl}(A) \Leftrightarrow a \in \varphi(\mathcal{U})$ ,  $\varphi(x) \in L_1(A)$ ,  
 so  $\varphi \in L_1(A_0)$  for some  $\substack{A_0 \subseteq A \\ \text{finite}}$ , so (iv).

(ii):  $a \in \text{acl}(\text{acl}(A)) \Rightarrow a \in \text{acl}(A \cup \bar{b})$ ,

for some  $\bar{b} \subseteq \text{acl}(A) \Rightarrow \text{tp}(\bar{b}/A)$  algebraic

and  $\text{tp}(a/\bar{b})$  algebraic  $\stackrel{\text{remark}}{\Rightarrow} \text{tp}(a\bar{b}/A)$  is algebraic

$\Rightarrow \text{tp}(a/A)$  algebraic  $\Rightarrow$

Example  $T = \text{ACF}_p$ ,  $A \subseteq \mathcal{U} \models T$ .  $\text{acl}(A) =$

the algebraic closure (in  $\mathcal{U}$ ) of the field generated by  $A$ .



# Measuring definable sets and types.

The Cantor-Bendixson rank.

Def. Let  $X$ : compact  $T_2$  space.

$$\underbrace{X'}_{\text{CB-derivative}} = X \setminus \underbrace{\{\text{isolated points}\}}_{\text{open in } X} \Rightarrow X' \subseteq X \text{ closed}$$

Iteration:  $X^{(\alpha+1)} = (X^{(\alpha)})'$ ,  $X^{(\delta)} = \bigcap_{\alpha < \delta} X^{(\alpha)}$  where  $\delta \in \text{Lim}$ ,  
 $X^{(\infty)} = \bigcap_{\alpha \in \text{Ord}} X^{(\alpha)} = X^{(\beta)}$  for some  $\beta < |w(X)|^+$   
↑ the perfect core of  $X$       ↑ minimal cardinality of basis of  $\mathcal{C}X$

Def CB:  $X \rightarrow \text{Ord} \cup \{\infty\}$

$$\text{CB-rank } p = \begin{cases} \min \{\alpha \in \text{Ord} : p \notin X^{(\alpha+1)}\} & \text{if } p \notin X^{(\infty)} \\ \infty & p \in X^{(\infty)} \end{cases}$$

Now:  $X = S(A)$ : 0-dimensional (extremely disconnected).

Let:  $\text{Clopen}(X) = \{V \subseteq X : V \text{ clopen}\}$ .



Def.  $CB: \text{Clopen}(X) \rightarrow \text{Ord} \cup \{\infty\}$ : the smallest function (value-wise) s.t.:  $CB(V) \geq \alpha + 1 \Leftrightarrow \forall n < \omega \exists V_1, \dots, V_n \subseteq V$   $\overset{\text{clopen}}{\text{disjoint}}$   $CB(V_i) \geq \alpha$ . (also we define " $\geq \delta$ ")

Then  $CB(V) := \min \{ \alpha : \neg CB(V) \geq \alpha + 1 \}$

Properties Let  $U, V \subseteq X$  clopen.

(0)  $CB(U) = -1 \Leftrightarrow U = \emptyset$

(1)  $CB(U) = 0 \Leftrightarrow 0 < |U| < \aleph_0$

(2)  $U \subseteq V \Leftrightarrow CB(U) \leq CB(V)$

(3)  $CB(U \cup V) = \max \{ CB(U), CB(V) \}$

} Very easy

Pf. 3 Obv.  $CB(U \cup V) \geq \max \{ CB(U), CB(V) \}$ .

Now assume  $CB(U \cup V) \geq \alpha \Rightarrow \max \{ CB(U), CB(V) \} \geq \alpha$ .

Pf by ind on  $\alpha$ . Base and limit easy.

Successor step  $\alpha \rightarrow \alpha + 1$ . Assume  $CB(U \cup V) \geq \alpha + 1$ .

So  $\forall n \exists W_1, \dots, W_n \subseteq U \cup V$   $\overset{\text{clopen}}{\text{disjoint}}$   $\bigwedge_{i=1}^n CB(W_i) \geq \alpha$ .

By ind hyp.:  $\max\{CB(W_i \cap U), CB(W_i \cap V)\} \geq \alpha$ .

So  $\forall n \left( \exists W_1, \dots, W_n \subseteq U \bigwedge_{i=1}^n CB(W_i) \geq \alpha \right)$   
 (or the same for  $V$ )

$\Downarrow$

$CB(U) \geq \alpha+1$  or  $CB(V) \geq \alpha+1$ .

(4)  $CB(V) \geq \alpha+1 \iff (\exists V_n \subseteq V, n < \omega) \bigwedge_n CB(V_n) \geq \alpha$   
clopen disjoint

(5) For  $p \in X$   $CB(p) = \min\{CB(U) : p \in U \subseteq X\}$   
clopen

### Notation

In model theory:  $X = S(A)$ .

$CB(a/A) := CB(\text{tp}(a/A))$

$CB(a/A) = 0 \iff \text{tp}(a/A)$  is isolated

$p \in S(A) \rightsquigarrow CB(p) = CB_A(p)$

$\varphi \in L(A) \rightsquigarrow [\varphi] \subseteq S(A) \rightsquigarrow CB_A(\varphi)$

$\parallel$   
 $CB_A([\varphi] \cap S(A))$

Morley rank "CB on  $L(\mathcal{M})$ , in  $S(\mathcal{M})$ "

Def. RM:  $L(\mathcal{M}) \rightarrow \{-1\} \cup \text{Ord} \cup \{\infty\}$ :

the minimal function s.t. For  $U \subseteq \mathcal{M}^n$   
definable  
[  $\varphi(\bar{x}) \in L_n(\mathcal{M})$  identified with  $U = \varphi(\mathcal{M}) \subseteq \mathcal{M}^n$  ]

$$(1) U = \emptyset \Rightarrow \text{RM}(U) = -1$$

$$(2) \text{ If } U \neq \emptyset, \text{ then } \text{RM}(U) \geq \alpha + 1$$

$$\Leftrightarrow \forall n < \omega \exists V_1, \dots, V_n \subseteq U \bigwedge_{i \leq n} \text{RM}(V_i) \geq \alpha$$

def. disjoint

Def. For  $\varphi(\bar{x}) \in L(\mathcal{M})$ ,  $\text{RM}(\varphi) = \text{RM}(\varphi(\mathcal{M}))$

• For a type  $p(\bar{x})$  over  $\mathcal{M}$  (not necessarily complete)

$$\text{RM}(p) = \min \{ \text{RM}(\varphi) : p \vdash \varphi \}$$

$$= \min \{ \text{RM}(U) : p(U) \subseteq U \}$$

Remark (1)  $\text{RM}(\varphi) = \text{CB}_{\mathcal{M}}(\varphi)$

(2) For  $p \in S(\mathcal{M})$   $\text{RM}(p) = \text{CB}_{\mathcal{M}}(p)$

Here  $\mathcal{M}$  may be replaced with any

$\aleph_0$ -saturated  $M \prec \mathcal{M}$ .

Properties (1)  $\varphi \vdash \psi \Rightarrow RM(\varphi) \leq RM(\psi)$

(2)  $\rho \vdash \varphi \Rightarrow RM(\rho) \leq RM(\varphi)$

(3)  $RM(\varphi \vee \psi) = \max\{RM(\varphi), RM(\psi)\}$

(4)  $RM(\varphi) \geq \alpha + 1 \iff \left( \exists \varphi_n \vdash \varphi, n < \omega \right) \wedge_n RM(\varphi_n) \geq \alpha$   
pairwise contradictory

(5)  $\exists \delta \in Lim$ , then  $RM(\varphi) \geq \delta \iff (\forall \alpha < \delta) RM(\varphi) \geq \alpha$

Thm  $T$  is  $\aleph_0$ -stable  $\iff RM("x=x") < \infty$ .

Pf. ( $\Rightarrow$ ) (a.a.)  $\exists f$   $t_p(\bar{a}) = t_p(\bar{b})$ , then

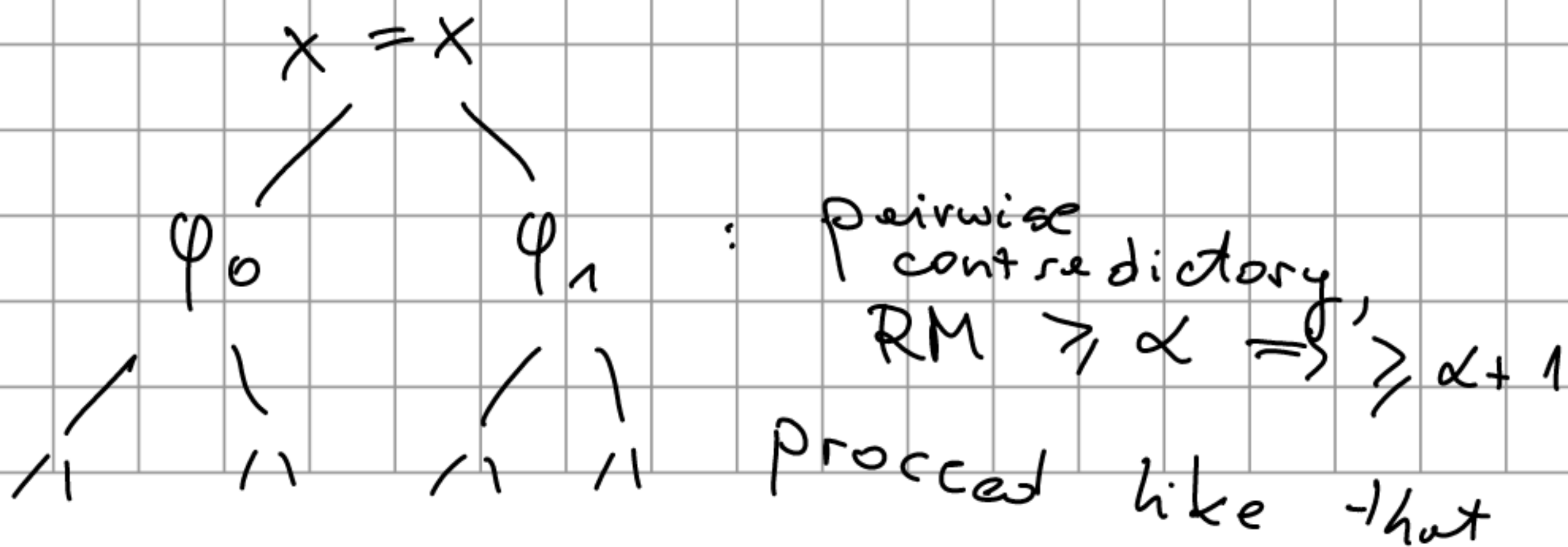
$$RM(\varphi(\bar{x}, \bar{a})) = RM(\varphi(\bar{x}, \bar{b}))$$

$$\exists f \in Aut(\mathcal{M}) \quad f(\bar{a}) = \bar{b} \Rightarrow f(\varphi(\mathcal{M}, \bar{a})) = \varphi(\mathcal{M}, \bar{b})$$

So  $|Rng(RM)| \leq 2^{|\mathcal{T}|} \Rightarrow \exists \alpha \in Ord \forall \varphi [RM(\varphi) \geq \alpha$

$\Rightarrow RM(\varphi) = \infty$ ].

Suppose  $RM("x=x") = \infty \Rightarrow \geq \alpha + 1$



We get  $2^{n_0}$  many types over table set  $A$   
 $\Rightarrow T$  is not  $n_0$ -stable.

11.04.2022

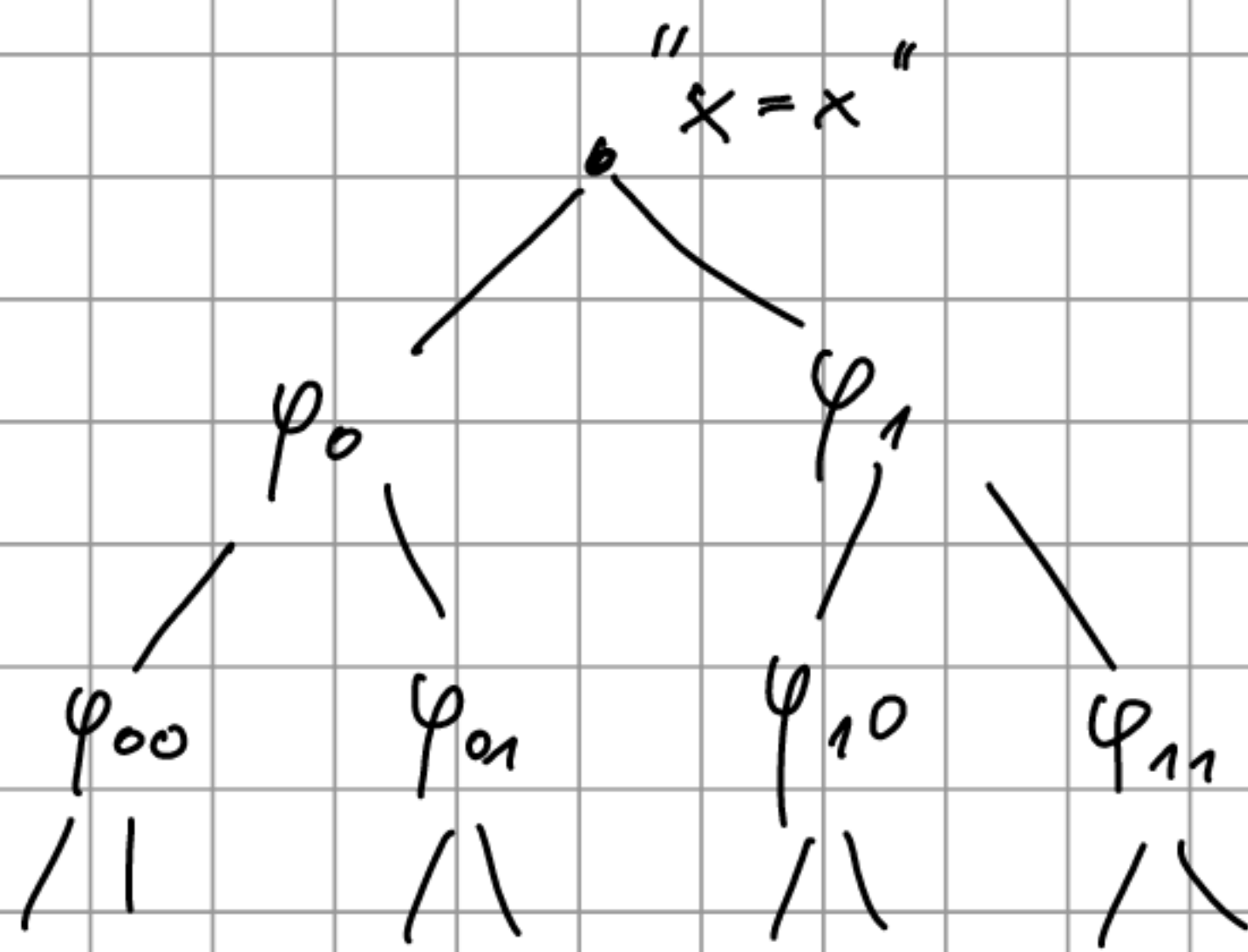
Thm  $T$  is  $\aleph_0$ -stable  $\iff$   $RM(x=x) \in Ord$

Pf. " $\implies$ " last time. " $\impliedby$ " Suppose  $T$  is not  $\aleph_0$ -stable.

$|S(A)| = 2^{\aleph_0}$ ,  $S(A)$ : a Polish space

$\implies S(A)^{(\omega)} \neq \emptyset \implies$  get a binary tree of formulas

$\{ \varphi_\eta(x) : \eta \in 2^{<\omega} \}$



Let  $\alpha = \min \{ RM(\varphi_\eta) : \eta \in 2^{<\omega} \}$ .

If  $\eta \subseteq \nu \in 2^{<\omega}$ , then  $RM(\varphi_\nu) = \alpha$ .

$RM(\varphi_\eta) \not\geq \alpha + 1$ , so:

(\*)  $\exists m \in \mathbb{N} \rightarrow \exists \psi_1 \dots \psi_m \vdash \varphi_\eta \wedge_{i=1}^m RM(\psi_i) \geq \alpha$

Take  $n$  s.t.  $2^n > m$ .  $\varphi_\nu$ ,  $\eta \subseteq \nu \in 2^{<\omega}$ ,  $|\nu| = |\eta| + n$   
contradicts (\*). ~~is~~

Def. Multiplicity of  $\varphi(\bar{x}) \in L(\mathcal{M})$  s.t.  $\text{RM}(\varphi(\bar{x})) < \infty$ :

$\text{Mlt}(\varphi) =$  the largest  $m \in \mathbb{N}$  s.t.

$$\exists \psi_1 \dots \psi_m \bigwedge_{i=1}^m \text{RM}(\psi_i) \geq d$$

Properties •  $\text{RM}(\varphi_1) = \text{RM}(\varphi_2)$  and  $\varphi_1(\mathcal{M}) \cap \varphi_2(\mathcal{M}) = \emptyset$ ,

then  $\text{Mlt}(\varphi_1 \vee \varphi_2) = \text{Mlt}(\varphi_1) + \text{Mlt}(\varphi_2)$

• If  $\text{RM}(\varphi_1) < \text{RM}(\varphi_2) < \infty$ , then  $\text{Mlt}(\varphi_1 \vee \varphi_2) = \text{Mlt}(\varphi_2)$

Example If  $\varphi(\bar{x})$  is algebraic, then  $\text{Mlt}(\varphi) = |\varphi(\mathcal{M})|$

$$\text{RM}(\varphi) = 0$$

Def. Assume  $p(\bar{x})$ : a type with  $\text{RM}(p(\bar{x})) < \infty$ .

$$\text{Mlt}(p(\bar{x})) = \min \{ \text{Mlt}(\varphi(\bar{x})) : p \vdash \varphi \text{ and } \text{RM}(\varphi) = \text{RM}(p) \}$$

Def.  $p(\bar{x})$  is stationary, if  $\text{Mlt}(p(\bar{x})) = 1$

Remark Assume  $p(\bar{x})$ : a type over  $A$ . Then  $\exists q \in S(A)$ ,

$$p(\bar{x}) \subseteq q(\bar{x}) \text{ s.t. } \text{RM}(p) = \text{RM}(q)$$

Pf. Let  $q_0 = \{ \varphi(\bar{x}) \in L(A) : \text{RM}(p \cup \{ \neg \varphi \}) < \text{RM}(p) \}$

•  $q_0 \supseteq p$ , if  $\varphi \in p$ , then  $RM(p \cup \{\neg\varphi\})$   
 $= -1 < RM(p)$

consistent

•  $q_0$ : a type:  $\varphi_1, \dots, \varphi_n \in q_0$ ,  $RM(p \cup \{\neg\varphi_i\}) < \alpha$   $RM(p)$

Choose  $\psi$  with  $p \vdash \psi$  and  $RM(\psi) = \alpha$ ,

$\psi_i$  with  $p \vdash \psi_i$  and  $RM(\psi_i \wedge \neg\varphi) < \alpha$ .

Wlog  $\psi = \bigwedge_{i=1}^n \psi_i$

⋮

Let  $q_0 \subseteq q \in S(A)$ , then  $RM(q) = \alpha$ . Clearly,  
 $RM(q) \leq \alpha$ . If  $RM(q) < \alpha \Rightarrow q \vdash \varphi$  with

$RM(\varphi) < \alpha$ . By compactness  $\exists$  finite  $q' \subseteq q$ ,  $q' \vdash \varphi$

$\Leftrightarrow \bigwedge q' \vdash \varphi$ ,  $\chi(\bar{x}) \vdash \varphi(\bar{x})$  so  $RM(\chi(\bar{x})) < \alpha$ ,  
 $\underbrace{\chi(\bar{x}) \in q(\bar{x})}$

then  $RM(p \cup \{\chi(\bar{x})\}) < \alpha$

$\downarrow$   
 $\neg \chi(\bar{x}) \in q_0(\bar{x}) \subseteq q(\bar{x})$   
 $\chi(\bar{x}) \in q(\bar{x})$

$\downarrow$



Example  $\overbrace{RM(p)=0}^{(\Rightarrow) p: \text{algebraic}}$ , then  $Mlt(p) = |p(\mathcal{M})|$

Example  $T = ACF_p$ ,  $K \subseteq \mathcal{M}$ ,  $\varphi \in S_n(K)$ ,  $p = t_p(\bar{a}/K)$   
subfield

a)  $p$  algebraic  $\Leftrightarrow RM(p)=0$ ,

$(W) = \overline{I(\bar{a}/K)} \neq \{0\}$ , " $W(x)=0$ "  $\in p(x)$ ,

$Mlt(p) = Mlt(W(x)=0) = \# \{ \text{roots of } W \text{ in } \mathcal{M} \}$ .

b)  $p$ : transcendental, then  $I(\bar{a}/K) = \{0\}$ ,

$RM(p) = 1 = RM(x=x)$

$Mlt(p) = Mlt(x=x) = 1$

So  $p$ : stationary.

c)  $p = t_p(\bar{a}/K)$ ,  $\bar{a} = \langle a_1, \dots, a_n \rangle \in \mathcal{M}$ . By q.e.

it is determined by  $I(\bar{a}/K) \triangleleft K[X_1, \dots, X_n]$

$\{ W(\bar{x}) \in K[\bar{x}] : W(\bar{a}) = 0 \}$ .

Let  $V_p = V(I(\bar{a}/K)) = \bigcap_{W \in I(\bar{a}/K)} Z(W) =$

$= \bigcap_{i=1}^l Z(W_i) \leftarrow \text{definable on } \mathcal{M} \text{ over } K, (W_1, \dots, W_l)$   
 $\underbrace{\quad}_{\{ \bar{b} \in \mathcal{M}^n : W(\bar{b}) = 0 \}}$

$\bar{a} \in V_p \Rightarrow p(\bar{x}) \vdash "x \in V_p"$

$$RM(p) = RM(V_p) = \dim V_p$$

$Mlt(p) = Mlt(V_p) =$  The number of irreducible components of  $V_p$  of  $\dim = \dim V_p$

$$\text{In } T = Th(ACF_p) : RM(\bar{x} = \bar{x}) < \omega$$


---

(Order) indiscernible sets:

Let  $(I, \leq)$ : a linear ordered set of indices.

Def.  $\{\bar{a}_i : i \in I\} \subseteq \mathcal{M}$ : order indiscernible over

$A \subseteq \mathcal{M}$ , if  $\forall k \forall i_1 < \dots < i_k \in I$   $tp(\bar{a}_{i_1} \dots \bar{a}_{i_k} / A) = tp(\bar{a}_{j_1} \dots \bar{a}_{j_k} / A)$   
 $j_1 < \dots < j_k \in I$

Recall: 1) Assume  $p(\bar{x})$ : a non-algebraic type over  $A \subseteq \mathcal{M}$ . Then  $\exists \{\bar{a}_n : n < \omega\} \subseteq p(\mathcal{M})$   
 infinite order indiscernible

2) (stretching) Assume  $\{a_i : i \in I\}$  order indiscernible /  $A$ ,

$I$ : infinite,  $(J, \leq)$ : a linear ordering. Then

$\exists \{b_j : j \in J\}$ : order ind. /  $A$  s.t.  $\forall k \forall i_1 < \dots < i_k \in I$   
 $\forall j_1 < \dots < j_k \in J$

$$tp(\bar{a}_{i_1} \dots \bar{a}_{i_k} / A) = tp(\bar{b}_{j_1} \dots \bar{b}_{j_k} / A)$$

Pf. (1) by Ramsey thm.

(2) Let  $b_j, j \in J$ : new constant symbols.

$$T^* = T(A) \cup \{ \varphi(b_{j_1}, \dots, b_{j_k}) : \varphi(\bar{x}) \in L_k(A), j_1 < \dots < j_k \in J \\ \text{and } \forall i_1 < \dots < i_k \in I \models \varphi(a_{i_1}, \dots, a_{i_k}) \}.$$

$T^*$ : consistent.  $b_{j_1}, \dots, b_{j_k} \xrightarrow{\text{interpret as}} a_{i_1}, \dots, a_{i_k}$

has a model  $M$

for any  $i_1 < \dots < i_k \in I$

$$M = \left( M, \underbrace{a^M}_{\prod T(A)}, b_j^M \right)_{\substack{a \in A \\ j \in J}} \models T^*$$

$$T(A) \Rightarrow \exists f: (M, a^M)_{a \in A} \xrightarrow{\cong} (M, a)_{a \in A} \\ a^M \xrightarrow{f} a$$

Let  $b_j = f(b_j^M)$ .  $\{ b_j : j \in J \} \subseteq M$  is good.

Example (1)  $M = (\mathbb{Q}, \leq) \leftarrow \mathbb{Q}$  is order indisc (indexed by itself)

(2)  $T = \text{ACF}_p, M \models T, \{ a_i : i \in I \} \subseteq M$ , alg-indepen. over  $K \subseteq M$  subfields, then it is indisc /  $K$  in  $M$

$$\begin{aligned} & \updownarrow \\ & I(a_{i_1}, \dots, a_{i_k} / K) = \neq 0 \end{aligned}$$



25.04.22 Pf. c.d.  $\sigma = \tau_1 \circ \tau_2 \circ \dots \circ \tau_l =$  product of transpositions of consecutive numbers.

Let  $\sigma_t = \tau_1 \circ \dots \circ \tau_t$ ,  $t = 1, \dots, l$ ,  $\sigma_0 = \text{id}$ . Then

$\models \varphi(\bar{a}_{i_{\sigma_0}})$  and  $\models \varphi(\bar{a}_{i_{\sigma_l}})$ . For some

$0 \leq t < l$ :

$\models \varphi(\bar{a}_{i_{\sigma_t}}) \wedge \neg \varphi(\bar{a}_{i_{\sigma_{t+1}}})$

Let  $\sigma' = \sigma_t$ ,  $\sigma'' = \sigma_{t+1} = \sigma' \circ \tau_{t+1}$

Then  $\models \varphi(a_{i_{\sigma'(1)}}, \dots, a_{i_{\sigma'(k)}})$

$\psi(a_{i_1}, \dots, a_{i_k})$  (by renaming)

But  $\models \neg \varphi(a_{i_{\sigma'(j(1))}}, \dots, a_{i_{\sigma'(j(k))}})$ ,

so  $\models \neg \psi(a_{i_{\tau(1)}}, \dots, a_{i_{\tau(k)}}) \wedge \psi(a_{i_1}, \dots, a_{i_k})$

( $\forall i_1 < i_2 < \dots < i_k \in \bar{I}$ )

e.g.  $\tau = (3, 4)$  and  $k > 4$ . Choose  $i_1 < i_2 < i_5 < \dots < i_k$ .

Let  $\chi(x_3, x_4) = \psi(a_{i_1}, a_{i_2}, x_3, x_4, a_{i_5}, \dots, a_{i_k}) \in L(\bar{a}_j) \upharpoonright \bar{I}$

dense  $\bar{I}$

Will show:  $|S(\bar{a}_z)| > \kappa$ ,  $|\bar{a}_z| \leq \kappa$ .

Namely: let  $i < i' \in (i_2, i_5)_I$ . Then

$$\text{tp}(a_i / \bar{a}_z) \neq \text{tp}(a_{i'} / \bar{a}_z) \leftarrow \text{enough}$$

$$i_1 < i_2 < i < j < i' < i_5 < \dots < i_k.$$

$$\text{Then } \models \varphi(a_i, a_j), \models \neg \varphi(a_{i'}, a_j)$$

because  $i < j$

because  $i' > j$

$$\text{Then } \varphi(x, a_j) \in \text{tp}(a_i / \bar{a}_z),$$

$$\neg \varphi(x, a_j) \in \text{tp}(a_{i'} / \bar{a}_z). \quad \downarrow$$

Remark ( $T$ : stable,  $\varphi(x, \bar{y}) \in L$ ). There is

$\kappa < \omega$   $\forall I \subseteq \mathcal{M}$   $\forall \bar{a} \subseteq \mathcal{M}$  one of the  
indiscernible  
infinite

$$\text{sets } I_{\bar{a}}^+ = \{c \in I : \models \varphi(c, \bar{a})\}$$

$$I_{\bar{a}}^- = \{c \in I : \models \neg \varphi(c, \bar{a})\}$$

has  $\leq \kappa$  elements.

---

Pf. (a.c.) If there's no such  $n$ , then

$\forall n \exists I_n \exists \bar{a}_n \quad |I_{n, \bar{a}_n}^+|, |I_{n, \bar{a}_n}^-| > n$ . Let

$\{c_i, i < \omega\}$ : new constant symbols.

$\bar{d}$ :  $|\bar{d}| = |\bar{y}|$ .

Let  $T' = T \cup \underbrace{\{ \{c_i : i < \omega\} \text{ is indiscernible in } T \}}_I$

$\cup \{ \{ \bar{I}_{\bar{d}}^+ = \{c_{2i} : i < \omega\} \} \cup \{ \bar{I}_{\bar{d}}^- = \{c_{2i+1} : i < \omega\} \} \}$ .

wlog  $|I| = \aleph$

$T'$  is consistent, so it has a model  $M'$ .

wlog  $M' \upharpoonright_L \mathcal{M}$ . So  $I \subseteq \mathcal{M}$  indiscernible,

$I_{\bar{d}}^+, I_{\bar{d}}^- \subseteq \mathcal{M} \Rightarrow |S(I)| = 2^\aleph > \aleph$ .

Pf.

$|I_{\bar{d}}^+|, |I_{\bar{d}}^-| = \aleph$ . For any  $I' \subseteq I$  with

$|I'| = |I \setminus I'| = \aleph \quad \exists f \in \text{Aut}(\mathcal{M}) [f[I] = I,$

$f(I') = I_{\bar{d}}^+, f(I \setminus I') = I_{\bar{d}}^-]$ . Let  $\bar{a}_{I'} = f^{-1}(\bar{d})$ .

If  $I' \neq I''$ , then  $\text{tp}(\bar{a}_{I'} / I) \neq \text{tp}(\bar{a}_{I''} / I)$ .

# $\aleph_1$ -categorical theories

Examples. 1.  $ACF_p$ ,

2.  $Th(V, +, k)$   $k \in K \leftarrow$   $\begin{matrix} \text{ctble} \\ \text{field} \end{matrix}$   
 $\uparrow$   
inf. vec. space

3.  $Th(\mathbb{N}, S)$ ,  $Th(\mathbb{Z}, S)$ ,  $Th(\mathbb{N}, =)$

4.  $Th(G, +)$ ,  $G$ : torsion free divisible abelian  $\uparrow$  "no structure"

5.  $Th(\mathbb{Z}_p^{\aleph_0}, +)$

6.  $Th(G)$  where  $G$ : an algebraic group

Theorem If  $\kappa > \aleph_0$  and  $T$  is  $\kappa$ -categorical, then  $T$  is  $\aleph_0$ -stable.

Lemma  $\forall \kappa \geq \aleph_0 \exists M \models T, \|M\| = \kappa \forall A \subseteq M, |A| < \aleph_0$

$|\{p \in S(A) : p(M) \neq \emptyset\}| \leq \aleph_0$ .

Pf. thm (lemma  $\Rightarrow$  thm) (A.c.) Suppose

$T$  is not  $\aleph_0$ -stable.  $\exists N \models T$   $|S(N)| > \aleph_0$   
 $\uparrow$   
ctble

But  $T$ :  $\kappa$ -categorical.  $N \cong N_1$  s.t.  $\|N_1\| = \aleph_1$ ,

$|\{p \in S(N) : p(N_1) \neq \emptyset\}| = \aleph_1$ .



$N_1 \prec N_\kappa \leftarrow$  of power  $\kappa$ . Let  $M_\kappa$ : a model from the lemma. Then  $M_\kappa \cong N_\kappa$   $\checkmark$ .

Pf. (lemma)  $T \subseteq T^S$ : the skolemization in  $L^S \supseteq L$ .

Let  $I = \{a_n : n < \omega\}$ : an infinite order indisc.

set in  $T^S$ .  $I \subseteq J = \{a_\alpha : \alpha < \kappa\}$  (stretching).

Then  $J \subseteq N^S \models T^S$ . Let  $M^S = \mathcal{H}(J) \prec N^S$ , i.e.

$M^S = \{t^{N^S}(\vec{j}) : t(\vec{x}) : \text{a term in } L^S, \vec{j} \subseteq J\}$

Will show that  $M^S$  satisfies the conditions  $\uparrow$  on the

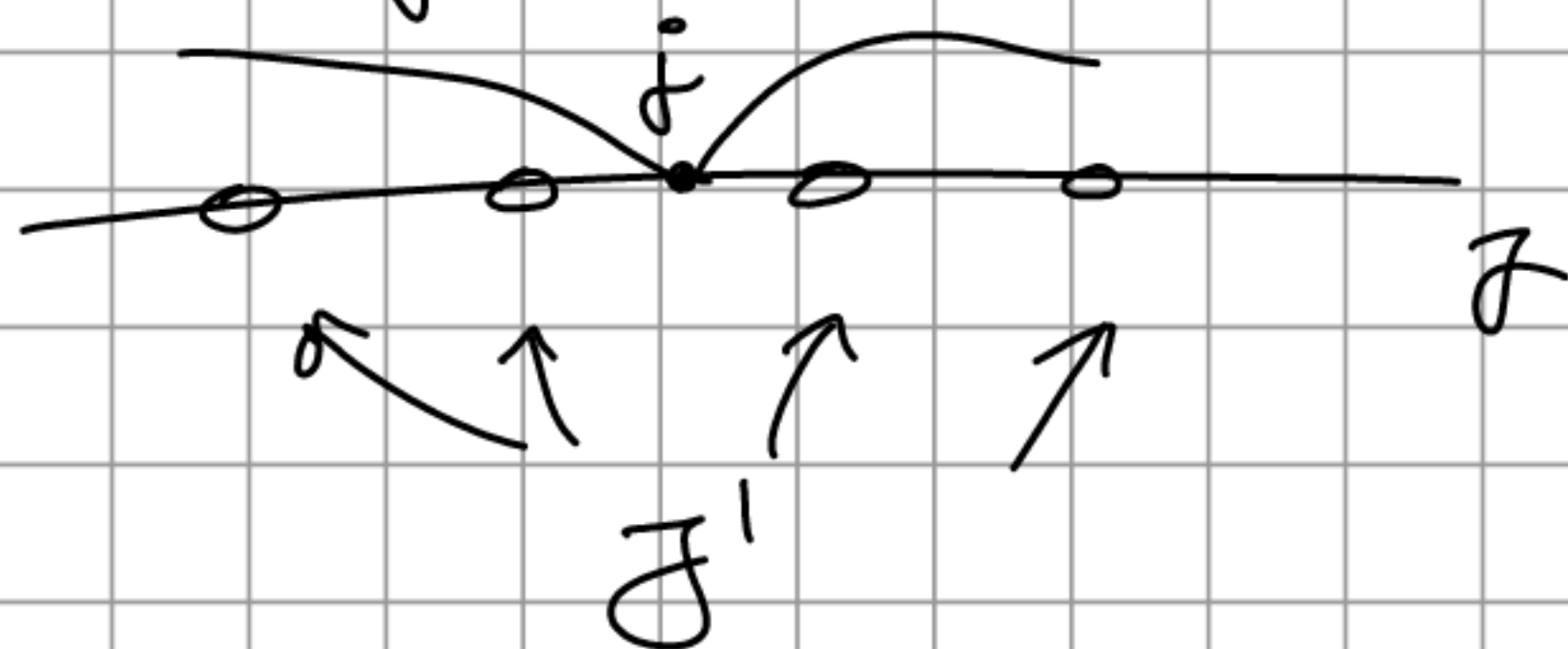
lemma for  $T^S$  (then  $M := M^S \upharpoonright L$  satisfies conditions for  $T$ )

Let  $A \subseteq \mathcal{H}(J)$ . Wlog  $A = \mathcal{H}(J')$  for some  $J' \subseteq J$ .

$M^S \ni a = t(j_1, \dots, j_k)$  for  $\overset{\text{some}}{j_1 < \dots < j_k} \in J$ .

$\sim$  eq. rel. on  $J$ :

$j \sim j'$   $\stackrel{\text{def.}}{\iff}$   $j$  and  $j'$  determine the same cut in  $J'$ .



$|J/\sim|$ : ctble as  $|J'|$  ctble. Now

$$(j_1, \dots, j_k) \sim (j_1', \dots, j_k') \stackrel{\text{def}}{\Leftrightarrow} \bigwedge_{1 \leq i \leq k} j_i \sim j_i'$$

again  
ctbly many  
classes

Let  $a, a' \in M^s$ . (\*) If  $t = t'$  and  $(j_1, \dots, j_k) \sim (j_1', \dots, j_k')$   
 $t(j_1, \dots, j_k) = t'(j_1', \dots, j_k')$  then  $\text{tp}^{M^s}(a/A) = \text{tp}^{M^s}(a'/A)$ .

(it obvs. implies the lemma).  $\exists j'$

Pf (\*):  $\varphi(x) \in L(A)$ ,  $b = \bar{t}(j'')$ .

$$\varphi(x, b), b \in A$$

$$\varphi(x, \bar{y}) \in L$$

$\bar{j} \sim \bar{j}' + \text{order indiscernibility of } \bar{j}$

$$\models \varphi(a, b) \Leftrightarrow \models \varphi(t(\bar{j}), \bar{t}(\bar{j}'')) \Leftrightarrow \models \varphi(t(\bar{j}'), \bar{t}(\bar{j}''))$$

$$\Leftrightarrow \models \varphi(a', b)$$



### Remark

Let  $T: \mathcal{M}_0$ -stable,  $p \in S(A)$ ,  $\text{RM}(p) < \infty$ ,  $\text{Mit}(p) = 1$ .

Then  $\forall B \supseteq A \exists! p_B \in S(B)$   $\text{RM}(p_B) = \text{RM}(p)$

Def.  $I = \{a_\alpha : \alpha < \beta\} \subseteq \mathcal{M}$  is a Morley sequence  
in  $\mathcal{P} \in S(A)$  if  $\forall \alpha < \beta \quad a_\alpha \overset{\mathcal{P}}{F} \mathcal{P}_{A_{a_\alpha}} \in S(A_{a_\alpha})$   
 $\mathcal{P} \in S(A)$   
 $\mathcal{P} \text{ : Mlt}(1)$

Remark A Morley sequence in  $\mathcal{P}$  is indiscernible over  $A$ .

Pf. Enough to show order-indiscernibility. Wlog  
 $I = \{a_\alpha : \alpha < \beta\}$ ,  $\beta = \kappa \geq \aleph_0$ . Induction on  $k < \omega$ :

$$\forall \alpha_1 < \dots < \alpha_k < \kappa \quad \text{tp}(a_{\alpha_1} \dots a_{\alpha_k} / A) = \text{tp}(a_{\beta_1} \dots a_{\beta_k} / A)$$

$$\beta_1 < \dots < \beta_k$$

Step  $k \rightarrow k+1$ ,  $\alpha_{k+1} > \alpha_k$ ,  $\beta_{k+1} > \beta_k$ .

$$\mathcal{P} \in \text{tp}(a_{\alpha_{k+1}} / A_{a_{\alpha_1} \dots a_{\alpha_k}}) \subseteq \mathcal{P}_{A_{a_{\alpha_{k+1}}}}$$

the same

RM, Mlt=1

$$\Rightarrow \text{tp}(a_{\alpha_{k+1}} / A_{a_{\alpha_1} \dots a_{\alpha_k}}) = \mathcal{P}_{A_{a_{\alpha_1} \dots a_{\alpha_k}}}$$

Likewise  $\text{tp}(a_{\beta_{k+1}} / A_{a_{\beta_1} \dots a_{\beta_k}}) = \mathcal{P}_{A_{a_{\beta_1} \dots a_{\beta_k}}}$

Consider  $f: A_{a_{\alpha_1} \dots a_{\alpha_k}} \rightarrow A_{a_{\beta_1} \dots a_{\beta_k}}$  s.t.

$f|_A = \text{id}$ ,  $f(a_{\alpha_i}) = a_{\beta_i}$ ,  $i=1, \dots, k$ . Then  $f$  is elementary.

and  $f(\underbrace{p_{A_{a_{\alpha_1} \dots a_{\alpha_k}}}}_p) = p_{A_{a_{\beta_1} \dots a_{\beta_k}}}$

and has the same RM as  $\text{RM}(p)$

$f \cup \langle a_{\alpha_{k+1}}, a_{\beta_{k+1}} \rangle$  elementary

So  $\text{tp}(a_{\alpha_1} \dots a_{\alpha_{k+1}} / A) = \text{tp}(a_{\beta_1} \dots a_{\beta_{k+1}} / A)$ .  $\square$

Thm (Morley, Shelah) If  $T: \aleph_0$ -stable and  $\kappa \geq \aleph_0$  then  $T$  has a saturated model of power  $\kappa$ .

9.05.2021 Thm (Morley, Shelah) T:  $\aleph_0$ -stable  $\Rightarrow$

T has a saturated model of power  $\kappa$

Pf.  $M = \bigcup_{\alpha < \kappa} M_\alpha \leftarrow$  elementary chain of models of T of power  $\kappa$   
 $\|M_\alpha\| = \kappa$

•  $M_\delta = \bigcup_{\alpha < \delta} M_\alpha$  when  $\delta \in \text{Lim}$

•  $\alpha \rightarrow \alpha+1 : M_{\alpha+1} \succ M_\alpha$  s.t.

(a)  $\forall A \subseteq M_\alpha$  finite  $\forall p \in S(A)$   $p(M_{\alpha+1}) \neq \emptyset$

(b)  $\forall A \subseteq M_\alpha$  finite  $\forall p \in S(A)$  stationary  $\exists I \subseteq M_{\alpha+1}$  ( $|I| = \kappa \wedge$

$I$  is a Morley sequence in  $p$ )

Claim M is saturated

• M is  $\aleph_0$ -saturated (easy)

• M is  $\kappa$ -saturated: (a.c.) let  $A \subseteq M$ ,

$|A| < \kappa$ ,  $p \in S(A)$ ,  $p(M) = \emptyset$ . Choose

A and p so that  $(RM(p), \text{Mlt}(p))$  is

lexicographically minimal. Then  $\text{Mlt}(p) = 1$ .

pf. let choose  $\varphi \in \mathcal{P}$  with  $\text{RM}(\mathcal{P}) = \text{RM}(\varphi)$ ,  
 $\text{Mlt}(\mathcal{P}) = \text{Mlt}(\varphi)$ .

- If  $\text{Mlt}(\varphi) > 1$ , then

(\*)  $\exists \psi(x) \in L(\mathcal{M})$   $\text{RM}(\varphi \wedge \psi) = \text{RM}(\varphi \wedge \neg \psi) = \text{RM}(\varphi)$

So  $\psi(x) = \psi(x, \bar{c})$ ,  $\varphi(x) = \varphi(x, \bar{a})$ . Choose

$\bar{c}' \subseteq M$  s.t.  $\text{tp}(\bar{c}'/\bar{a}) = \text{tp}(\bar{c}/\bar{a}) \in S_k(\bar{a})$ , ( $k = |\bar{c}'|$ )

(we can choose it by (a))

(\*) holds for  $\psi(x, \bar{c}')$  in place of  $\psi$

Let  $A' = A \cup \bar{c}' \subseteq M$ ,  $|A'| < \kappa$

$S(A)$

$S(A')$

$\subseteq$   
 $\mathcal{P}$



$\mathcal{P}_1$   
 $\mathcal{P}_2$   
 $\vdots$   
 $\mathcal{P}_l$

s.t.  $\text{RM}(\mathcal{P}_i) = \text{RM}(\mathcal{P})$

( $1 \leq \text{Mlt}(\varphi)$ )

Also (\*)  $\Rightarrow$   $\text{Mlt}(\varphi) = \text{Mlt}(\varphi \wedge \psi') + \text{Mlt}(\varphi \wedge \neg \psi')$

Look at  $\mathcal{P}_1$ : either  $\varphi \wedge \psi \in \mathcal{P}_1$  or  $\varphi \wedge \neg \psi \in \mathcal{P}_1$

$(\text{RM}(\mathcal{P}_1), \text{Mlt}(\mathcal{P}_1)) \prec_{\text{lex}} (\text{RM}(\mathcal{P}), \text{Mlt}(\mathcal{P}))$

$\mathcal{P}_1(M) \subseteq \mathcal{P}(M) = \emptyset \Rightarrow \mathcal{P}_1(M) = \emptyset \downarrow$

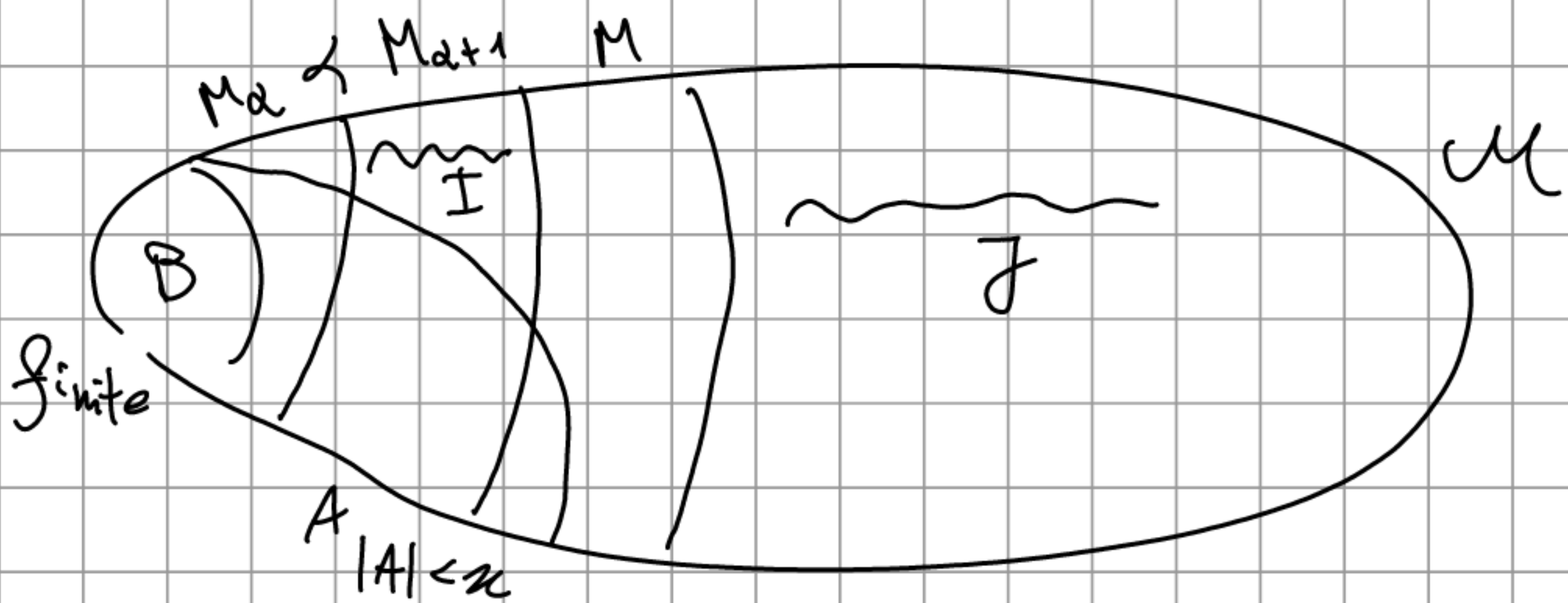
Therefore  $\text{Mlt}(\varphi) = 1$ . Choose a finite  $B \subseteq A$

s.t.  $\text{RM}(\varphi) = \text{RM}(\varphi|_B)$ ,  $\text{Mlt}(\varphi) = \text{Mlt}(\varphi|_B)$   
 (enough that  $\varphi \in L(B)$ )

By (b)  $\exists I \subseteq M$ : a Morley sequence in  $\mathcal{P}'$ ,

$|I| = \kappa$ . Let  $\mathcal{J} = \{a_\alpha : \alpha < \kappa\}$ : a  $M$ -sequence  
 in  $\mathcal{P}'_{AI} \in \mathcal{S}(AI)$

Then  $I \cup \mathcal{J}$  is a Morley sequence in  $\mathcal{P}'_{AI}$ .



Let  $\chi(x) \in \mathcal{P} \subseteq \mathcal{P}'_{AI} \Rightarrow \mathcal{J} \subseteq \chi(\mathcal{U})$ ,  
 infinite

However  $I \cup \mathcal{J}$  indiscernible  $\Rightarrow \{i \in I : \models \chi(i)\}$  is  
 also cofinite in  $I$

$I \cup \mathcal{J} = (I \cup \mathcal{J})^+ \cup (I \cup \mathcal{J})^-$   
 finite

(by the lemma  
 from prev. lecture)

$$\left| \bigcup_{\chi \in \mathcal{P}} I_{\chi}^{-} \right| < \kappa \Rightarrow \left| \bigcap_{\chi \in \mathcal{P}} I_{\chi}^{+} \right| = \kappa \Rightarrow \left| \bigcap_{\chi \in \mathcal{P}} I_{\chi}^{+} \right| \neq \emptyset.$$

( $|\mathcal{P}| < \kappa$ ) Any  $c \in \bigcap_{\chi \in \mathcal{P}} I_{\chi}^{+}$  realises  $p$  in  $M$  ▀

So: if  $\kappa > \aleph_0$ ,  $T: \kappa$ -categorical

$\Downarrow \cup$

$T: \aleph_0$ -stable

$\Downarrow$

$\exists M \models T \quad \|M\| = \kappa$   
saturated

$S$ -isolation ( $S = \text{"set"}$ )

Def. (1)  $p \in S(A)$  is  $S$ -isolated if  $\exists B \subseteq A$   
finite

$p|_B \vdash p$ .

(2)  $M$  is  $S$ -atomic over  $A$  if

$\forall \bar{a} \subseteq M$  tp  $(\bar{a}/A)$  is  $S$ -isolated  
finite

(3)  $M$  is an  $S$ -model, if  $\forall A \subseteq M$   $\forall p \in S(A)$   
finite  $p(M) \neq \emptyset$   
 $\parallel$   
 $\aleph_0$ -saturated

(4)  $M$  is  $S$ -prime over  $A$  if  $M$  is  $S$ -model

and  $\forall N \supseteq A$   $\exists f: M \xrightarrow[A]{\cong} N$  ( $f|_A = \text{id}_A$ )  
 $\hat{M}$   $\aleph_0$ -saturated



Remark  $T$ :  $\aleph_0$ -stable,  $p \in S(B)$ ,  $B \subseteq A$   
finite

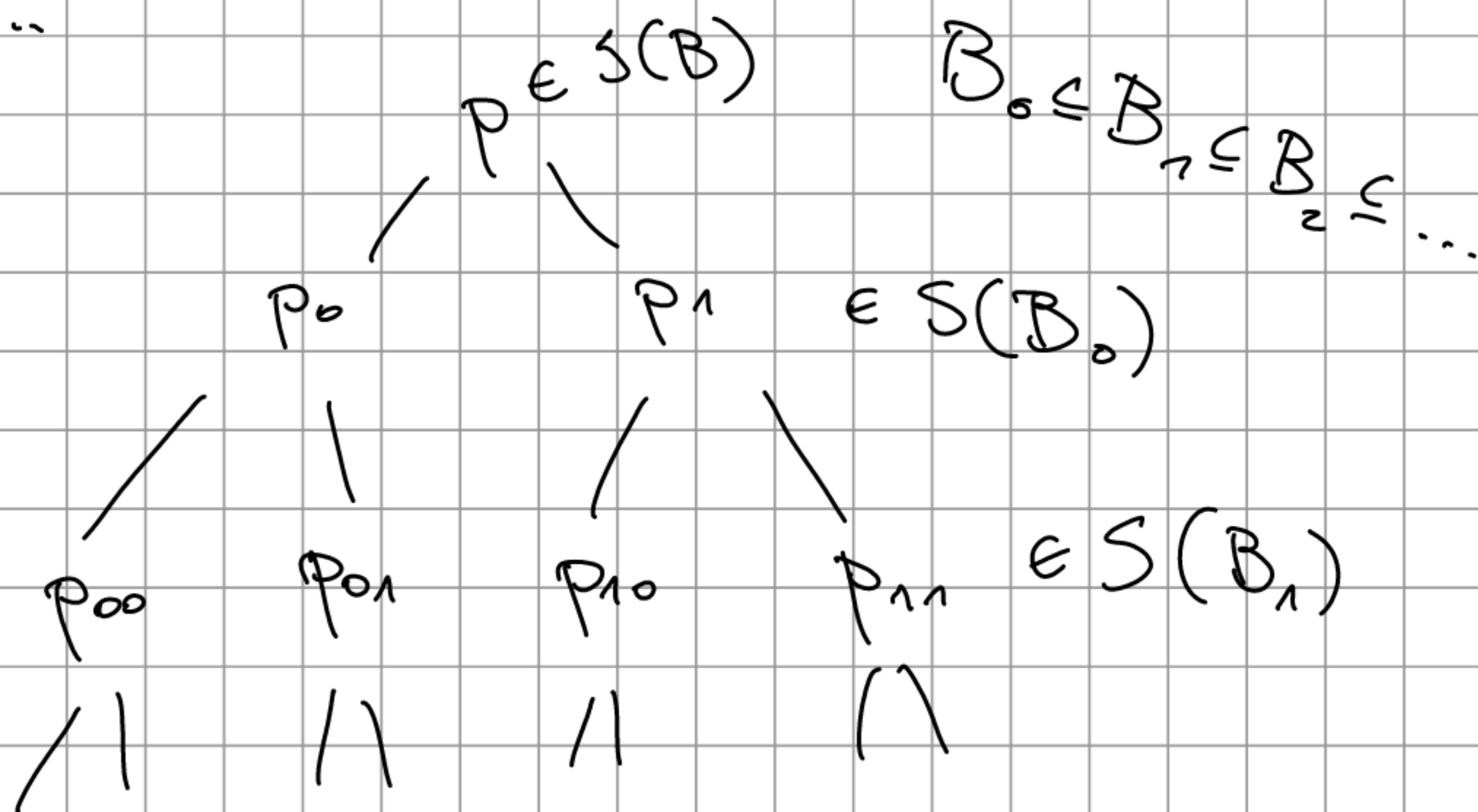
$\Rightarrow \exists q \in S(A)$   $q$ : s-isolated  
 $p \subseteq q$

Proof (A.c.) Suppose there's no such  $q$ .

(1)  $p \not\perp$  a type in  $S(A)$ . So there's  $B_0 \subseteq A$   
 $B_0 \supseteq B$   
 and  $p_0 \neq p_1 \in S(B_0)$  extending  $p$ .

(2)  $p_i \not\perp$  a type in  $S(A)$ . So: there is  $B_1 \subseteq A$   
 $B_1 \supseteq B_0$   
 and  $p_{00}, p_{01}, p_{10}, p_{11} \in S(B_1)$   
 pairwise distinct

(3) ....



Let  $B_\omega = \bigcup_n B_n \subseteq A$ : dble set, but  $|S(B_\omega)| \geq \aleph_0$ . ▣

Property of  $s$ -isolation:

$\text{tp}(ab/A)$   $s$ -isolated  $\Leftrightarrow$   $\text{tp}(a/A)$  is  $s$ -isolated and  $\text{tp}(b/Aa)$  is  $s$ -isolated.

(exercise)

Corollary  $T: \aleph_0$ -stable  $\Rightarrow \forall A \stackrel{\mu}{=} \exists M \supseteq A$   
 $\uparrow$   
 $s$ -prime over  $A$

Proof (sketch)  $M = A \cup \{a_\alpha : \alpha < \aleph_0\}$  s.t.:

(1)  $\text{tp}(a_\alpha/Aa_{<\alpha})$  is  $s$ -isolated

(2)  $M$  is  $\aleph_0$ -saturated

it terminates at some point

Then  $M$  is  $s$ -prime

The model  $M$  constructed this way is  $s$ -constructible/ $A$  and  $s$ -primary/ $A$ , it's unique up to  $\frac{\aleph_0}{A}$ .

Corollary  $T: \kappa_0$ -stable,  $M: s$ -prime  $/A$

$\Rightarrow M: s$ -atomic

Proof Let  $N: s$ -primary  $/A$

$\Downarrow$  property of  
 $s$ -isolation

$N: s$ -atomic

$M: s$ -prime  $/A \Rightarrow \exists f: M \xrightarrow[A]{=} N$

$\Downarrow$

$M: s$ -atomic

Def  $(M, N)$  is a Vaughtian pair for  $T$ , if

$N \not\equiv M \models T$  and for some  $\varphi(x) \in L_1(N)$

non-algebraic  
consistent

$\varphi(N) = \varphi(M)$

(here  $(M, N, \varphi)$ : Vaughtian triple)

Lemma 1 Assume  $T: \kappa_0$ -stable (enough  $T$ : small). If

$T$  has a Vaughtian pair, then there's  $V_p(M, N)$

s.t.  $M, N$ : cble and saturated.

Proof Let  $(M_0, N_0, \varphi)$ : Vaughtian triple.

$\varphi(x, \bar{a})$   
 $\uparrow$   
 $N_0$

Let  $L' = L \cup \{\bar{a}\} \cup \{P(x)\}$   
 new constant symbols      new predicate symbol

$T'$  complete theory in  $L'$  s.t.

(0)  $T' \supseteq \text{Th}(N_0, \bar{a}) = \text{Th}(M_0, \bar{a})$  and for

any  $M' \models T'$ :

(1)  $N := P(M') \upharpoonright_L \prec M := M' \upharpoonright_L$

(2)  $\bar{a}^{M'} \in P(M') : \bigwedge_i P(a_i) \in T'$

(3)  $(M, N, \varphi)$ : a V. triple:

$\varphi(x, \bar{a}^{M'})$

-  $[\forall x (\varphi(x) \rightarrow P(x))] \in T'$

-  $[\exists x \neg P(x)] \in T'$

Ad (1): Let  $\varphi(\bar{x}, y) \in L$ .

$T' \ni [\forall \bar{x} [\bigwedge_i P(x_i) \wedge \exists y \varphi(\bar{x}, y) \rightarrow \exists y (P(y) \wedge \varphi(\bar{x}, y))]]$

Fact  $\exists M' \models T'$  ( $M := M' \upharpoonright_L$  and  $N := P(M') \upharpoonright_L$ )

over both stable and saturated

Pf (fact)  $M' = \bigcup_{n < \omega} M'_n$ : models of  $T'$   
 elem. chain

•  $M'_0$ : arbitrary

•  $n \rightsquigarrow n+1$ :  $M'_{n+1} \supseteq M'_n$  such that:

(i)  $\forall A \subseteq M'_n$   $\forall p \in S^L(A)$   $p(M'_{n+1}) \neq \emptyset$   
finite

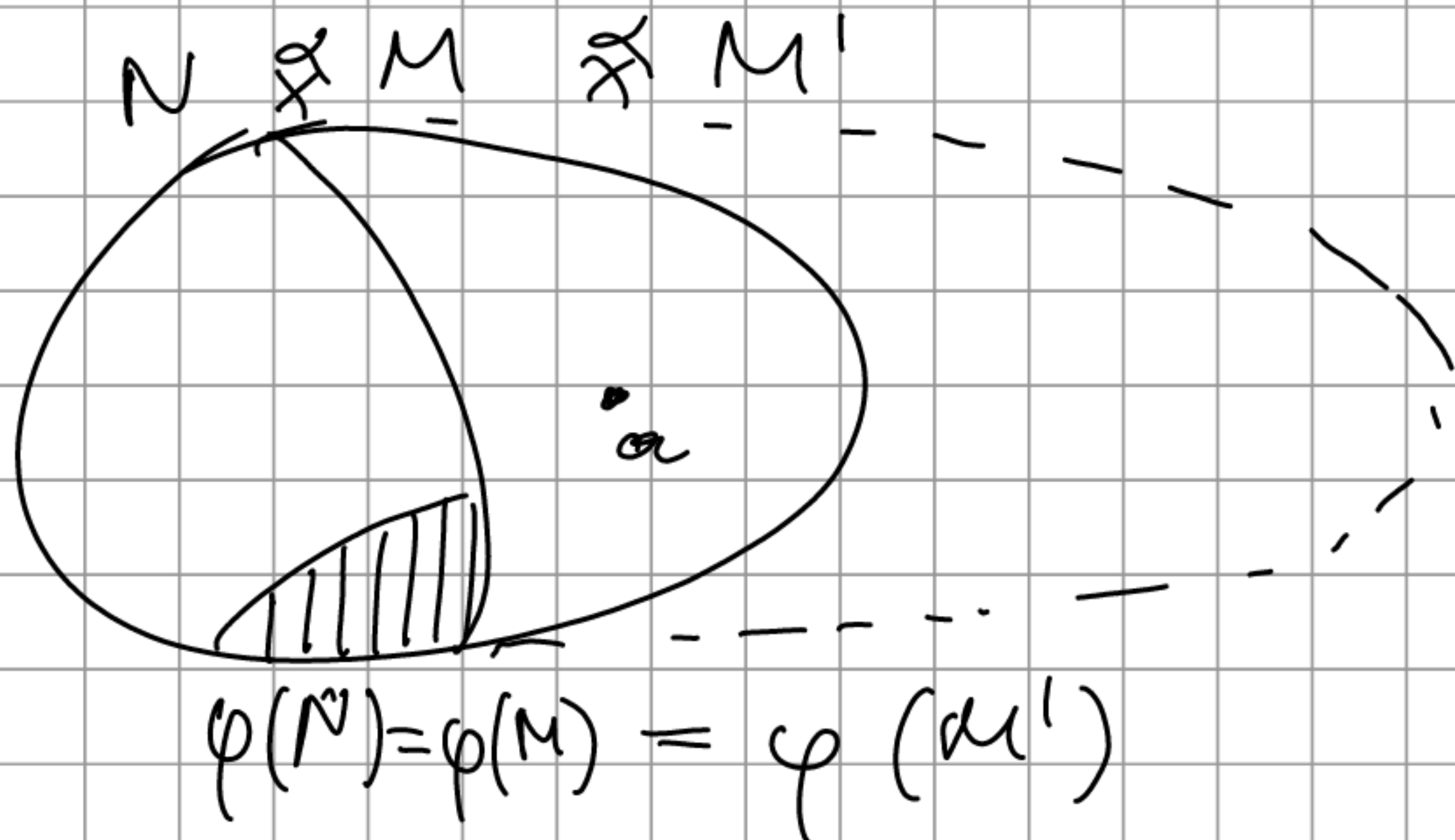
(ii)  $\forall A \subseteq P(M'_n)$   $\forall p \in S^L(A)$   $p(x) \cup \{p(x)\}$  is  
finite realised in  $M'_{n+1}$ .

Fact, Lemma 1 ~~is~~

### Lemma 2 (stretching Vaughtian pair)

Let  $(M, N, \varphi)$ : V. triple,  $M, N$ :  $\mathcal{N}_0$ -saturated,

$T$ :  $\mathcal{N}_0$ -stable. Then  $\exists M' \not\cong M$  ( $M', N, \varphi$ ) is a V. triple  
 $\uparrow$   
 $\mathcal{N}_0$ -saturated



Proof Let  $a \in M \setminus N$ ,  $p = \text{tp}(a/N)$ ,  $\text{RM}(p) = 1$ .

So  $p \subseteq q \in S(M)$ ,  $\text{RM}(q) = \text{RM}(p)$   
 $\uparrow$   
 unique

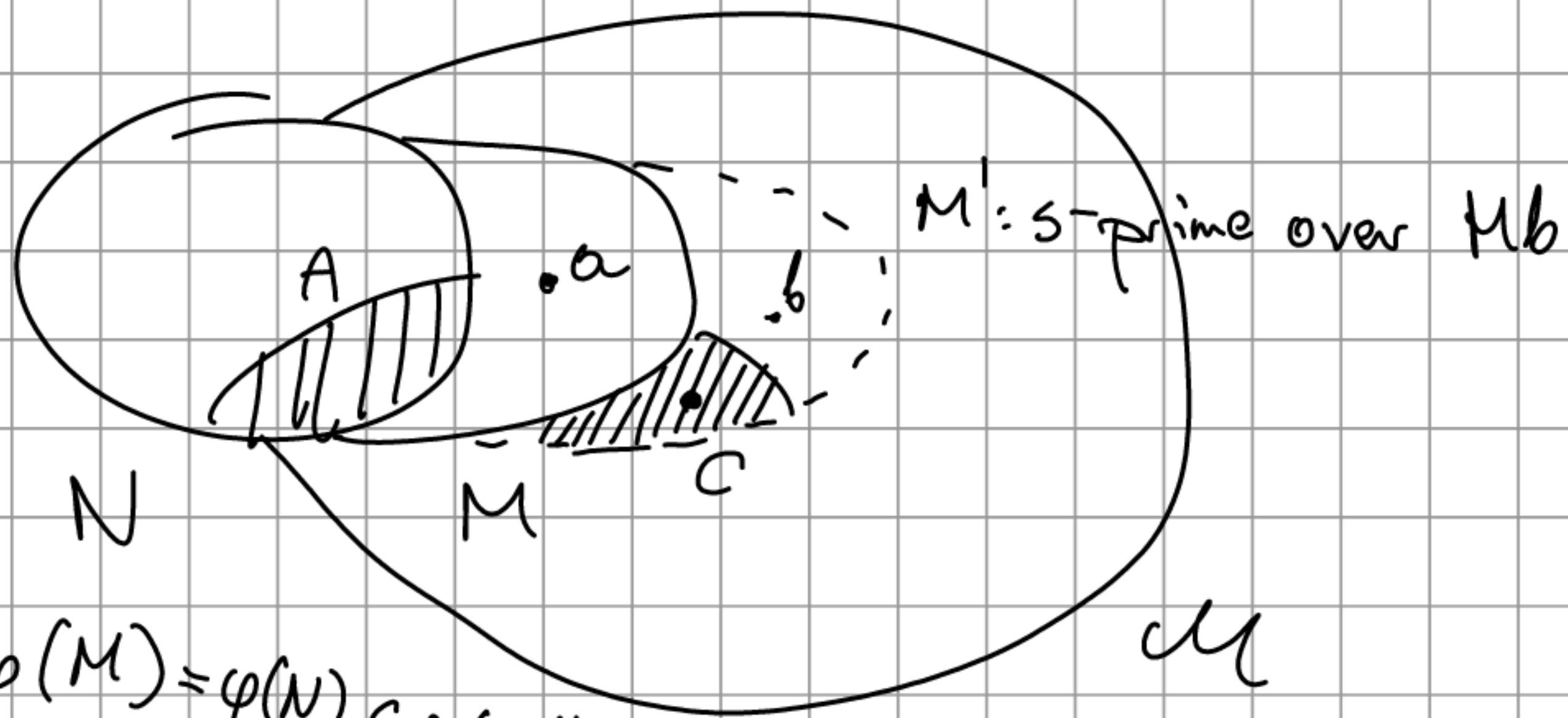
16.05.2022

Lemma 2 (Stretching Voughtian pair)

$T: \mathcal{A}_0$ -stable,  $(M, N, \varphi): \text{Voughtian triple}$

$\Rightarrow \exists M' \xrightarrow{\mathcal{A}_0\text{-saturated}} M$   $(M', N, \varphi): \text{Vt.}$

Proof



$A = \varphi(M) = \varphi(N) \subseteq \varphi(M')$

$tp(b/M) \geq tp(a/M), RM(b/A) = RM(b/M) = RM(a/N)$

Claim:  $\varphi(M) = \varphi(M')$

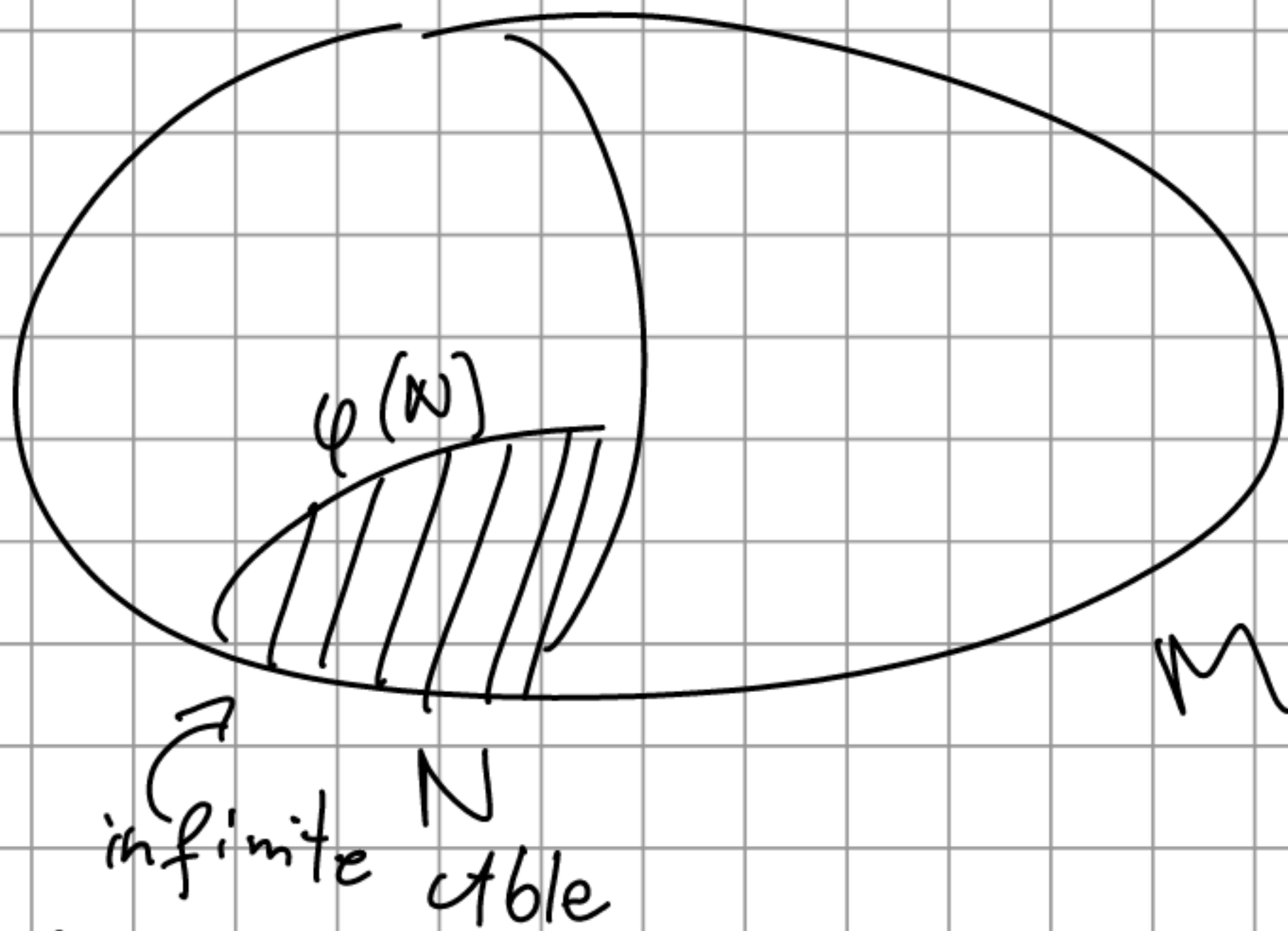
Corollary Suppose  $\kappa > \mathcal{A}_0$ ,  $T: \kappa$ -categorical. □

$T$  has no Voughtian pair and  $T: \mathcal{A}_0$ -stable.

Proof  $T: \mathcal{A}_0$ -stable: proved earlier.

No Voughtian pair: (A.c.) suppose  $(M, N, \varphi): \text{Vt.}$

By lemma 1 wlog  $M, N$  are  $\aleph_0$ -saturated,  $N$  cble  
 by lemma 2 wlog  $\|M\| = \kappa$ , so  $M$  is not  
 saturated



$$\varphi(x) = \{\varphi(x)\} \cup \{x \neq a\}_{a \in \varphi(N)} \Rightarrow \text{not realised in } M.$$

But  $T: \aleph_0$ -stable  $\rightsquigarrow \exists M' \|M'\| = \kappa$   
 saturated

$$M \not\cong M' \quad \Downarrow$$

Theorem If  $T: \aleph_0$ -stable without a Vaughtian pair.

Then  $\forall \kappa > \aleph_0$   $T: \kappa$ -categorical.

Corollary (Morley thm, 1964) Let  $\kappa > \aleph_0$ ,  $\mathcal{D}$

(1)  $T: \kappa$ -categorical

(2)  $T: \aleph_1$ -categorical

Lemma 3 ( $T: \mathcal{A}_0$ -stable with no Veightian pair).

$\exists$  a strongly minimal formula  $\varphi(x, \bar{c})$ ,  $\bar{c} \subseteq M \neq T$   
prime

$$RM(\varphi) = \text{Mlt}(\varphi) = 1$$

Proof Problem from list 6 ~~□~~

Lemma 4 ( $T: \mathcal{A}_0$ -stable, no v. pair) Assume

$M \neq T$ ,  $\varphi(x) = \varphi(x, \bar{a})$ , then:  
s.m.  $\uparrow$   $\downarrow$   $\uparrow$   $\downarrow$   
 $M$

(a)  $M$  is prime and minimal /  $\varphi(M) \cup \bar{a}$

(b) If  $\|M\| > \mathcal{A}_0$ , then  $\dim(\varphi(M) / \bar{a}) = \|M\|$

Explanation 1.  $A \subseteq M \not\subseteq \mathcal{U}$ ,  $M$ : minimal /  $A$

$$\Leftrightarrow \neg \exists M' \not\subseteq M$$

2.  $\text{acl}$ -dimension: Assume  $\mathcal{U} = \varphi(M)$ , where

$\varphi(x, \bar{a})$ : s.m.,  $\text{acl}_{\bar{a}}: \mathcal{P}(\mathcal{U}) \rightarrow \mathcal{P}(\mathcal{U})$ ,

$$\text{acl}_{\bar{a}}(A) = \text{acl}(A\bar{a}) \cap \mathcal{U}$$

3.  $(\mathcal{U}, \text{acl}_{\bar{a}})$  is a pregeometry, i.e.

$$\bullet \text{cl}(\text{cl}(A)) = \text{cl}(A) \supseteq A$$



$$\bullet A \subseteq B \Rightarrow \mathcal{d}(A) \subseteq \mathcal{d}(B)$$

$$\bullet \mathcal{d}(A) = \bigcup_{\substack{A_0 \subseteq A \\ \text{finite}}} \mathcal{d}(A_0)$$

$$\bullet a \in \mathcal{d}(Ab) \setminus \mathcal{d}(A) \Rightarrow b \in \mathcal{d}(Aa)$$

$\leadsto$  Basis  $B \subseteq \mathcal{U}$   $\mathcal{d}$ -independent set,  $\dim(\mathcal{U}) = |B|$ .

In the case of  $\text{ad}_{\bar{a}}$  on  $\varphi(\mathcal{U})$  additionally:

$$\text{Let } p(x) \in S(\bar{a}) \cap [\varphi(x)] \text{ i.e. } \text{RM}(p) = \text{MH}(p) = \underline{1}$$

s.m.

Then an  $\text{ad}_{\bar{a}}$ -independent set  $I \subseteq \varphi(\mathcal{U})$

is a Marley sequence in  $\mathcal{P} \Rightarrow$  indiscernible over  $\bar{a}$ .

Pf (lemma 4)

(a) Suppose  $M$  is <sup>prime and</sup> not minimal /  $\varphi(\mathcal{U}) \cup \bar{a}$ .

$$\text{So } \varphi(\mathcal{U}) \cup \bar{a} \subseteq N \not\subseteq M$$

$\uparrow$   
prime /  $\varphi(\mathcal{U}) \cup \bar{a}$

But then  $(M, N, \varphi) =$  a V. triple  $\downarrow$

(b) Let  $I \subseteq \varphi(\mathcal{U})$ . Then  $\bar{a} \cup \varphi(\mathcal{U}) \subseteq \text{ad}_{\bar{a}}(I) \cup \bar{a} \subseteq M$   
an  $\text{ad}_{\bar{a}}$ -basis of  $\varphi(\mathcal{U})$

$$\|M\| \stackrel{(\bar{a})}{=} |\bar{a} \cup \varphi(M)| \stackrel{\substack{\uparrow \\ \varphi(M) \\ \text{uncountable}}}{=} |I| = \dim(\varphi(M)/\bar{a})$$

Proof of Morley Thm  $\mathcal{M}_r$

Let  $\kappa > \aleph_0$ ,  $M, N \models T$  of power  $\kappa$ .

$M_0$ : prime

$\bar{a} \quad \varphi(x, \bar{a})$ : s.m. (lemma 3)

By lemma 4:  $M$ : prime & minimal /  $\varphi(M, \bar{a}) \cup \bar{a}$

$N$ : ——— || ——— /  $\varphi(N, \bar{a}) \cup \bar{a}$

Let  $I \subseteq \varphi(M)$  bases.  
 $J \subseteq \varphi(N)$

Let  $p \in S(\bar{a}) \cap [\varphi(x)]$ : a s.m. type

$I, J$ : Morley sequences in  $p$ . By L4b

$|I| = |J| = \kappa$ .  $f: I\bar{a} \rightarrow J\bar{a}$  s.t.

$f = g \upharpoonright I\bar{a} = \text{id}_{I\bar{a}}$

$f$  is elementary

$f: I \xrightarrow[\text{onto}]{1-1} J$

$\Downarrow$

$f: \text{ad}(I\bar{a}) \xrightarrow{\cong} \text{ad}(J\bar{a})$

$\cup$   
 $\varphi(M)\bar{a}$

$\cup$   
 $\varphi(N)\bar{a}$

$$f \upharpoonright_{\varphi(M)\bar{a}} : \varphi(M)\bar{a} \xrightarrow{\cong} \varphi(N)\bar{a}$$

$\cap$

$$f' : M \xrightarrow{\cong} N$$

(lemme 4a)



Thm (Baldwin, Lachlan, 1971) Assume  $T$ :  $\mathcal{L}$ -categorical

not  $\aleph_0$ -categorical. Then:

(1)  $T$  has  $\aleph_0$  many stable models

(2) Every model of  $T$  is homogeneous.

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$\mathcal{N}_1$ -categorical theories:

- $\mathcal{N}_0$ -stable
- have prime model
- no V. pairs
- s.m. sets
- M. sequences, "bases"

## Stable formulas

$T$ : fixed complete theory with infinite models

$L \ni \varphi = \varphi(\bar{x}, \bar{y})$   
stable

distinguished variables

parameters variables

Def Let  $\delta(x, y) \in L$ .

(1)  $\delta(x, c)$ ,  $c \subseteq \mathcal{M}$ : an instance of  $\delta$

(2)  $\varphi(x)$  is positive  $\delta$ -formula if

$\models \varphi \leftrightarrow$  [positive] boolean combination of instances of  $\delta$

(3)  $(A \subseteq \mathcal{M})$   $\delta$ -type = a type consisting of  $\delta$ -formulas,

$$L_\delta(A) = \{ \delta\text{-formulas over } A \}$$

(4)  $S_\delta(A) = \{ \text{complete } \delta\text{-types over } A \}$   
(ultrafilters in  $L_\delta(A)$ )

Remark Let  $\varphi \in L(A)$ ,  $A \subseteq M$ . Then

$\varphi$  is a  $\delta$ -formula  $\Leftrightarrow \models \varphi \Leftrightarrow$  boolean comb. of instances of  $\delta$  over  $M$

Pf  $(\Rightarrow)$   $\mathcal{M} \models \varphi(\bar{x}) \Leftrightarrow \bigwedge_i \bigvee_j \delta(x, c_{ij})$   $\varepsilon_{ij} \in S_{\delta, A}$   
 $\mathcal{M}$

$$\mathcal{M} \models (\exists y_{ij}) (\varphi(\bar{x}) \Leftrightarrow \bigwedge_i \bigvee_j \delta(x, y_{ij}))$$

$$M \models \text{---} \quad \parallel \quad \text{---}$$

So  $c_{ij}$  may be taken from  $M$ .

Example  $T$ : the theory of a single equivalence relation  $E$  with two infinite classes. Let  $\delta(x, y) = \bar{E}(x, y)$ .

$$\models x = x \leftrightarrow E(x, a) \vee E(x, b)$$

is a  $\delta$ -formula /  $\emptyset$ , where  $a, b \in \mathcal{M} \neq T, \models \neg E(a, b)$

Def (1)  $\delta(x, y)$  has order property, if  
 $\exists_{i < \omega} a_i, b_i \in \mathcal{M} \quad \forall i, j < \omega \models \delta(a_i, b_j) \Leftrightarrow i \leq j$

(2)  $\delta(x, y)$  is stable if it does not have the order property

Lemma 1 (1)  $\varphi(x, y), \psi(x, z)$ : stable

$\Rightarrow \neg \varphi(x, y), (\varphi \vee \psi)(x, yz), (\varphi \wedge \psi)(x, yz)$   
are stable

(2) Let  $\psi(y, x) = \varphi(x, y)$ , then  
 $\varphi$  stable  $\Leftrightarrow \psi$  stable

(3)  $\varphi$ : stable  $\Leftrightarrow \exists n < \omega \neg \exists a_i, b_i (i \leq n)$

$$\bigwedge_{i, j} \models \varphi(a_i, b_j) \Leftrightarrow i \leq j$$

Pf. exercise

Def Let  $p \in S(M)$  or  $p \in S_\delta(M)$ .

A  $\delta$ -definition of  $p$ : a formula  $\psi(y) \in L(M)$

s.t.  $\forall c \in M (\delta(x, c) \in p \Leftrightarrow \models \psi(c))$

(i.e.  $\{c \in M : \delta(x, c) \in p\} = \psi(M)$ )

Lemma 2 Assume  $\delta(x, y)$  is stable,  $p \in S(M)$

or  $p \in S_\delta(M)$ . Then:

(1)  $p(x)$  has a  $\delta$ -definition  $\psi(y)$  that

is a positive  $\delta^*$ -formula, where  $\delta^*(y, x) = \delta(x, y)$

(2)  $A \subseteq M$  and  $M$  is  $|A|^+$ -saturated, then

$\exists c_1, c_2, \dots \in M \quad c_i \models p \upharpoonright_{A \setminus c_i}$

and the  $\delta$ -definition of  $p$  is equivalent

to a positive boolean combination of

formulas  $\delta(c_i, y)$ ,  $i < \omega$ .

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Lemma 2 Assume  $\delta$  is stable,  $\varphi \in S(M)$  or  $\varphi \in S_{\sigma}(M)$ . Then

(1)  $\varphi(x)$  has a  $\delta$ -definition  $\psi(y)$ , that is a positive  $\delta$ -formula

(2) If  $A \subseteq M \leftarrow |A|^+$ -saturated, then  $\exists c_1, c_2, \dots \in M \bigwedge_i c_i \models \varphi \upharpoonright_{A \setminus \{c_i\}}$   
and  $\psi$  from (1) is a positive Boolean combination of formulas  $\delta(c_i, x), i < \omega$

Proof Let  $N < \omega$  s.t.

$$(i) \neg (\exists c_i, a_i, i \leq N) \left( \bigwedge_{i < j} \models \delta(c_i, a_j) \Leftrightarrow i < j \right)$$

$$(ii) \neg (\exists c_i, b_i, i \leq N) \left( \bigwedge_{i < j} \models \neg \delta(c_i, b_j) \Leftrightarrow i < j \right)$$

Let  $c^* \models \varphi$ . So:  $\delta(x, a) \in \varphi \Leftrightarrow \delta(c^*, a)$  for  $a \in M$ .

We shall define  $c_i \in M, A_i, B_i \subseteq M, i < \omega$  as follows:

•  $c_0 \in M$  arbitrary ( $c_0 \models \varphi \upharpoonright_A$ )

• Suppose we have  $c_0, \dots, c_n, A_0, \dots, A_n, B_0, \dots, B_n$ . We shall find  $c_{n+1}, A_{n+1}, B_{n+1}$ .

•  $A_{n+1} \subseteq M$  s.t.  $\forall W \subseteq \{0, \dots, n\}$  if  $(\exists a \in M) \left( \bigwedge_{j \in W} \models \delta(c_j, a) \text{ and } \models \neg \delta(c^*, a) \right)$

then  $\exists$  such an  $a_W \in A_{n+1}$

•  $B_{n+1} \subseteq M$  s.t.  $\forall W \subseteq \{0, \dots, n\}$  if  $(\exists b \in M) \left( \bigwedge_{j \in W} \models \neg \delta(c_j, b) \text{ and } \models \delta(c^*, b) \right)$

then  $\exists$  such an  $b_W \in B_{n+1}$ .

• for  $b \in B_{\leq n+1} \models \delta(c^*, b)$ , for  $a \in A_{\leq n+1} \models \neg \delta(c^*, a)$   
*we may assume*

•  $c_{n+1} \in M$  s.t.  $(\forall a \in A_{\leq n+1}) (\forall b \in B_{\leq n+1}) (\models \delta(c_{n+1}, b) \text{ and } \models \neg \delta(c_{n+1}, a))$

[for (2):  $c_{n+1} \models \varphi \upharpoonright_{A_{\leq n+1} \cup B_{\leq n+1}}$  is enough]

• So for  $a \in A_{\leq n}, b \in B_{\leq n} \models \neg \delta(c_n, a) \wedge \delta(c_n, b)$

• if  $0 \leq i_0 < \dots < i_n < \omega$  and  $\exists a \in M$

$$\models \delta(c_{i_0}, a) \wedge \delta(c_{i_1}, a) \wedge \dots \wedge \delta(c_{i_n}, a) \wedge \neg \delta(c^*, a)$$

then  $\exists d_0, \dots, d_n \in M (\forall 0 \leq j, r \leq n) (\models \delta(c_{i_j}, d_r) \Leftrightarrow j \leq r)$



pf Let  $W_r = \{i_0, \dots, i_{r-1}\} \subseteq \{i_0, \dots, i_n\}$ . Let  $d_r = a_{w_r} \in A_{(i_{r-1})+1}^2$  s.t.  
 $\models \delta(c_{i_0}, d_r) \wedge \dots \wedge \delta(c_{i_{r-1}}, d_r) \wedge \neg \delta(c^*, d_r)$ ,  $\models \neg \delta(c_j, d_r)$  for all  $j > i_{r-1}$

• Symmetrically: if  $0 \leq i_0 < \dots < i_n < \omega$  and  $\exists b \in M$

$$\models \neg \delta(c_{i_0}, b) \wedge \dots \wedge \neg \delta(c_{i_n}, b) \wedge \delta(c^*, b)$$

then  $\exists d_0, \dots, d_n \in M$   $\forall 0 \leq j, r \leq n$  ( $\models \neg \delta(c_{i_j}, d_r) \Leftrightarrow j < r$ )

• Therefore:

(a) If  $a \in M$  and  $\exists W \subseteq \{0, \dots, 2N\}$  s.t.  $\bigwedge_{i \in W} \models \delta(c_i, a)$ , then  $\models \delta(c^*, a)$   
 $|W| = N+1$

(b) If  $a \in M$  and  $\exists W \subseteq \{0, \dots, 2N\}$  s.t.  $\bigwedge_{i \in W} \models \neg \delta(c_i, a)$ , then  $\models \neg \delta(c^*, a)$   
 $|W| = N+1$

• Let  $a \in M$ .  $\delta(x, a) \in \mathcal{P} \Leftrightarrow \models \delta(c^*, a) \Leftrightarrow \bigvee_{\substack{W \subseteq \{0, \dots, 2N\} \\ |W| = N+1}} \bigwedge_{i \in W} \delta(c_i, a)$

put  $\gamma = \bigvee_{i \in W} \delta(c_i, a)$

• So  $\forall a \in M$  ( $\delta(x, a) \in \mathcal{P} \Leftrightarrow \models \bigvee_{i \in W} \delta(c_i, a)$ ) :  $\alpha$ . delta definition of  $\mathcal{P}$ .

Lemma  $\blacksquare$

Corollary Let  $\delta \in L$ .  $\square$ :

(1)  $\delta(x, y)$ : stable

(2)  $\forall M \forall \mathcal{P} \in \mathcal{S}(M)$   $\mathcal{P}$  has a  $\delta$ -definition

(3)  $(\forall \lambda \geq \lambda_0) (\forall A \subseteq M) (|S_\delta(A)| \leq \lambda \Rightarrow \exists \mathcal{P} \in \mathcal{S}(M) \text{ s.t. } \mathcal{P} \text{ has a } \delta\text{-definition})$

(4)  $(\exists \lambda \geq \lambda_0) (\forall \mathcal{P} \in \mathcal{S}(M) \text{ s.t. } \mathcal{P} \text{ has a } \delta\text{-definition}) \Rightarrow (\forall \lambda \geq \lambda_0) (\forall A \subseteq M) (|S_\delta(A)| \leq \lambda)$

Proof (1)  $\Rightarrow$  (2): by lemma 2

• (2)  $\Rightarrow$  (3)  $A \subseteq M \text{ s.t. } |S_\delta(A)| \leq |S_\delta(M)| \leq \|M\| \leq \lambda$   
 $\|M\| \leq \lambda$

$\uparrow$   
 there are  $\leq \|M\|$   
 $\delta$ -definitions /  $M$

• (3)  $\Rightarrow$  (4) trivial

(4)  $\Rightarrow$  (1) by contradiction: suppose  $\delta$  has order property. Let  $\lambda \geq \aleph_0$ .

We shall find  $A$  s.t.  $|A| \leq \lambda$  and  $|S_\delta(A)| > \lambda$

•  $(\exists a_i, b_i, i < \omega)(\neq \delta(a_i, b_j) \Leftrightarrow i < j)$ , choose  $(I, \leq) \neq \text{DLO}, |I| > \lambda$

where  $\exists J \subseteq I : J$  dense. By compactness  $|J| \leq \lambda$

$$(\exists a_i, b_i, i \in I) \bigwedge_{i, j \in I} (\neq \delta(a_i, b_j) \Leftrightarrow i < j)$$

•  $\delta$  defines an order:

$$0 < \frac{a_i}{b_i} < 0 \leq 0 < \dots$$

$i \in I \geq j$

Let  $A = \{b_j : j \in J\}$ , then  $|A| \leq \lambda$  and  $|S_\delta(A)| > \lambda$ .

When  $i_1 < i_2 \in I$  then  $\text{tp}_\delta(a_{i_1}/A) \neq \text{tp}_\delta(a_{i_2}/A)$ ,  
just choose  $j \in J$  s.t.  $i_1 < j < i_2$ .

Corollary (1)  $T$ -stable  $\Leftrightarrow (\forall \delta \in L)(\delta$ -stable in  $T)$

(2)  $T$ -stable and  $\lambda^{|T|} = \lambda$ , then  $T$ - $\lambda$ -stable

Proof (1)  $\Leftrightarrow$   $T$ - $\lambda$ -stable. Then  $\forall A \in \mathcal{M}, |A| \leq \lambda, |S_\delta(A)| \leq |S(A)| \leq \lambda$ , so by

previous corollary  $\delta$ -stable.

•  $(\Leftarrow)$  later...

(2)  $|T| = \aleph_0$ , assume  $\lambda = \lambda^{\aleph_0}$ . Let  $|A| \leq \lambda$ ,  ~~$|S(A)| \leq \lambda$~~

$$S(A) \xrightarrow{\lambda^{-1}} \langle p|_\delta : \delta \in L \rangle \in \prod_{\delta \in L} S_\delta(A)$$

$$|\prod_{\delta \in L} S_\delta(A)| \leq \lambda^{\aleph_0} = \lambda \rightsquigarrow |S(A)| \leq \lambda$$

$(\Leftarrow)$  Assume  $\forall \delta \in L \delta$ -stable in  $T$ . Then  $|S(A)| \leq |\prod_{\delta \in L} S_\delta(A)| \leq \lambda^{\aleph_0}$

$\Downarrow$   
 $T$  stable in  $\lambda$  if  $\lambda^{\aleph_0} = \lambda$

Def (1)  $\delta(x, y) \in L$  has strict order property (SOP)

if  $(\exists b_i, i < \omega) (\delta(\mathcal{M}, b_0) \neq \delta(\mathcal{M}, b_1) \neq \dots)$

$$[\Leftrightarrow \delta(b_0, \mathcal{M}) \neq \delta(b_1, \mathcal{M}) \neq \dots]$$

↑  
alternative definition

$T$  has SOP  $\Leftrightarrow (\exists \sigma \in L) (\sigma \text{ has SOP})$

(2)  $\delta$  has independence property (IP)

if  $(\exists b_i, i < \omega) \{ \delta(\mathcal{M}, b_i) : i < \omega \}$  is independent

$T$  has IP  $\Leftrightarrow (\exists \sigma \in L) (\sigma \text{ has IP})$

Theorem (Shelah)

$T$  has OP  $\Leftrightarrow T$  has SOP or  $T$  has IP  
(is unstable)

Examples •  $DLO_0 = Th(\mathbb{Q}, \leq)$  has SOP :  $\varphi(x, y) = x \leq y$   
has SOP,

but  $DLO_0$  has NIP

• Random graph has IP and NSOP

~~TYPES AND DEFINABLE~~

TYPES & TYPE-DEFINABLE SETS.

$p(\bar{x})$ : a type  $\leadsto p(\mathcal{M}) = \{ a \in \mathcal{M}^n : a \models p \}$  : a type-definable set (over  $A$ )  
 $|\bar{x}| = n$  (over  $A$ )  
 $= \bigcap_{\varphi \in p} \varphi(\mathcal{M})$

$$p = \{ \varphi_i : i \in I \} \rightarrow \left( \bigwedge_{i \in I} \varphi_i \right) (\mathcal{M})$$

Let  $p(\bar{x}), q(\bar{x})$ : types,  $(p \wedge q)(\bar{x}) = (p \cup q)(\bar{x})$ ,

$$(p \vee q)(\bar{x}) = \{ (\bigwedge p_0) \vee (\bigwedge q_0) (\bar{x}) : p_0 \in p, q_0 \in q, \text{ finite} \}$$

Fact (1)  $(p \wedge q)(\mathcal{M}) = p(\mathcal{M}) \cap q(\mathcal{M})$

(2)  $(p \vee q)(\mathcal{M}) = p(\mathcal{M}) \cup q(\mathcal{M})$

$$(\exists \bar{y} p)(\bar{x}, \bar{y}) = \exists \bar{y} \bigwedge_i \varphi_i(\bar{x}, \bar{y}) \stackrel{\text{def}}{=} \left\{ \exists \bar{y} \bigwedge p_0(\bar{x}, \bar{y}) : p_0 \in P_{\text{finite}} \right\}$$

where  $p = \bigwedge_{i \in I} \varphi_i$

$$(3) (\exists \bar{y} p)(\mathcal{M}) = \pi_{\bar{x}} \left( p(\mathcal{M}) \right)$$

$\mathcal{M}^{\bar{x}} \times \mathcal{M}^{\bar{y}}$

$$(4) (\forall \bar{y} p)(\mathcal{M}) = \mathcal{M}^{\bar{x}} \setminus \pi_{\bar{x}}(\mathcal{M}^{\bar{x}\bar{y}} \setminus p(\mathcal{M}))$$

Remark  $\mathcal{M} \setminus p(\mathcal{M})$  is usually not type-definable

Let  $\Delta = \{ \varphi_0(x, y) \dots \varphi_n(x, y) \} \in L$

↑  
tuples

↓  
 $\Delta$ -formula,  $\Delta$ -type

Sketch trick There's formula  $\delta(x, \bar{y})$  s.t. "instances of formulas  $\Delta$ "  
= "instances of  $\delta$ " over any set  $A$  of power  $\geq 2$ .

Proof  $\delta = \delta(x, y, z_0, \dots, z_n) =$

$$\begin{cases} \varphi_0(x, y), & \text{when } z_0 = z_1 \\ \varphi_1(x, y), & \text{when } z_0 \neq z_1 \wedge z_0 = z_2 \\ \varphi_2(x, y), & \text{when } z_0 \neq z_1, z_2 \wedge z_0 = z_3 \\ \vdots \\ \varphi_n(x, y), & \text{when } z_0 \neq z_1, \dots, z_n \end{cases}$$

•  $\delta(\mathcal{M}, a, \bar{c}) = \varphi_i(\mathcal{M}, a)$  for some  $i \in \{0, \dots, n\}$

• For every  $a \in A$ , every  $i \in \{0, \dots, n\}$   $\exists \bar{c} \in A$   $\varphi_i(\mathcal{M}, a) = \delta(\mathcal{M}, a, \bar{c})$

■

$\varphi \in S_\delta(A) \ [\varphi \in S_\Delta(A)] \ (\Delta \text{ possibly infinite})$

$\bullet \text{CB}_\delta(\varphi) = \text{CB in } S_\delta(A) \ (\text{CB-delta rank, CB-local rank})$

$\bullet \varphi \in L_\delta(A) \rightsquigarrow \text{CB}_{\delta,A}(\varphi) = \text{CB}(S_\delta(A) \cap [\varphi])$

$\bullet \text{CB}_\delta \rightsquigarrow R_\delta: L(\mathcal{M}) \longrightarrow \{ \neg 1 \} \cup \text{Ord} \cup \{ \infty \}$   
local  $\delta$ -rank  
local  $\Delta$ -rank

The smallest function s.t.  $\forall \alpha \in \text{Ord} \cup \{ \neg 1 \}$

$R_\delta(\varphi) \geq \alpha + 1 \iff \forall n < \omega \exists \psi_1, \dots, \psi_n \bigwedge_i R_\delta(\varphi \wedge \psi_i) \geq \alpha$   
 $\delta$ -formulas  
pairwise contr.

[Alternatively]

$R_\delta(\varphi) = \text{CB}_{\delta, \mathcal{M}}(\{ p \in S_\delta(\mathcal{M}) : p \cup \{ \varphi \} \text{ consistent} \})$

$R_\delta(p) = \min \{ R_\delta(\varphi) : p \vdash \varphi \}$

[Alternatively]

$R_\delta(p) = \text{CB}_{\delta, \mathcal{M}}(\{ q(x) \in S_\delta(\mathcal{M}) : p \cup q \text{ consistent} \})$

Remark (a)  $R_{\delta(x,y)}: L_x(\mathcal{M}) \xrightarrow{\text{onto}} (\{ \neg 1 \} \cup \text{initial segment of ordinals} \cup \{ \infty \})$   
proper  
possibly

(b)  $R_\delta(\varphi) \geq \alpha + 1 \iff \exists \psi_1, \psi_2, \dots \bigwedge_i R(\varphi \wedge \psi_i) \geq \alpha$   
pairwise contr.  
 $\delta$ -formulas

(c)  $R_\delta(\varphi \vee \psi) = \max \{ R_\delta(\varphi), R_\delta(\psi) \}$

(d)  $p(x): \text{a type / } A \Rightarrow \exists q(x): \text{a complete type / } A \text{ with } p \subseteq q \implies R_\delta(p) = R_\delta(q)$

Example

$R_\delta = R_{\{ \delta, \neg \delta \}} \xrightarrow{\Delta} R_{\delta'}$   
Shelah trick (for some  $\delta'$ )

$\rightsquigarrow$  We assume that " $\delta$  is closed under negation":  
 i.e.  $\forall \bar{a} \exists \bar{b} \models \delta(x, \bar{a}) \iff \neg \delta(x, \bar{b})$

(1)  $R_\delta(\varphi(x, \bar{a})) \geq k + 1 \iff \forall m < \omega \exists \psi_1, \dots, \psi_m \bigwedge_i R_\delta(\varphi(x, \bar{a}) \wedge \psi_i) \geq k$   
 $\delta$ -formulas  
pairwise contr.

$$\Leftrightarrow \forall n \exists p_1, \dots, p_n \in S_\delta(\mathcal{M}) \wedge R_\delta(p_i \cup \{\varphi\}) \geq k$$

distinct

$$\left[ \begin{array}{l} p_i \neq p_j \text{ for } i \neq j \\ \exists c_{ij} \delta(x, c_{ij}) \wedge \neg \delta(x, c_{ji}) \end{array} \right]$$

only a def. of  $\psi_i$

enough to have this in the condition

$$\Leftrightarrow \forall m \exists \langle c_{ij} : 1 \leq i \neq j \leq n \rangle \left\{ \bigwedge_{\substack{i, j \\ i \neq j}} \delta(x, c_{ij}) \wedge \neg \delta(x, c_{ji}) \in p_i \right\} \text{ and } R_\delta(\varphi(x, \bar{a}) \wedge \psi_i(x, \bar{c})) \geq k$$

$\psi_i(x, \bar{c})$ : a  $\delta$ -formula

[Notice:  $\psi_i(x, \bar{c})$ ,  $i=1, \dots, m$  are explicitly pairwise  $\perp$ ]

Lemma For every  $\varphi(x, \bar{y})$ ,  $k < \omega$  there is a type  $\Phi_{\varphi, k}(\bar{y})$  s.t.  
 $\forall \bar{a} \in \mathcal{M} R_\delta(\varphi(x, \bar{a})) \geq k \Leftrightarrow \models \Phi_{\varphi, k}(\bar{a})$

PP Induction on  $k$ .

$k=0$   $R_\delta(\varphi(x, \bar{a})) \geq 0 \Leftrightarrow \models \exists x \varphi(x, \bar{a})$  so  $\Phi_{\varphi, 0}(\bar{y}) = \{ \exists x \varphi(x, \bar{y}) \}$

$k \rightarrow k+1$   $R_\delta(\varphi(x, \bar{a})) \geq k+1 \Leftrightarrow \models \exists \bar{c} \bigwedge_{1 \leq i \leq n} \Phi_{\varphi, \psi_i, k}(\bar{a}, \bar{c})$   
 $\models \Psi_n(\bar{a})$  for a type  $\Psi_n(\bar{y})$

$$\Phi_{\varphi, k+1}(\bar{y}) = \bigcup_n \Psi_n(\bar{y})$$

□

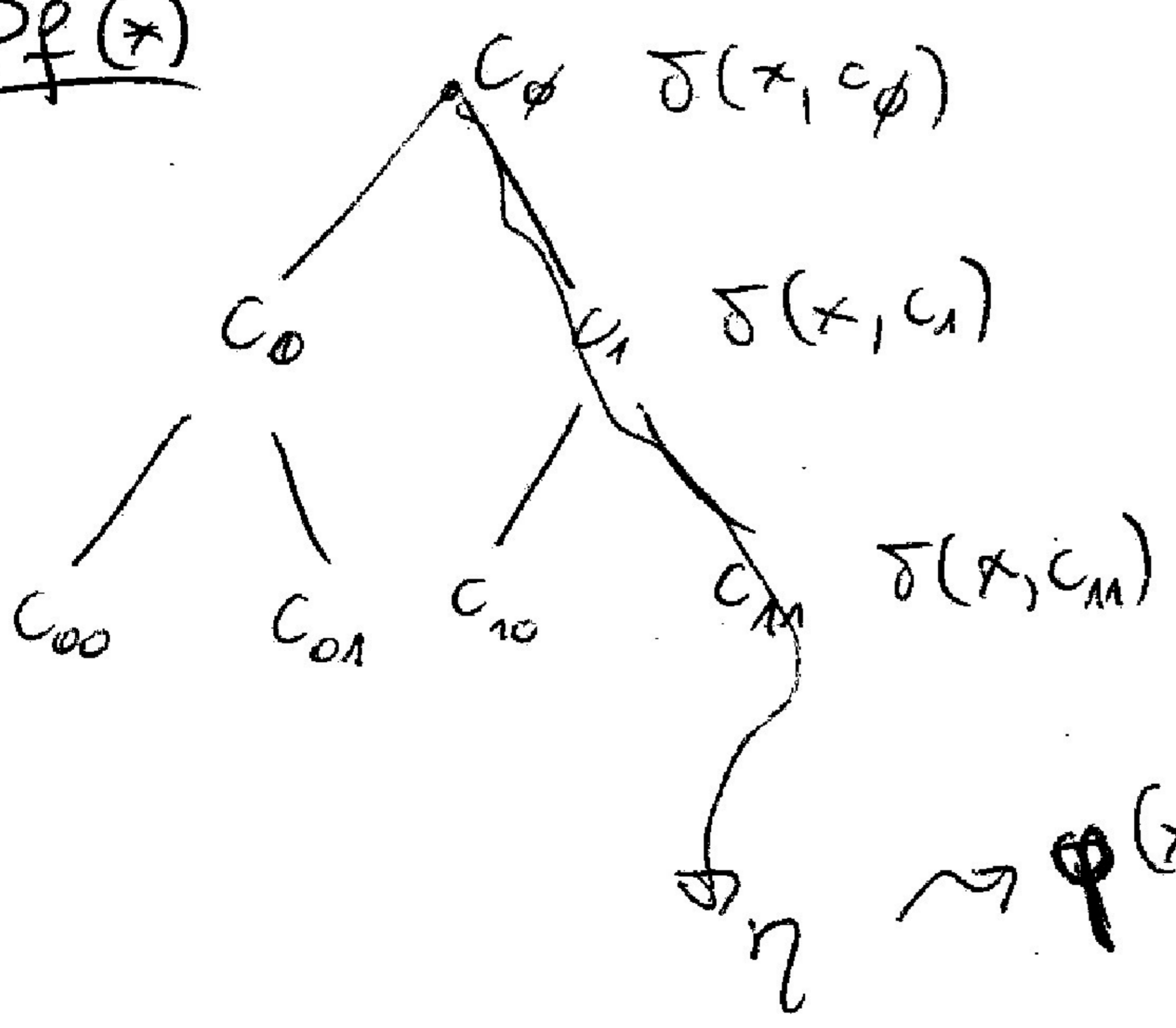
Corollary (1)  $R_\delta(\varphi) \geq \omega \Leftrightarrow R_\delta(\varphi) = \infty$

(2)  $\delta$  is stable  $\Leftrightarrow R_\delta(x=x) < \omega$

here  $\begin{cases} \chi^0 = x \\ \chi^1 = \neg x \end{cases}$

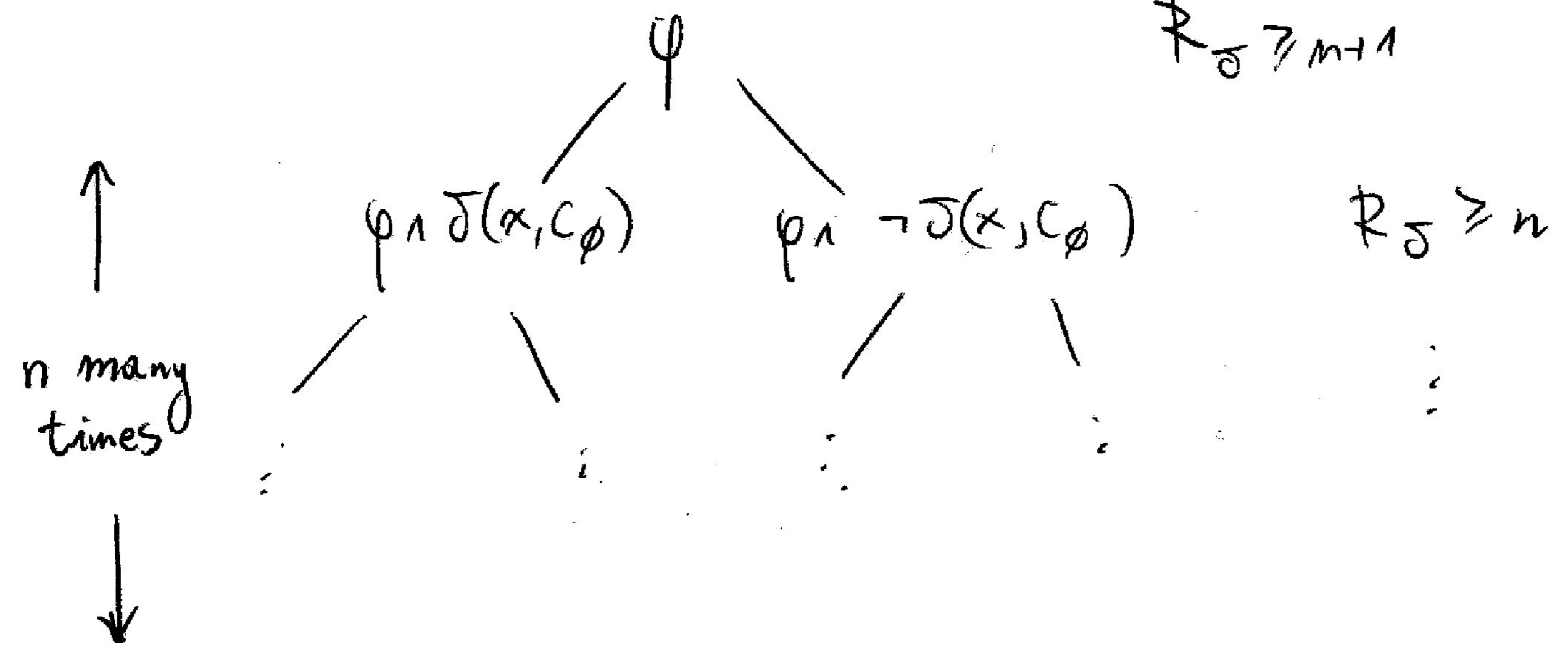
Proof (1)  $\Rightarrow$  (\*):  $\exists \langle c_\eta, \eta \in 2^{<\omega} \rangle \forall \eta \in 2^{<\omega} p_\eta(x) = \{ \varphi \} \cup \{ \delta(x, c_\eta) \} : n < \omega \}$  is consistent

PP (\*)



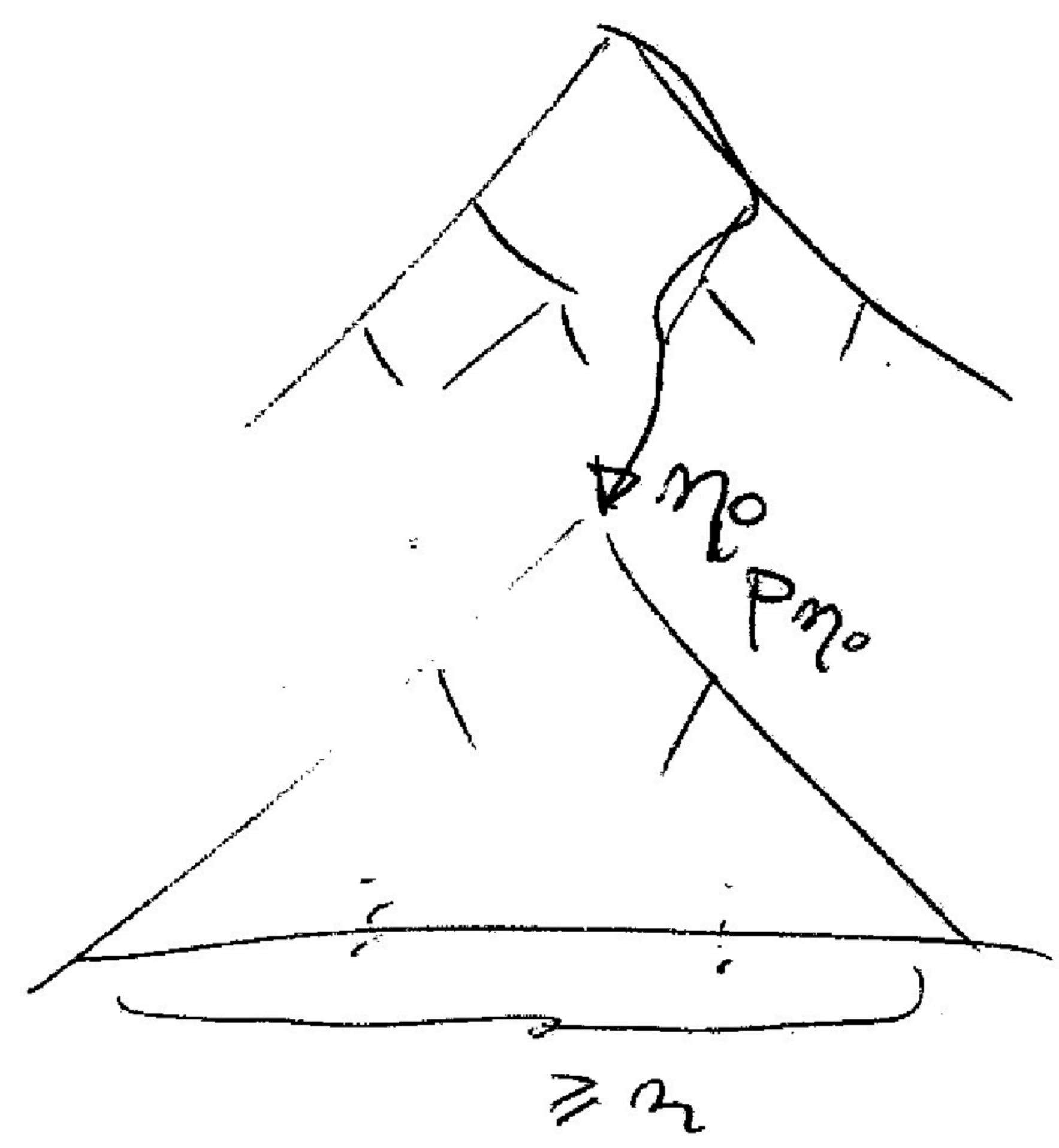
goal

Enough to show  $\forall m \exists c_\eta, \eta \in 2^{<m} \forall \eta \in 2^m p_\eta(x)$  consistent. But this follows from  $R_\delta(\varphi) \geq m+1$



(\*) □

Proof of  $\Rightarrow$  in (1): (A.c.) suppose  $R_\delta(\varphi) < \infty$ . For  $\eta \in 2^{<\omega}$   $R_\delta(p_\eta(x)) < \infty$ .  
 Let  $\eta_0 \in 2^{<\omega}$  s.t.  $\alpha := R_\delta(p_{\eta_0})$  is minimal.



So  $\forall m \exists \varphi_1, \dots, \varphi_m$   $\delta$ -formulas pairwise  $\perp$   $\bigwedge_i R_\delta(p_{\eta_0} \wedge \varphi_i) \geq \alpha$

PF (2) ~~unstable  $\Leftrightarrow R_\delta(\Rightarrow)$  (A.c.)~~  $R_\delta(x \neq x) \geq \omega \stackrel{(1)}{\Rightarrow} R_\delta(x=x) = \infty$   
 the proof of (1) get a tree of instances of  $\delta \Rightarrow |S_\delta(\{c_\eta : \eta \in 2^{<\omega}\})| = 2^{\aleph_0}$   
 $\Rightarrow \delta$  unstable

( $\Leftarrow$ )  $R_\delta(x=x) < \omega$ ,  $A$ : a set,  $|A| \leq \kappa$ , shall prove  $|S_\delta(A)| \leq \kappa$

(a)  $p \in S_\delta(A) \rightsquigarrow \exists \varphi \in \mathcal{P} R_\delta(p) = R_\delta(\varphi) \leq \kappa < \omega$  (for some)

There are  $\leq \kappa$ -many such  $\varphi \in \mathcal{L}(A)$

(b) For  $\varphi \in \mathcal{L}_\delta(A)$   $| \{ p \in S_\delta(A) : \varphi \in \mathcal{P} \text{ and } R_\delta(p) = R_\delta(\varphi) \} | < \aleph_0$

□

Corollary  $R_{\delta}(p) \geq k+1 \Leftrightarrow \forall m \exists \varphi_1, \dots, \varphi_m \text{ } \delta\text{-formulas pairwise } \perp \text{ } R_{\delta}(p \cup \{\varphi_i\}) \geq k$   
any type

BINARY RANKS:  $R_{\delta,2}$

$$R_{\delta,2}(\varphi) \geq \alpha+1 \Leftrightarrow \exists c [R_{\delta,2}(\varphi \wedge \delta(x,c)) \geq \alpha \text{ and } R_{\delta,2}(\varphi \wedge \neg \delta(x,c)) \geq \alpha]$$

Lemma (1)  $R_{\delta,2}(\varphi(x,\bar{a})) \geq k \Leftrightarrow \models \Theta_{\varphi,k}(\bar{a})$  for a formula  $\Theta_{\varphi,k}(\bar{y})$

(2)  $R_{\delta,2}(\varphi(x,\bar{a})) = k \Leftrightarrow \models \Theta_{\varphi,k}(\bar{a}) \wedge \neg \Theta_{\varphi,k+1}(\bar{a})$

(3)  $R_{\delta,2}(\varphi) \geq \omega \Leftrightarrow R_{\delta,2}(\varphi) = \infty$

(4)  $R_{\delta,2}(x=x) < \infty \Leftrightarrow \delta \text{ is stable}$

Comments (a) Usually  $R_{\delta,2}(\varphi \vee \psi) = \max\{R_{\delta,2}(\varphi), R_{\delta,2}(\psi)\}$ .

(b) So usually it is not true that  $\forall p \forall A \exists q \in S(A) R_{\delta,2}(p) = R_{\delta,2}(q)$

Definability again

Lemma (Sketch) Let  $\delta(x,y)$ : stable,  $p(x) \in S_{\delta}(A)$ . Then

$$\exists \chi \in L(A) \forall \bar{a} \in A \models \chi(\bar{a}) \Leftrightarrow \delta(x,\bar{a}) \in p(x)$$

a  $\delta$ -definition

Proof  $R_{\delta,2}(p) = k < \omega$ . Choose  $\psi \in p$  with  $R_{\delta,2}(\psi) = k$ . Then for  $\bar{a} \in A$

$$\delta(x,\bar{a}) \in p \Leftrightarrow R_{\delta,2}(\psi \wedge \delta(x,\bar{a})) \geq k \Leftrightarrow \underbrace{\Theta_{\psi \wedge \delta,k}(\bar{a})}_{\chi(\bar{a})}$$

Corollary  $[\delta: \text{stable}, a \in \mathcal{M}_{A \neq}] \delta(a,A) = \{b \in A : \models \delta(a,b)\} = \{b \in A : \models \chi(b)\}$   
 for some  $\chi(y) \in L(A)$

Proof  $p = \text{tp}_{\delta}(a/A) \rightsquigarrow \chi$  from the lemma





$M \models L, T = Th(M), ER(T) = \{E(\bar{x}, \bar{y}) \in L : M \models "E \text{ is an equivalence relation}"\}$   
 $E \rightsquigarrow S_E : \text{new sort symbol.}$

$L \rightsquigarrow L^{eq} : \text{many sorted language.}$

$L^{eq} :$

- the standard sort  $S_ =$
- $S_E, E \in ER(T)$  (those are imaginary sort symbols)
- symbols of  $L$  refer to  $S_ =$  ~~(the standard sort)~~
- new function symbols of  $L^{eq} :$ 
  - $E \in ER(T) \rightsquigarrow F_E(\bar{x})$   
 $\begin{matrix} \text{"} E(\bar{x}, \bar{y}) \text{"} & & \text{of sort } S_E & \text{of } S_ = \end{matrix}$

$M \models T \rightsquigarrow M^{eq} = \bigcup_{E \in ER(T)} S_E^{M^{eq}}, \text{ where } S_E^{M^{eq}} = \{ \bar{a} / E : \bar{a} \in M^n \}$   
 $S_ =^{M^{eq}} = M$   
 the home sort of  $M^{eq}$  standard

$M_E := S_E^{M^{eq}}$ , elements of  $M_E$ : imaginary elements, imaginaries where  $E \neq x=y$

$F_E^{M^{eq}}(\bar{a}) = \bar{a} / E$ . The symbols of  $L$  are naturally interpreted in  $M$ .

~~$T^{eq} = Th(M^{eq})$~~

Fact (1) For  $\varphi(\bar{x}) \in L$  and  $\bar{a} \in M^n$   $M \models \varphi(\bar{a}) \Leftrightarrow M^{eq} \models \varphi(\bar{a})$

Def  $T^{eq} := \text{Cr}(T \cup \{ "F_E : S_ = \xrightarrow{\text{onto}} S_E" \}_{E \in ER(T)} \cup \{ F_E(\bar{x}) = F_E(\bar{y}) \Leftrightarrow E(\bar{x}, \bar{y}) \}_{E \in ER(T)} )$

Fact (1)  $M \models T \Leftrightarrow M^{eq} \models T^{eq}$

(2)  $\forall M^* \models T^{eq} \exists M \models T (M^* \stackrel{\text{essentially}}{=} M^{eq} \text{ and } M^*_ = M)$   
 i.e.  $\exists f: M^* \xrightarrow{\cong} M^{eq}$   
 $f \upharpoonright_M = \text{id}_M$

(3)  $T^{eq}$ : complete

(4)  $\exists E(1), \dots, E(k) \in ER(T), \varphi(x_1, \dots, x_k) \in L^{eq}, x_i \in E(i)$

then there's  $\psi(y_1, \dots, y_k) \in L$  s.t.  $T^{eq} \vdash (\forall y_1, \dots, y_k) (\psi(y_1, \dots, y_k) \leftrightarrow$   
of sort  $S =$

$$\varphi(F_{E(1)}(y_1), \dots, F_{E(k)}(y_k))$$

Corollary (1)  $\forall M \prec \mathcal{M} \quad M^{eq} \prec \mathcal{M}^{eq}$

(2)  $\mathcal{M}^{eq} = dcl^{eq}(\mathcal{M}_=), M^{eq} = dcl^{eq}(M_=)$

(3)  $A \subseteq \mathcal{M} = \mathcal{M}_=$  is definable [with parameters] in  $M^{eq}$

$\Leftrightarrow A$  is definable [with parameters] in  $\mathcal{M}$

(4)  $\mathcal{M}^{eq}$  is  $\kappa$ -saturated and strongly  $\kappa$ -homogeneous

Proof of fact (1), (2): obvious

(3) Let  $M^*, N^* \models T^{eq} \stackrel{(2)}{\Rightarrow} M^* = M^{eq}, N^* = N^{eq}$  for some  $M, N \models T$

$T^{complete} \Rightarrow M \equiv N \stackrel{(4)}{\Rightarrow} M^{eq} \equiv N^{eq}$

(4) Induction on  $\varphi$ :

- $\varphi$ : atomic
  - the connectives
- } easy

•  $\varphi: \exists x_0 \varphi_0(x_0, x_1, \dots, x_k)$ . By the ind. hyp.  $\varphi_0$  is "equivalent" <sup>in  $T^{eq}$</sup>  to  $\varphi_0(\bar{y}_0, \dots, \bar{y}_k)$   
of sort  $S_{E_0}$  So:  $T^{eq} \vdash \exists x_0 \varphi_0(x_0, F_{E(1)}(\bar{y}_1), \dots, F_{E(k)}(\bar{y}_k))$

||| (by axioms of  $T^{eq}$ )

$$\exists \bar{y}_0 \varphi_0(F_{E(0)}(\bar{y}_0), \dots, F_{E(k)}(\bar{y}_k))$$

|||

$$\exists \bar{y}_0 \psi(\bar{y}_0, \dots, \bar{y}_k)$$

Proof of corollary (1) Goal:  $M^{eq} \prec \mathcal{M}^{eq}$ . Take  $\bar{a}^{eq} \in M^{eq} \models \varphi(\bar{a}^{eq}), \bar{a} = \bar{b}/E$

by fact (4)  $M^{eq} \models \varphi(\bar{a}^{eq}) \Leftrightarrow M \models \psi(\bar{b}) \Leftrightarrow M \models \psi(\bar{b}) \Leftrightarrow \mathcal{M} \models \psi(\bar{b})$

$\Leftrightarrow \mathcal{M}^{eq} \models \psi(\bar{b}) \Leftrightarrow \mathcal{M}^{eq} \models \varphi(\bar{a})$

Fact Assume  $E \in ER(T^{eq})$ . Then there is  $\psi \in ER(T)$  on  $\mathcal{M}^n$  and 3

$\exists f: \mathcal{M}^n \xrightarrow{\text{onto}} \text{dom}(E)$  s.t.  $\forall a, b \in \mathcal{M} \quad (\mathcal{M}^{eq} \models \psi(\bar{a}, b) \leftrightarrow f(\bar{a}) E f(b))$   
 $\uparrow$   
 $0$ -definable in  $T^{eq}$ . So:  $\exists g: \mathcal{M} \xrightarrow{1-1 \text{ onto}} \mathcal{M}^{eq}_E$  in  $(\mathcal{M}^{eq})^{eq}$   $g$ :  $0$ -definable

Def Assume  $X \subseteq \mathcal{M}^{eq}$  and  $c \in \mathcal{M}^{eq}$ .  $c$  is a name (code) of  $X$  if

$\forall f \in \text{Aut}(\mathcal{M}^{eq}) \quad (f[X] = X \Leftrightarrow f(c) = c)$

Remark (1)  $X \subseteq \mathcal{M}^{eq} \xrightarrow{\text{def}} X$  has a name  $c \in \mathcal{M}^{eq}$

(2) If  $c$ : name of  $X$  then  $X = \psi(\mathcal{M}^{eq}, c)$  for some  $\psi \in L^{eq}$

Proof (1)  $1^\circ X \subseteq \mathcal{M} \xrightarrow{\text{def}} X = \psi(\mathcal{M}, \bar{a})$ .  $E_\psi \in ER(T)$ ,

$E_\psi(\bar{y}_0, \bar{y}_1) = \forall x (\psi(x, \bar{y}_0) \leftrightarrow \psi(x, \bar{y}_1))$

So for  $\bar{a}_0, \bar{a}_1 \in \mathcal{M} \quad \models E_\psi(\bar{y}_0, \bar{y}_1) \Leftrightarrow \psi(\mathcal{M}, \bar{a}_0) = \psi(\mathcal{M}, \bar{a}_1)$

$c = \bar{a} / E_\psi = F_{E_\psi}(\bar{a}) \in \mathcal{M}_{E_\psi}$   
 $\uparrow$   
 good, name for  $X$

$2^\circ X \subseteq \mathcal{M}^{eq} \xrightarrow{1^\circ} \text{a name of } X \text{ in } \mathcal{M}^{eq} \xrightarrow{c \in (\mathcal{M}^{eq})^{eq}} \text{get } c' \in \mathcal{M}^{eq} \text{ a name of } X$   
 $(\mathcal{M}^{eq})^{eq} = \mathcal{M}^{eq}$

(2)  $X \subseteq \mathcal{M}^{eq}$ ,  $c$ : a name of  $X$  in  $\mathcal{M}^{eq}$ . By def.  $\forall f \in \text{Aut}(\mathcal{M}^{eq})$   
 $(f[X] = X \Leftrightarrow f(c) = c)$

hence  $X$  is invariant under  $\text{Aut}(\mathcal{M}^{eq}/c)$

$\Downarrow$   
 $X$  definable over  $c$

Example  $X = \psi(\mathcal{M}, \bar{a})$ ,  $c = \bar{a} / E_\psi$  a name of  $X$

$x \in X \Leftrightarrow \exists \bar{y} (\psi(x, \bar{y}) \wedge F_{E_\psi}(\bar{y}) = c)$

$\delta(x, c)$   
 defines  $X$

From now on we work in  $\mathcal{M}^{eq}, T^{eq}$   
 $\mathcal{M}^*$

Definition Assume  $X \subseteq \mathcal{M}$ .  $X$  is definable almost over  $A \subseteq \mathcal{M}$   
definable

$\Leftrightarrow$  the set  $\{f[X] : f \in \text{Aut}(\mathcal{M}/A)\}$  is finite

Lemma  $X$  is definable almost over  $A \Leftrightarrow X$  is definable over  $\text{acl}^{eq}(A)$

Proof  $(\Leftarrow) X = \varphi(\mathcal{M}, \bar{a})$   $f[X] = \varphi(\mathcal{M}, f(\bar{a}))$ ,  $f \in \text{Aut}(\mathcal{M}/A)$   
 $\bar{a} \in \text{acl}^{eq}(A)$       finitely many possibilities

$(\Rightarrow) X = \varphi(\mathcal{M}, \bar{b})$ ,  $|\{f[X] : f \in \text{Aut}(\mathcal{M}/A)\}| < \aleph_0$ . A type over  $A$  in  $\bar{y}_1, \dots, \bar{y}_k$

$\{\varphi(\mathcal{M}, \bar{y}_i) \neq \varphi(\mathcal{M}, \bar{y}_j) \}_{1 \leq i \neq j \leq k+1} \cup \{\bar{y}_i \in \text{tp}(\bar{b}/A)\}_{1 \leq i \leq k+1}$

This type has to be inconsistent. So for some  $\Theta(\bar{x}) \in \text{tp}(\bar{b}/A)$   
 $\text{tp}(\bar{b}/A)$

$\Theta \cup \{\bigwedge_{1 \leq i \leq k+1} \Theta(\bar{y}_i)\}$  is inconsistent with  $T$ .

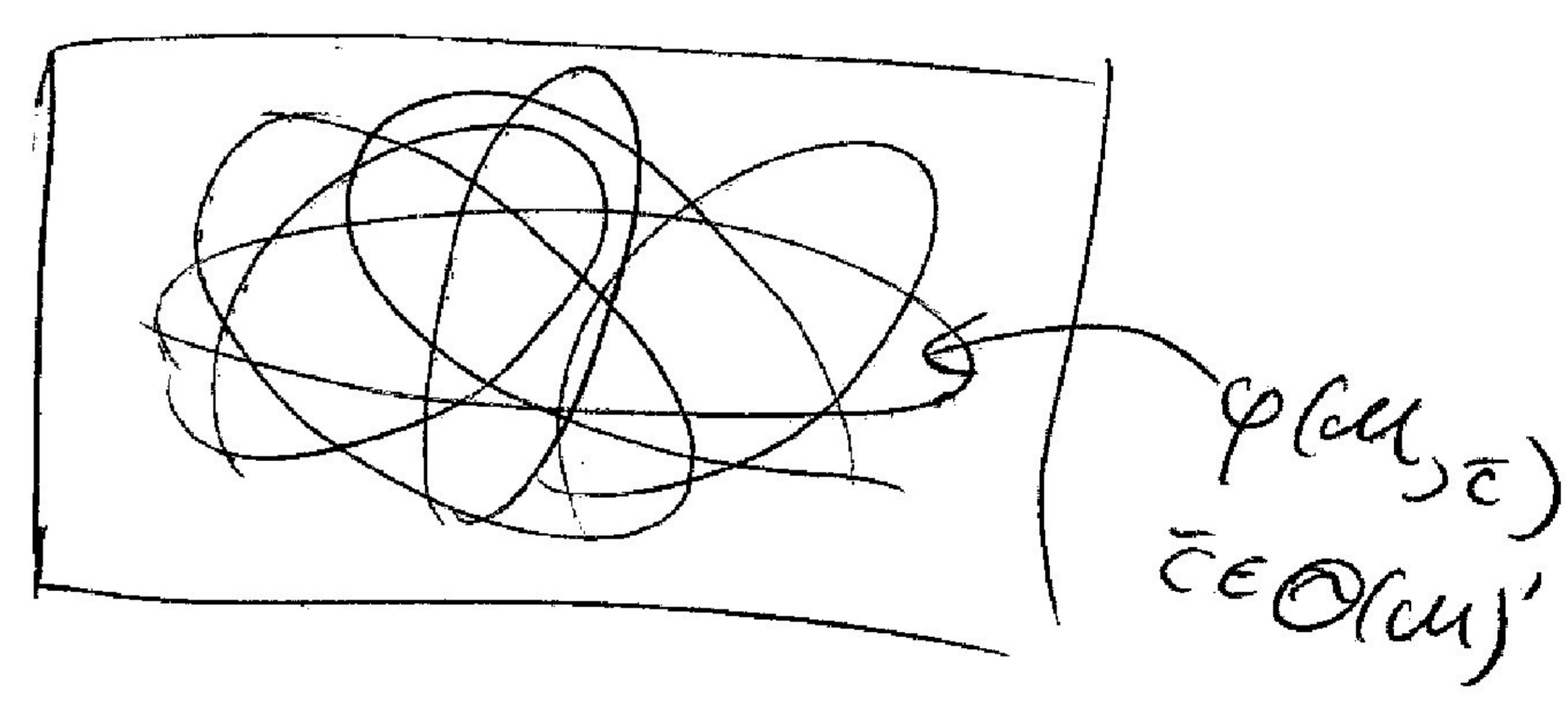
So:  $|\{\varphi(\mathcal{M}, \bar{c}) : \bar{c} \in \Theta(\mathcal{M})\}| \leq k$

$E(x, x') : \forall \bar{c} \in \Theta(\mathcal{M}) (\varphi(x, \bar{c}) \leftrightarrow \varphi(x', \bar{c}))$   
 $\Theta(\mathcal{M})$   
 $L(A)$

$E$  defines in  $\mathcal{M}^{eq}$  an equiv. rel. with finitely many classes.

$E \in \text{FE}(A)$

formulas over  $A$  defining in  $\mathcal{M}$  an equiv. rel. with finitely many classes



$E$ -classes are atoms of the algebra of sets generated by  $\Theta(\mathcal{M})$

$X =$  union of some  $E$ -classes  $E_1, \dots, E_n$

$\uparrow$  definable in  $\mathcal{M} \Rightarrow$  have names  $c_1, \dots, c_n \in \mathcal{M}^{eq}$   
 $\Rightarrow c_1, \dots, c_n \in \text{acl}^{eq}(A)$

So  $X$  is definable over  $\text{acl}^{eq}(A)$

# FORKING OF TYPES

Example  $p \in S(A)$ ,  $A \subset B$ ,  $RM(p) < \infty$

1)  $q \in S(B)$

1°  $RM(p) = RM(q)$ : " $q$  is a free extension of  $p$ "

2°  $RM(q) < RM(p)$ : " $q$  is a degenerated extension of  $p$ "  
 forking

Def (1)  $\{\varphi(x, a_i) : i < \omega\}$  is  $k$ -contradictory ( $k < \omega$ ) if every  $k$ -element subset is inconsistent (with  $T$ )

(2)  $\varphi(x, a)$  divides over  $A$  if

$$\exists k < \omega \exists \{\varphi(x, a_i) : i < \omega\} : k\text{-contradictory} \wedge \bigwedge_{i < \omega} tp(a_i/A) = tp(a/A)$$

(3) A type  $p(x)$  (over  $M$ ) divides over  $A$  if  $p \vdash \varphi(x, \bar{a})$  for some  $\varphi(x, \bar{a})$  that divides over  $A$

Remark In (2) can request  $a_0 = a$  and  $\langle a_i, i < \omega \rangle$  is indiscernible order

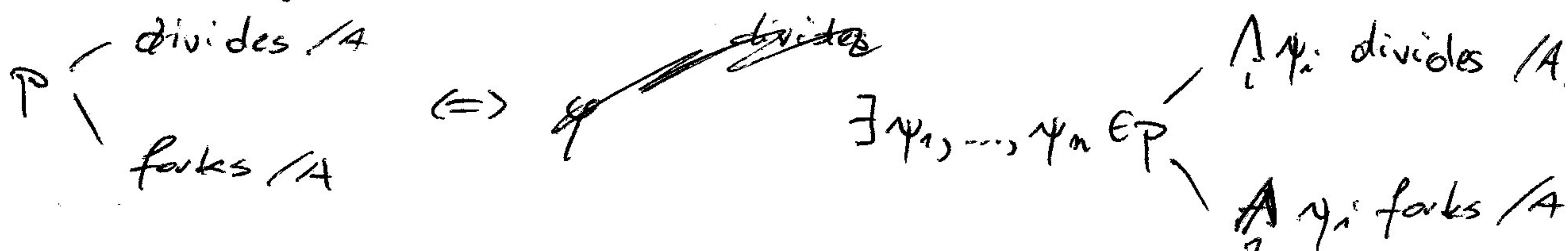
Def (1)  $\varphi(x, \bar{a})$  forks over  $A$  if  $\varphi(x, \bar{a}) \vdash \bigvee_{i \leq n} \varphi_i(x, \bar{b}_i)$   
 for some  $\varphi_i(x, \bar{b}_i)$ ,  $i \leq n$   
 divides over  $A$

(2) A type  $p(x)$  forks /  $A$  if  $p(x) \vdash \underbrace{\varphi(x, \bar{a})}_{\text{forks / } A}$  for some  $\varphi(x, \bar{a})$

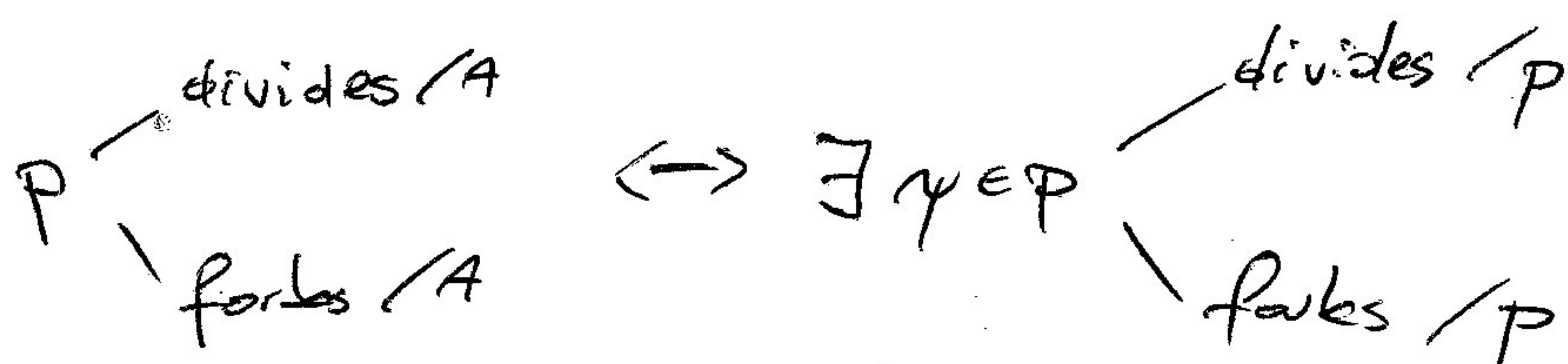
Properties (1) dividing  $\Rightarrow$  forking

(2)  $\exists \varphi \vdash \psi$ , then  $\left\{ \begin{array}{l} \psi \text{ divides } /A \Rightarrow \varphi \text{ divides } /A \\ \psi \text{ forks } /A \rightarrow \varphi \text{ forks } /A \end{array} \right.$

(3) Assume  $p$  is a type. Then



(4)  $p \in S(B)$ , then



(5)  $p$  is a type /  $B$  and  $p$  does not fork /  $A \Rightarrow \exists q \in S(B)$ ,  $q \supseteq p$ ,  $q$  does not fork /  $A$

Pf (5) Let  $\Gamma = \{ \neg \psi(x, \bar{b}) : \psi \in L, \bar{b} \in B \text{ \& } \psi(x, \bar{b}) \text{ forks } /A \}$

$p \cup \Gamma$  is consistent type over  $B$

If not then  $p \cup \Gamma_0$  is inconsistent, thus  $p \vdash \bigvee_{i=0}^k \psi_i(x, \bar{b}_i)$   
 $\{\neg \psi_0, \dots, \neg \psi_m\}$

But each of  $\psi_i$  forks /  $A$ , thus  $\bigvee_{i=0}^k \psi_i(x, \bar{b}_i)$  forks /  $A$ ,

thus  $p$  forks /  $A$ .  $\downarrow$

Let  $q(x) \in S(B)$ . Then  $q$  does not fork /  $A$ .  
 $p \cup \Gamma \subseteq q$

From now on: assume  $T$  is stable.

ORKING AND RANKS

Remark Assume  $\varphi(x, \bar{b})$  forks /  $A$ . Then:

(1)  $\exists \Delta_0 \subseteq L \forall \Delta_0 \subseteq \Delta \subseteq L \forall p \in S(A) R_{\Delta}(p \cup \{\varphi(x, \bar{b})\}) < R_{\Delta}(p)$   
 finite finite

(2)  $RM(p) < \infty$ , then  $RM(p \cup \{\varphi(x, b)\}) < RM(p)$  ②

Proof (1), (2)  $\varphi \vdash \bigvee_{i=0}^k \varphi_i(x, b_i)$ . Let  $\Delta_0 = \{\varphi_i(x, y_i) : i \leq k\} \leftarrow \text{good}$ .  
divides A

Let  $\Delta_0 \subseteq \Delta \subseteq L$  and assume  $R_\Delta(p) < \infty$ . Will show that

$R_\Delta(p \cup \{\varphi(x, b)\}) < R_\Delta(p)$ . (a.c.) Suppose  $R_\Delta(p \cup \{\varphi(x, b)\}) = R_\Delta(p)$ .  
 Then there's  $q \in S(A \cup \{b, b_i : i \leq k\})$   $R_\Delta(q) = R_\Delta(p)$ . Then

$q \supseteq \varphi(x, b) \vdash \bigvee_i \varphi_i(x, b_i)$ , thus (wlog)  $\varphi_0(x, b_0) \in q$ ,  
 but  $p_0$  divides  $A$ . By the def. of dividing we  
 get  $\{\varphi_0(x, b_0^i) : i < \omega\} = L$ -contradictory.

and  $tp(b_0^i/A) = tp(b_0/A)$ .

$R_\Delta(p \cup \{\varphi_0(x, b_0^i)\}) = R_\Delta(p)$  for  $i < \omega$

extend to  $q_i \in S(C)$ ,  $R_\Delta(q_i) = R_\Delta(p)$   
 where  $C \supseteq A \cup \{b, b_0^i : i < \omega\}$

$q_i \upharpoonright \Delta \in S_\Delta(C)$ ,  $\{q_i \upharpoonright \Delta : i < \omega\}$  is infinite (by L-contradictionality)  
 and  $R_\Delta(p \cup (q_i \upharpoonright \Delta)) = R_\Delta(p)$  for all  $i$  ~~of  $\varphi$~~

(because there can be only finitely many  
 such types) □

Corollary If  $p \in S(A)$ , then  $p$  dnf  $A$ .

~~Assume  $\varphi(x, y)$  is~~

Desell Lemma Assume  $\varphi(x, y) \in L$ . Then  $\exists n_\varphi < \omega \forall I$ : infinite indiscernible  $\forall a$  ( $|p(I, a)| < n_\varphi$ )  
 or  $\neg \varphi(I, a) < n_\varphi$

Def. Let  $I \subseteq \mathcal{M}$ ,  $A \subseteq \mathcal{M}$ .  
infinite indiscernible  $\leftarrow$  permanent assumption

$A_{av}(I/A) = \{\varphi(x) \in L(A) : \varphi(I) \text{ infinite}\} \in S(A)$   
 the average type of  $I$  over  $A$  cofinite in  $I$



Lemma If  $\varphi(x): a$  formula over  $\mathcal{M}$ ,  $a \in \text{acl}(A)$  and  $\models \varphi(a)$ , then  $\varphi \text{ dnf } / A$  (3)

Def.  $I$  is based on  $A$  iff  $\forall B \text{ Aut}(I/B) \text{ dnf } / A$ .  
 $(\Leftrightarrow \text{Aut}(I/\mathcal{M}) \text{ dnf } / A)$

Remark For  $\alpha < \beta \in \text{Ord}$  let  $a_\alpha \models \text{Aut}(I/A \upharpoonright a_{<\alpha})$ . Then:

(1)  $I \cup \{a_\alpha : \alpha < \beta\}$  is indiscernible

(2)  $\{a_\alpha : \alpha < \beta\}$  is indiscernible  $/ A$

Proof Hint: (a) If  $f \in \text{Aut}(\mathcal{M}/I)$ ,  $A \subseteq \mathcal{M}$ , then  $f(\text{Aut}(I/A)) = \text{Aut}(I/f(A))$

(b) If  $a \notin I$ , then  $I \cup \{a\}$  is indiscernible  $\Leftrightarrow a \models \text{Aut}(I/I)$

Lemma If  $a \notin \text{acl}(A)$ , then  $\exists I$ : indiscernible  $/ A$ ,  $a \in I$   
 based on  $A$

Proof Choose  $M \supseteq Aa$ , where  $\kappa = \max(|A|, \aleph_0)$ .  
 $\uparrow$   
 $\kappa^+$ -saturated

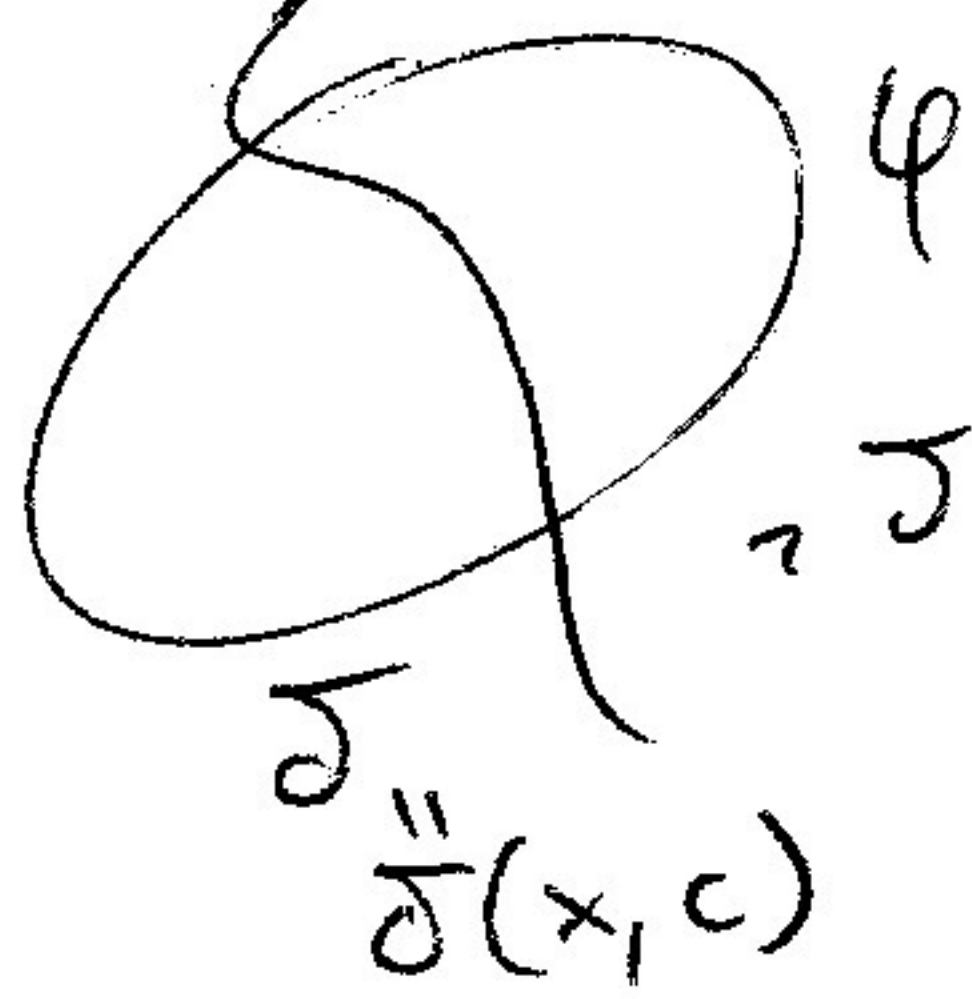
Let  $p = \text{tp}(a/A) \subseteq q \in S(M)$ . Then  $\forall \Delta \subseteq L$  finite  $\text{Mlt}_\Delta(q) = \Delta$ .

(non forking)  
 dnf  $/ A$

Then  $\exists B \subseteq M$ ,  $|B| \leq \omega$ ,  $|B \setminus A| \leq \omega$

$\forall \Delta \subseteq L$  finite  $\left\{ \begin{array}{l} \text{Mlt}_\Delta(q) = \text{Mlt}_\Delta(q|_B) \\ R_\Delta(q) = R_\Delta(q|_B) \end{array} \right.$

in  $\mathcal{M}$ :



$\delta$ :  $\Delta$ -formula  
 $R_\Delta(\varphi \wedge \delta) = R_\Delta(\varphi \wedge \neg \delta) = R_\Delta(\varphi)$

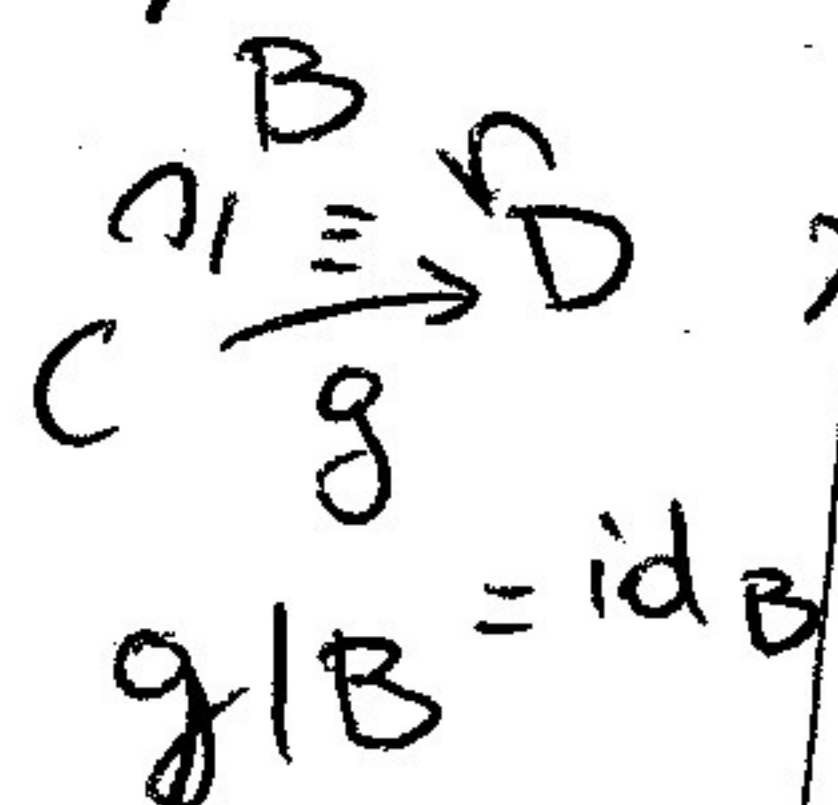
Let  $a_n \models q|_B a_n$  for  $n=0, 1, \dots, \omega$

choose  $c' \in M$  with  $\text{tp}(c'/a) = \text{tp}(c/a)$

Claim  $I = \{a_n : n < \omega\}$  is indiscernible  $/ B$

Then  $R_\Delta(\varphi \wedge \delta(x, c')) = R_\Delta(\varphi \wedge \neg \delta(x, c')) = R_\Delta(\varphi)$

[hint: as with Morley sequence]  
 crucial point: if

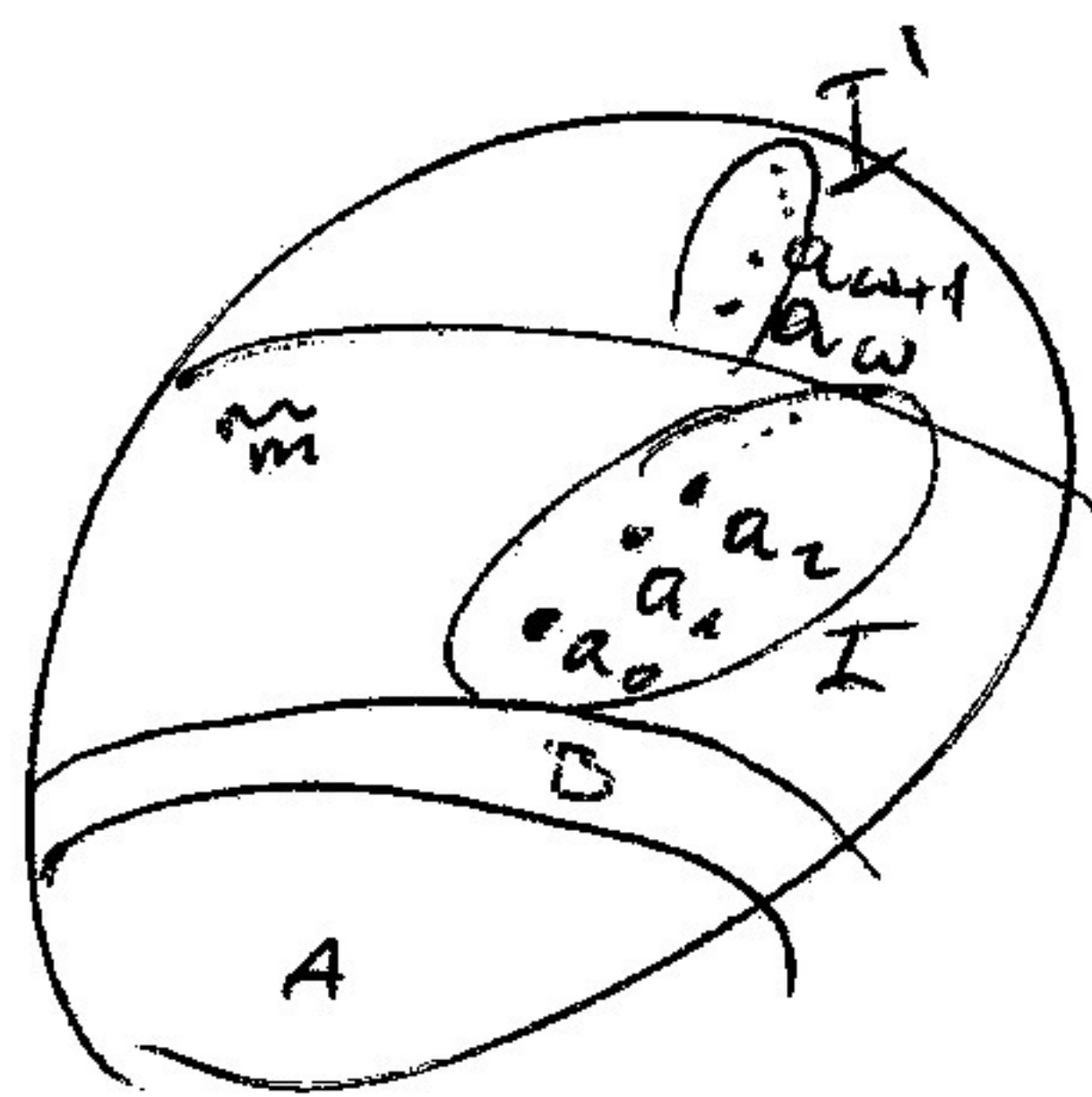


$\varphi$  (or)  $\varphi$

then  $g(q|_C) = q|_D$

Claim  $q = \text{Av}(I/M)$ . ~~Let  $C \in \mathcal{M}$  and  $\varphi(x, c) \in \text{Av}(I/C)$~~  (4)

Pf. Let  $\varphi(x, m) \in q$ . Let  $a_{\omega+k} \in B_{a_{\omega+k}}^m$  for all  $k=0, 1, \dots < \omega$



$J = I \cup I'$ : also indiscernible /  $\mathcal{B}$

But  $I' \in \varphi(J, m)$ , thus infinite in  $J$ , thus cofinite in  $J$ , thus cofinite in  $I$ , thus  $\varphi(x, m) \in \text{Av}(I/M)$   $\square$

Claim  $I$  is based on  $A$ .

Pf Let  $C \in \mathcal{M}$  and  $\varphi(x, c) \in \text{Av}(I/C)$ . Want:  $\varphi(x, c)$  def /  $A$ .

Let  $c' \in M$  with  $\text{tp}(c'/A) = \text{tp}(c/A)$ . So  $\varphi(x, c)$  def /  $A \Leftrightarrow \varphi(x, c')$  def /  $A$ ,  
 $\uparrow$   
 $\kappa^+$ -saturated

$\varphi(x, c') \in \text{Av}(I/c') \subseteq \text{Av}(I/M) = q$  nf ext. of  $p$

So  $\varphi(x, c')$  def /  $A$   $\square$

Choose  $f \in \text{Aut}(M/A)$  with  $f(a_0) = a$ . So  $a \in f[I] \leftarrow$  an  $A$ -indisc. set based on  $A$ .

Lemma  $\square$

Lemma Assume  $|B| > \kappa := (|A|+2)^{\kappa_0}$ . Then  $\exists I \subseteq B$  infinite, indiscernible /  $A$

Proof For a type  $p(x)$  let  $\vec{R}(p) = \langle (R_\Delta(p), M_{\text{tp}_\Delta(p)}) : \Delta \subseteq L \text{ finite} \rangle$

$\forall p \in S(A) \exists A_0 \subseteq A$  c.t.b.e.  $\vec{R}(p) = \vec{R}(p|_{A_0})$ .

Let  $B_0 \subseteq B$  c.t.b.e. Let  $p \in S(A \cup B_0)$  where  $B_0 \subseteq B$  s.t. c.t.b.e.

$p(B) = \{b \in B : b \models p\}$  has power  $> \kappa$  and

$\vec{R}(p)$  is minimal possible (lexicographically) (over all  $B_0, p$ )

$\forall B_1 \subseteq B$  c.t.b.e.  $\exists p' \in S(A \cup B_1)$  ( $\vec{R}(p') = \vec{R}(p)$  &  $|p'(B)| > \kappa$ )  $\square$

If  $\mu$  is a measure let  $\mathcal{P} = \{p' \in S(AB_n) : p \subseteq p'\}$

$$p' \supseteq p \Rightarrow \vec{R}(p') \subseteq \vec{R}(p)$$

$$\exists p' \in \mathcal{P} \text{ with } |\mathcal{P}'(B)| > \aleph \text{ (as } p(B) = \bigcup_{\substack{p' \in \mathcal{P} \\ |P| \leq \aleph}} p'(B) \text{)}$$

$$\text{Then } \vec{R}(p') = \vec{R}(p)$$

Uniqueness of  $p'$ :  $Mlt_{\Delta}(p') = Mlt_{\Delta}(p)$  for every  $\Delta \in L$  finite

If there's other  $p''$  like that then

$$p' \neq p'' \Rightarrow \exists \Delta \text{ formula } \Rightarrow Mlt_{\Delta}(p') < Mlt_{\Delta}(p)$$

Let  $I = \{b_n : n < \omega\} \in B$  st.  $\forall n \ p_n = tp(b_n / AB, b_{<n})$  is as in (\*)

clearly  $I$  is good.

Lemma Assume  $I = \{a_n : n < \omega\}$  indiscernible /  $A$ , based on  $A$ .

$$\text{Assume } \varphi(x, y) \in L \text{ and } \text{for } n < \omega \quad \varphi'(x, \bar{y}_{<n}) = \bigvee_{W \subseteq 2n} \bigwedge_{i \in W} \varphi(x, y_i)$$

Then for  $n \geq n_{\varphi}$   $\varphi'(x, \bar{a}_{<n})$  is almost over  $A$ .

[that is, it is over  $acl^{eq}(A)$ ]

$\uparrow$  recall  
 $\downarrow$   
 $\{f(\varphi'(x, \bar{a}_{<n})) : f \in \text{Aut}(M/A)\}$   
is finite

Theorem (Shelah, finite equivalence relation thm)

Assume  $p \neq q \in S(M)$  def /  $A$ . Then  $\exists E \in FE(A) \quad p(x) \cup q(y) \vdash \neg E(x, y)$

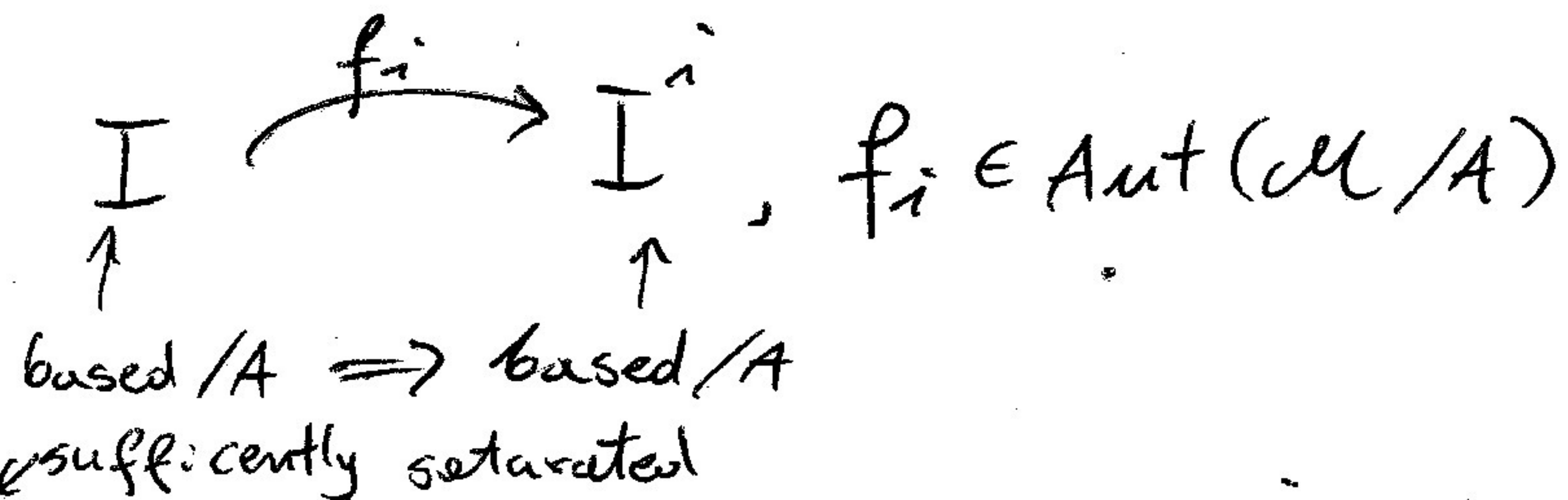
13.06.2022 Lemma  $I = \{a_n : n < \omega\}$  indiscernible /  $A$ , based on  $A$ ,  $\varphi(x, y) \in L^1$

$\varphi'(x, \bar{y}_{2n}) = \bigvee_{W \subseteq 2n} \bigwedge_{i \in W} \varphi(x, y_i)$ . Then for  $n \geq n_\varphi$ ,  $\varphi'(x, \bar{a}_{2n})$  is almost over  $A$   
 "  
 $\langle y_1, \dots, y_{2n-1} \rangle$   $|M| \geq n$

Proof (A.a.) Let  $\kappa = (|A|+2)^{\aleph_0}$ . Let  $\{\bar{a}_{2n}^i : i < (2^\kappa)^+\}$  s.t.

$tp(\bar{a}_{2n}^i / A) = tp(\bar{a}_{2n} / A)$  and  $\{\varphi'(x, \bar{a}_{2n}^i) : i < (2^\kappa)^+\}$  are pairwise non-equivalent.

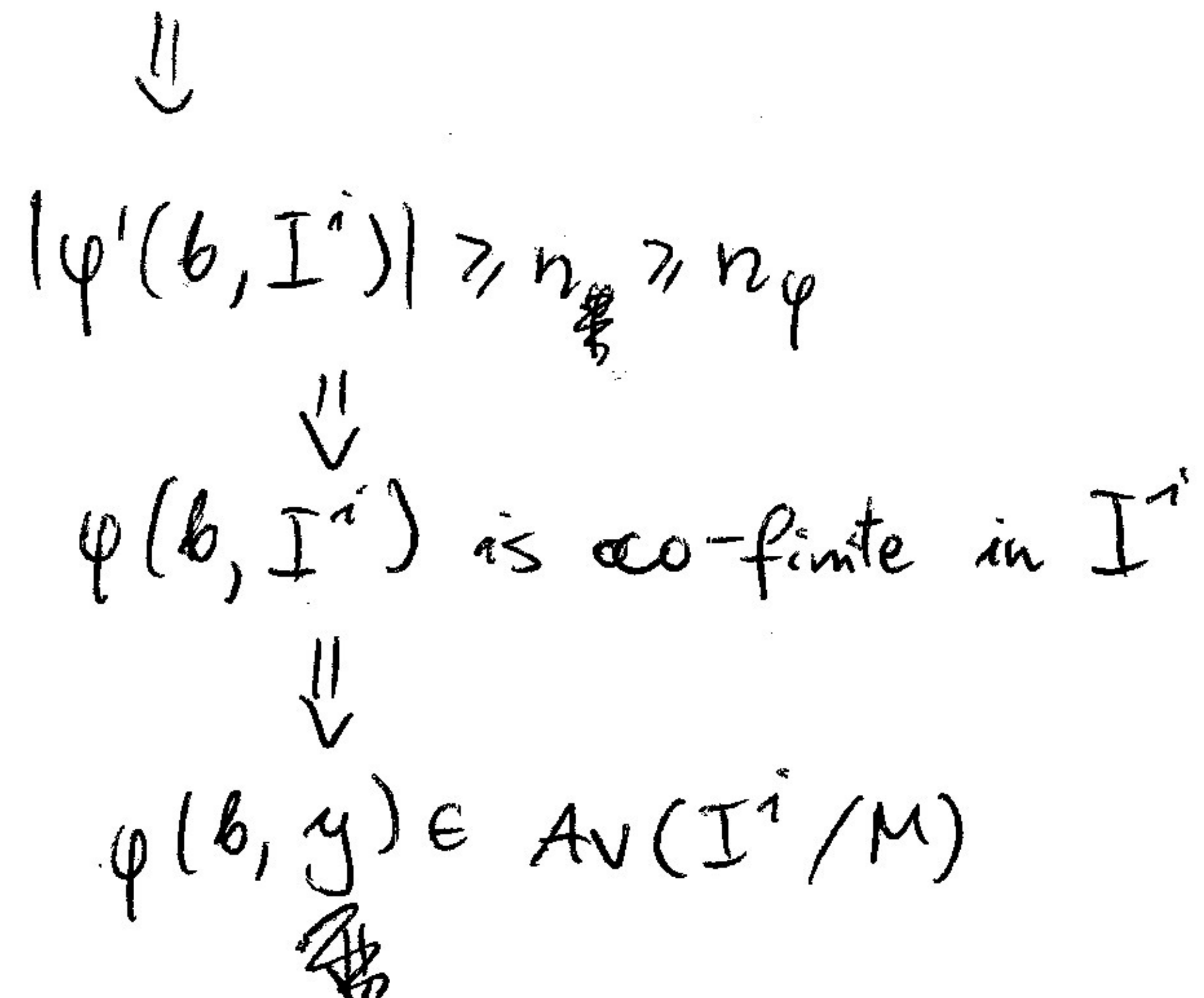
$$\bar{a}_{2n}^i \in I^i = \{a_n^i : i < \omega\}$$



Let  $M \supseteq A \cup \bigcup_i I_i$ . Let  $p_i = \text{Av}(I^i / M)$ ,  $i < (2^\kappa)^+$

Claim If  $i \neq j < (2^\kappa)^+$ , then  $p_i \neq p_j$

Pf Let  $b \in M$  s.t.  $\underbrace{\varphi'(b, \bar{a}_{2n}^i)} \wedge \neg \varphi(b, \bar{a}_{2n}^j)$



Likewise  $\neg \varphi(b, y) \in \text{Av}(I^j / M)$  □

Define  $\sim$  on  $M$ :

$$b \sim_0 b' \Leftrightarrow \exists y \text{ infinite, indiscernible } / A, \text{ both } \langle b y^n \rangle, \langle b' y^n \rangle \text{ are indiscernible } / A$$

Let  $\sim$  be the transitive closure of  $\sim_0$ .

$$\text{So: } b \sim b' \Leftrightarrow \exists b = b_0, \dots, b_n = b' \wedge b_i \sim_0 b_{i+1}$$

Subclaim  $b \sim b', p \in S(M)$  def / A,  $\Rightarrow \psi(x, b) \in p \Leftrightarrow \psi(x, b') \in p$  2

Pf May assume  $b \sim_0 b'$  (A.c.) <sup>finiteness  $\psi(x, y) \in L$</sup>  Suppose  $\psi(x, b) \in p \wedge \neg \psi(x, b') \in p$ .  
 Let  $c \models p$ .  $\underbrace{\psi(c, y) \in Av(J/Ac)}_{\text{consider this}}$  or  $\neg \psi(c, y) \in Av(J/Ac)$

Then there is  $b^* \in J$  s.t.  $\models \psi(c, b^*)$ .

Look at  $J' = \{b^*\} \cup J$ .  $b^* \neq b' \in J'$ : infinite indiscernible / A  
 and  $\underbrace{\psi(x, b^*) \wedge \neg \psi(x, b')} \in p$ , so  $p$  forks / A ⊥

but this forks / A (see problem from list 3)

Cases:

1°  $\sim$  has  $\leq \kappa$  classes on  $M$ .

Let  $B \subseteq M$ ,  $|B| \leq \kappa$ ,  $B$  contains representatives of all  $\sim$ -classes,  
 $A^M$

by previous claim, for  $i \neq j$   $p_i|B \neq p_j|B$ . So  $|S(B)| \geq (2^\kappa)^+$ ,  
 but  $|S(B)| \leq 2^\kappa$  ⊥

2°  $\sim$  has  $> \kappa$  classes on  $M$ .

Let  $\{b_\alpha : \alpha < \kappa^+\} \subseteq M$ , pairwise ~~non-~~ <sup>non-</sup> ~~non-~~ <sup>non-</sup> equivalent

By the previous lemma:  $\exists J \subseteq \{b_\alpha : \alpha < \kappa^+\}$   
 infinite, indiscernible / A

$b_\alpha, b_\beta \in J$  for some  $\alpha < \beta < \kappa^+$

$\Downarrow$   
 $b_\alpha \sim_0 b_\beta = b_\alpha \sim b_\beta$  ⊥

Thm (Shelah)  $p \neq q \in S(\mathcal{U})$ , both def / A

$\Downarrow$   
 $\exists E \in FE(A) \ p(x) \cup q(y) \vdash \neg \exists (x, y)$

Proof  $\psi(x, b) \in p, \neg \psi(x, b) \in q$ . Let  $I \subseteq \mathcal{U}$  indiscernible / A, based / A  
 with  $b \in I = \{a_n : n < \omega\}$

This implies (by the claim):

$$\forall c \in I (\varphi(x, c) \in p \ \& \ \neg \varphi(x, c) \in q)$$

$$[\text{or else } \varphi(x, b) \wedge \neg \varphi(x, c), \neg \varphi(x, b) \wedge \varphi(x, c)]$$

fork for every  $c \in I$

Let  $m \succ m_\varphi$ .  $\varphi'(x, \bar{a}_{\leq n})$  is almost / A,  $\neg \varphi'(x, \bar{a}_{\leq n}) \in q$

So  $\exists E \in FE(A)$  s.t.  $\varphi'(\mathcal{M}, \bar{a}_{\leq n}) =$  a union of E-classes.

$$p(x) \cup q(y) \vdash \varphi'(x, \bar{a}_{\leq n}) \wedge \neg \varphi'(y, \bar{a}_{\leq n}) \vdash \neg E(x, y) \quad \square$$

Def  $p \in S(A)$  is stationary if  $\forall B \supseteq A \exists! q \in S(B)$

Corollary (1)  $p \in S(\text{acl}^{eq}(A)) \Rightarrow p$ : stationary  $p \upharpoonright^{\text{nf}}$

(2)  $p \in S(M) \Rightarrow p$ : stationary

Proof (1) Let  $B \supseteq \text{acl}^{eq}(A)$ . Suppose  $p \upharpoonright^{\text{nf}} \in S(B)$ .

Then  $q_i \subseteq q_i' \in S(\mathcal{M})$   
 $\nwarrow$  still dnf /  $\text{acl}^{eq}(A)$

$$q_1' \neq q_2' \Rightarrow \exists E \in FE(\text{acl}^{eq}(A)) \ q_1'(x) \cup q_2'(y) \vdash \neg E(x, y) \quad (t)$$

Let  $a_1, \dots, a_k$ : names of E-classes.  $a_1, \dots, a_k \in \text{acl}^{eq}(\text{acl}^{eq}(A)) = \text{acl}^{eq}(A)$   
 $\uparrow$   
 $\mathcal{M}$

$$"x/E = a_i" \in L(\text{acl}^{eq}(A)), \ i = 1, \dots, k$$

for some  $1 \leq i < j \leq k$   $"x/E = a_i" \in q_1'$ ,  $"x/E = a_j" \in q_2'$

By (t):  $i \neq j$ . But  $q_1' \upharpoonright \text{acl}^{eq}(A) = p = q_2' \upharpoonright \text{acl}^{eq}(A)$ , so  $i = j$ .  $\downarrow$

(2)  $M = \text{acl}^{eq}(M)$

$\square$

Corollary (1)  $A \subseteq B$ ,  $p \in S(B)$ . Then  $p \text{ dnf } A \Leftrightarrow \forall \Delta \subseteq L \text{ finite } R_\Delta(p) = R_\Delta(p|A)$

(2)  $p \in S(A)$  is stationary  $\Leftrightarrow \forall \Delta \in L \text{ finite } \text{Mlt}_\Delta(p) = 1$

(3)  $p \in S(A)$  is stationary  $\Rightarrow (\forall \Delta \in L \text{ finite } \forall B \supseteq A \ R_{\Delta,2}(p|B) = R_{\Delta,2}(p))$   
 and  $p|B$  is definable over  $A$  (unique nf extension of  $p$  is  $S(B)$ )

Proof  $\Leftarrow$  clear.  $\Rightarrow$ :  $p \subseteq p' \in S(\text{acl}^{\text{eq}}(B) \cup \text{acl}^{\text{eq}}(A))$  ~~s.t.  $p' \text{ dnf } A$~~

Then (\*)  $\forall \Delta \subseteq L \text{ finite } R_\Delta(p') = R_\Delta(p)$  (exercise)

$$R_\Delta(p'|\text{acl}^{\text{eq}}(A)) = R_\Delta(p|A)$$

~~By~~ We can assume  $p' \text{ dnf } A$ , hence  $p' \text{ dnf } \text{acl}^{\text{eq}}(A)$

Therefore we may assume  $A = \text{acl}^{\text{eq}}(A)$ ,  $B = \text{acl}^{\text{eq}}(B)$

Shall prove  $\forall \Delta \subseteq L \text{ finite } R_\Delta(p') = R_\Delta(p'|\text{acl}^{\text{eq}}(A))$

Let  $q = p|_A$ .  $A = \text{acl}^{\text{eq}}(A)$ , so  $q$  is stationary and

$$q \in p \in S(B) = S(\text{acl}^{\text{eq}}(B)).$$

Let  $\Delta$ : finite. Want:  $R_\Delta(p) = R_\Delta(q)$   
 $\leftarrow$   
obvious

$\geq$ : Let  $k = R_\Delta(q)$ ,  $k \in \mathbb{N}$ , wlog  $\Delta = \{1, \dots, k\}$ , closed under

:  
□

Comment: similarly one can show that in this situation (i.e. when  $A = \text{acl}^{\text{eq}}(A)$ ,  $B = \text{acl}^{\text{eq}}(B)$ )  $\text{Mlt}_\Delta(p) = \text{Mlt}_\Delta(q)$

Pf (3) As in (1) ("lifting") we show that  $R_{\Delta,2}(p|B) = R_{\Delta,2}(p)$   
 for every  $\Delta \subseteq L$  (but easier, binary tree)  
 $\leftarrow$   
finite

By sketch proof of definability lemma  $p|B$  is definable  $A$  □

Pf (2)  $\Leftarrow$ : Let  $B \supseteq A$ . Suppose

$$S(B) \ni q \supseteq$$

$$\# \text{ of } p \in S(A)$$

$$S(B) \ni q' \supseteq$$

By (1)  $R_\Delta(q) = R_\Delta(q') = R_\Delta(p)$  for every  $\Delta \subseteq L$  finite

Let  $\delta(x, \bar{c}) \in q, \neg \delta(x, \bar{c}) \in q'$ . Let  $\Delta = \{\delta(x, \bar{c})\} \subseteq L$

$$R_\Delta(q) = R_\Delta(q') = R_\Delta(p) \Rightarrow \text{Mlt}_\Delta(p) > 1 \quad \Downarrow$$

$\Rightarrow$ : Let  $q \notin p \in S(A)$ : stationary.

$$\# \text{ of } q \in S(\mathcal{M}), \text{ so } \text{Mlt}_\Delta(q) = 1$$

by the comment  $\text{Mlt}_\Delta(q) = \text{Mlt}_\Delta(p)$

Corollary Assume  $p \in S(A), \text{RM}(p) < \infty, p \subseteq q \in S(B)$ . Then

$$q \text{ dnf } / A \Leftrightarrow \text{RM}(p) = \text{RM}(q)$$

Corollary Assume  $p \in S(\mathcal{M})$ .

$$(1) \text{RM}(p) < \infty \Rightarrow \text{Mlt}_{\text{RM}}(p) = 1$$

$$(2) \forall \Delta \subseteq L \text{ finite } \text{Mlt}_\Delta(p) = 1$$

Corollary  $p \in S(B)$ . Then  $p \text{ dnf } / A \Leftrightarrow p$  is definable /  $\text{ed}^{\text{eq}}(A)$

FORKING INDEPENDENCE

$A, B, C \subseteq \mathcal{M}, a \in \mathcal{M}$

Def  $a \perp_C B$  [ $a$  is independent from  $B$  over  $C$ ] if  $\text{tp}(a/B \cup C)$

Def  $A \perp_C B$  if  $\forall \bar{a} \subseteq A \bar{a} \perp_C B$  dnf /  $C$

$\perp$  : has properties