25.04.22 Pf. c.d. $\sigma = \tau_1 \circ \tau_2 \circ \ldots \circ \tau_l$: product of transpositions of consecutive numbers.

Let $\sigma_t = \tau_1 \circ \ldots \circ \tau_t$, $t = 1, \ldots, l$, $\sigma_0 = id$. Then $\models \varphi(\bar a_{i_{\sigma_0}})$ and $\models \varphi(\bar a_{i_{\sigma_l}})$. For some $0 \le t < l$:

$$\models \varphi(\bar a_{i_{\sigma_t}}) \wedge \neg \varphi(\bar a_{i_{\sigma_{t+1}}})$$

Let $\sigma' = \sigma_t$, $\sigma'' = \sigma_{t+1} = \sigma' \circ \tau_{t+1}$, $\tau_{t+1} =: \tau$

Then $\models \underbrace{\varphi(a_{i_{\sigma'(1)}}, \ldots, a_{i_{\sigma'(k)}})}_{\psi(a_{i_1}, \ldots, a_{i_k})}$ (by renaming)

But $\models \neg \varphi(a_{i_{\sigma'(\tau(1))}}, \ldots, a_{i_{\sigma'(\tau(k))}})$,

so $\models \neg \psi(a_{i_{\tau(1)}}, \ldots, a_{i_{\tau(k)}}) \wedge \psi(a_{i_1}, \ldots, a_{i_k})$
$(\forall i_1 < i_2 < \ldots < i_k \in I)$

e.g. $\tau = (3,4)$ and $k > 4$. Choose $i_1 < i_2 < i_5 < \ldots < i_k$.

Let $\chi(x_3, x_4) = \varphi(a_{i_1}, a_{i_2}, x_3, x_4, a_{i_5}, \ldots, a_{i_k}) \in L(\bar a_J)$ $J \subseteq_{\text{dense}} I$

Will show: $|S(\bar{a}_J)| > \varkappa$, $|\bar{a}_J| \leq \varkappa$.

Namely: let $i < i' \in (i_2, i_5)_I$. Then

$tp(a_i/\bar{a}_J) \neq tp(a_{i'}/\bar{a}_J)$ ← enough

$\exists$ $i_1 < i_2 < i < j < i' < i_5 < \ldots < i_k$.

Then $\models \varphi(a_i, a_j)$, $\models \neg\varphi(a_{i'}, a_j)$

because $i < j$ because $i' > j$

Then $\varphi(x, a_j) \in tp(a_i/\bar{a}_J)$,

$\neg\varphi(x, a_j) \in tp(a_{i'}/\bar{a}_J)$. $\lightning$

Remark (T: stable, $\varphi(x, \bar{y}) \in L$). There is

$n < \omega$ $\forall I \subseteq \mathcal{M}$ $\forall \bar{a} \subseteq \mathcal{M}$ one of the

indiscernible
infinite

sets $I^+_{\bar{a}} = \{c \in I : \models \varphi(c, \bar{a})\}$

$I^-_{\bar{a}} = \{c \in I : \models \neg\varphi(c, \bar{a})\}$

has $\leq n$ elements.

Pf. (a.c.) If there's no such n, then $\forall n \exists I_n \exists \bar{a}_n \; |I^+_{n,\bar{a}_n}|, |I^-_{n,\bar{a}_n}| > n$. Let $\{c_i, i<\omega\}$: new constant symbols.

$\bar{d}$: , $|\bar{d}| = |\bar{y}|$.

Let $T' = T \cup \{ "\underbrace{\{c_i : i<\omega\}}_{I}$ is indiscernible in $T" \}$

$\cup \{ "I^+_{\bar{d}} = \{c_{2i} : i<\omega\}" \} \cup \{ "I^-_{\bar{d}} = \{c_{2i+1} : i<\omega\}" \}$.

wlog $|I| = \varkappa$

T' is consistent, so it has a model M'.

Wlog $M' \restriction_L \prec \mathcal{M}$. So $I \subseteq \mathcal{M}$ indiscernible,

$\overset{\cup}{I}, \bar{d}$ $\quad I^+_{\bar{d}}, I^-_{\bar{d}} \subseteq \mathcal{M} \Rightarrow |S(I)| = 2^{\varkappa} > \varkappa$.

Pf. $|I^+_{\bar{d}}|, |I^-_{\bar{d}}| = \varkappa$. For any $I' \subseteq I$ with $|I'| = |I \setminus I'| = \varkappa$ $\exists f \in Aut(\mathcal{M}) \left[f[I] = I, f(I') = I^+_{\bar{d}}, f(I \setminus I') = I^-_{\bar{d}} \right]$. Let $\bar{a}_{I'} = f^{-1}(\bar{d})$.

If $I' \neq I''$, then $tp(\bar{a}_{I'}/I) \neq tp(\bar{a}_{I''}/I)$.

$\aleph_1$-categorical theories

Examples. 1. ACF_p,

2. $Th(V, +, k)_{k\in K}$ ← ctble field

inf. vec. space

3. $Th(\mathbb{N}, S)$, $Th(\mathbb{Z}, S)$, $Th(\mathbb{N}, =)$ ← "no structure"

4. $Th(G, +)$, G: torsion free divisible abelian

5. $Th(\mathbb{Z}_p^{\aleph_0}, +)$

6. $Th(G)$ where G: an algebraic group

Theorem If $\kappa > \aleph_0$ and T is κ-categorical, then T is $\aleph_0$-stable.

Lemma $\forall \kappa \geq \aleph_0 \; \exists M \models T, \; \|M\| = \kappa \; \forall A \subseteq M, \; |A| < \aleph_0$

$$|\{p \in S(A) : p(M) \neq \emptyset\}| \leq \aleph_0 .$$

Pf. thm (Lemma ⟹ thm) (A.c.) Suppose T is not $\aleph_0$-stable. $\exists_{\text{ctble}} N \models T \; |S(N)| > \aleph_0$

But T: κ-categorical. $N \prec N_1$ s.t. $\|N_1\| = \aleph_1$,

$|\{p \in S(N) : p(N_1) \neq \emptyset\}| = \aleph_1$.

$N_1 \prec N_\varkappa$ ← of power $\varkappa$. Let $M_\varkappa$: a model from the lemma. Then $M_\varkappa \cong N_\varkappa$ ↯.

<u>Pf.</u> (lemma) $T \subseteq T^S$: the skolemization in $L^S \supseteq L$.

Let $I = \{a_n : n < \omega\}$: an infinite order indisc. set in T^S. $I \subseteq \mathcal{J} = \{a_\alpha : \alpha < \varkappa\}$ (stretching).

Then $\mathcal{J} \subseteq N^S \models T^S$. Let $M^S = \mathcal{H}(\mathcal{J}) \prec N^S$, i.e.

$M^S = \{ t^{N^S}(\vec{j}) : t(\bar{x})$: a term in L^S, $\underset{\text{increasing}}{\vec{j}} \subseteq \mathcal{J} \}$

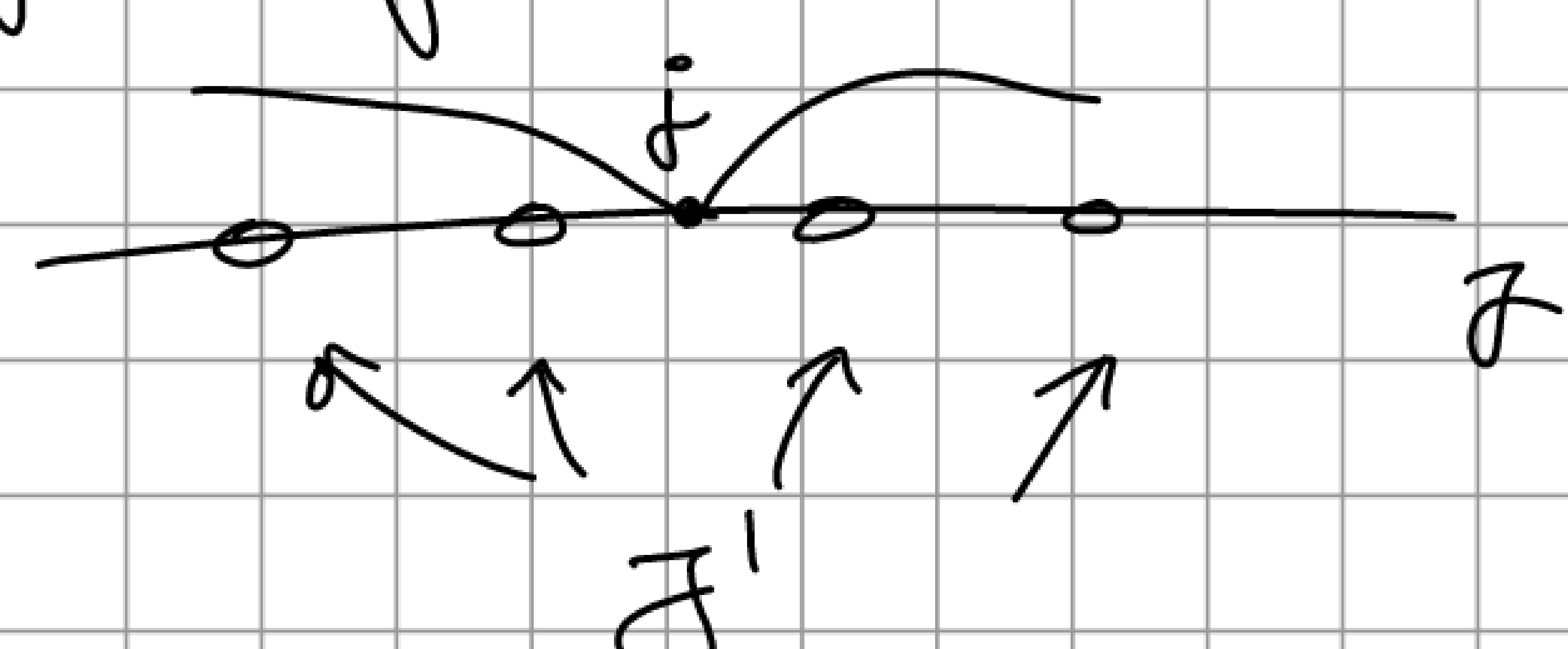
Will show that M^S satisfies the conditions on the lemma for T^S (then $M := M^S \restriction L$ satisfies conditions for T)

Let $\underset{\text{ctble}}{A} \subseteq \mathcal{H}(\mathcal{J})$. Wlog $A = \mathcal{H}(\mathcal{J}')$ for some $\mathcal{J}' \subseteq \mathcal{J}$.

$M^S \ni a = t(j_1, \dots, j_k)$ for some $j_1 < \dots < j_k \in \mathcal{J}$.

$\sim$ eq. rel. on $\mathcal{J}$:

$j \sim j' \overset{\text{def.}}{\iff}$ j and j' determine the same cut in $\mathcal{J}'$.



$|\mathcal{J}/\sim|$: ctble as $|\mathcal{J}'|$ ctble. Now

$$(j_1, \ldots, j_k) \sim (j_1', \ldots, j_k') \overset{\text{def}}{\iff} \bigwedge_{1 \leq i \leq k} j_i \sim j_i'$$

$<\ <$ $\qquad$ $<\ <$

again ctbly many classes

Let $\underset{t(j_1,\ldots,j_k)}{\underset{\parallel}{a}}, \underset{t'(j_1',\ldots,j_k')}{\underset{\parallel}{a'}} \in M^s$. $(\ast)$ If $t = t'$ and $(j_1, \ldots, j_k) \sim (j_1', \ldots, j_k')$ then $tp^{M^s}(a/A) = tp^{M^s}(a'/A)$.

(it obvs. implies the lemma).

Pf $(\ast)$: $\varphi(x) \in L(A)$, $b = \bar{t}(\underset{\cup\, \mathcal{J}'}{\bar{j}''})$.

$\varphi(x) = \varphi(x, b)$, $b \subseteq A$

$\varphi(x, \bar{y}) \in L$

$$\models \varphi(a, b) \iff \models \varphi(t(\bar{j}), \bar{t}(\bar{j}'')) \overset{\bar{j} \sim \bar{j}' + \text{order indiscernibility of } \mathcal{J}}{\iff} \models \varphi(t(\bar{j}'), \bar{t}(\bar{j}''))$$

$\iff \models \varphi(a', b)$ ∎

<u>Remark</u>

Let T: $\aleph_0$-stable, $p \in S(A)$, $RM(p) < \infty$, $Mlt(p) = 1$.

Then $\forall B \underset{\mathfrak{M}}{\underset{\cap\!\shortparallel}{\supseteq}} A$ $\exists!\, p_B \in S(B)$ $RM(p_B) = RM(p)$

Def. $I = \{a_\alpha : \alpha < \beta\} \subseteq \mathcal{M}$ is a Morley sequence in $p \in S(A)$ if $\forall \alpha < \beta$ $a_\alpha \models p_{Aa_{<\alpha}} \in S(Aa_{<\alpha})$

$$p_{Aa_{<\alpha}} \supseteq p,\ \mathrm{Mlt}(1), \quad p \in S(A)$$

Remark A Morley sequence in p is indiscernible over A.

Pf. Enough to show order-indiscernibility. Wlog $I = \{a_\alpha : \alpha < \beta\}$, $\beta = \kappa \geq \aleph_0$, $\kappa \in CN$. Induction on $k < \omega$:

$$\forall \alpha_1 < \dots < \alpha_k < \kappa, \ \beta_1 < \dots < \beta_k \quad tp(a_{\alpha_1} \dots a_{\alpha_k} / A) = tp(a_{\beta_1} \dots a_{\beta_k} / A)$$

Step $k \to k+1$, $\alpha_{k+1} > \alpha_k$, $\beta_{k+1} > \beta_k$.

$$p \subseteq tp(a_{\alpha_{k+1}} / Aa_{\alpha_1} \dots a_{\alpha_k}) \subseteq p_{Aa_{<\alpha_{k+1}}}$$

the same

RM, Mlt $= 1$ $\Longrightarrow tp(a_{\alpha_{k+1}} / Aa_{\alpha_1} \dots a_{\alpha_k}) = p_{Aa_{\alpha_1} \dots a_{\alpha_k}}$

Likewise $tp(a_{\beta_{k+1}} / Aa_{\beta_1} \dots a_{\beta_k}) = p_{Aa_{\beta_1} \dots a_{\beta_k}}$

Consider $f : Aa_{\alpha_1}\dots a_{\alpha_k} \longrightarrow Aa_{\beta_1}\dots a_{\beta_k}$ s.t.

$f|_A = id$, $f(a_{\alpha_i}) = a_{\beta_i}$, $i = 1, \dots, k$. Then f: element.

and $f(\underbrace{p_{Aa_{\alpha_1}\dots A_{\alpha_k}}}_{\supseteq\, p}) = p_{Aa_{\beta_1}\dots a_{\beta_k}}$

and has the same RM as RM(p)

$\Downarrow$

$f \cup \{\langle a_{\alpha_{k+1}}, a_{\beta_{k+1}} \rangle\}$ elementary

So $tp(a_{\alpha_1}\dots a_{\alpha_{k+1}}/A) = tp(a_{\beta_1}\dots \beta_{k+1}/A)$. ∎

<u>Thm</u> (Morley, Shelah) If T: $\aleph_0$-stable and $\kappa \geq \aleph_0$ then T has a saturated model of power κ.