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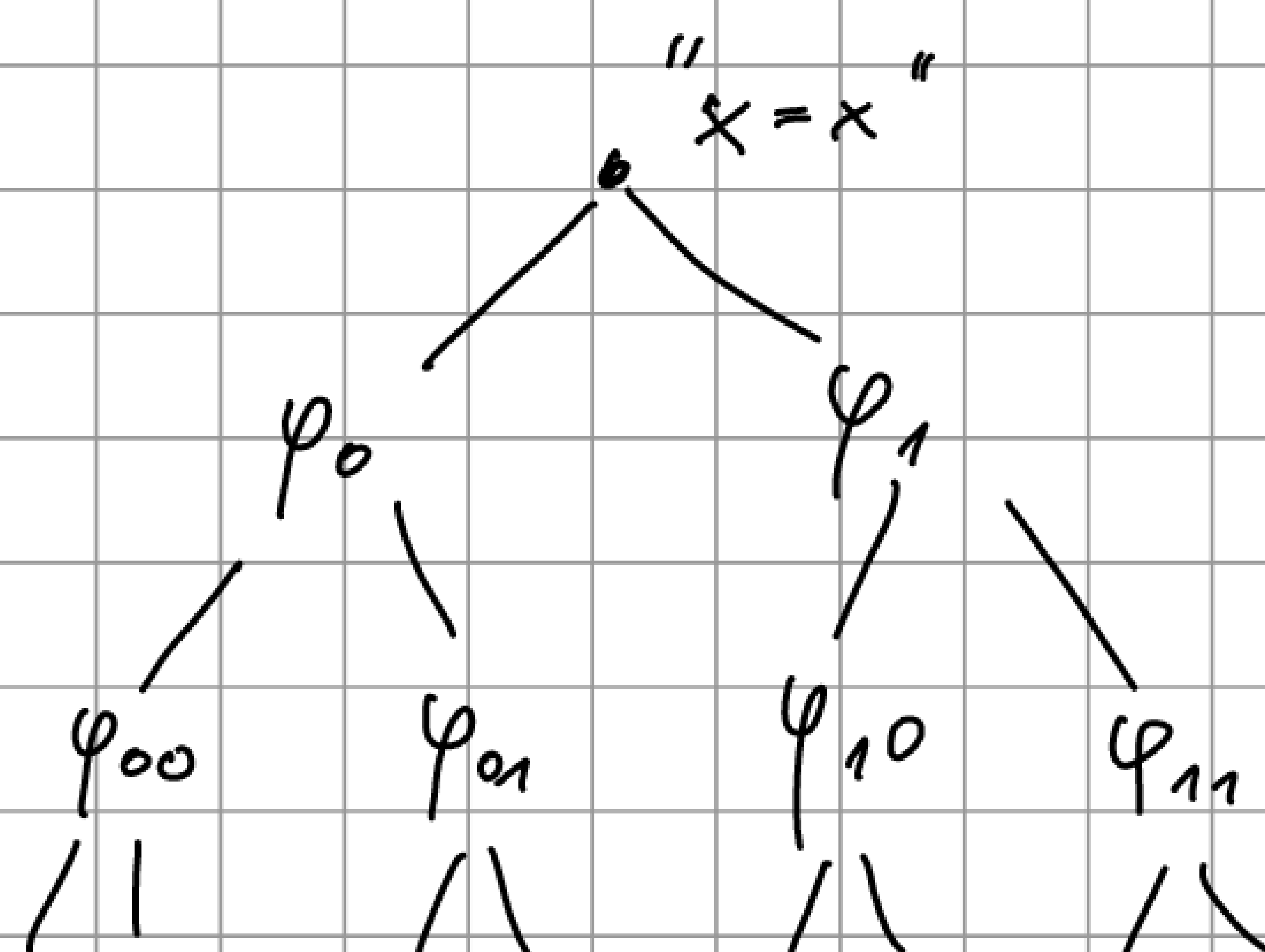
Thm T is $\aleph_0$-stable $\iff RM(x=x) \in \mathrm{Ord}$

Pf. "$\Rightarrow$" last time. "$\Leftarrow$" Suppose T is not $\aleph_0$-stable.

$|S(A)| = 2^{\aleph_0}$, $S(A)$: a Polish space

$\Rightarrow S(A)^{(\infty)} \neq \emptyset \Rightarrow$ get a binary tree of formulas

$$\{\varphi_\eta(x)^{\in L_1(A)} : \eta \in 2^{<\omega}\}$$



Let $\alpha = \min\{RM(\varphi_\eta) : \eta \in 2^{<\omega}\}$.

If $\eta \subseteq \nu \in 2^{<\omega}$, then $RM(\varphi_\nu) = \alpha$.

$RM(\varphi_\eta) \not\geq \alpha + 1$, so:

$$(*)\ \exists m \in \mathbb{N}\ \neg\exists \psi_1 \dots \psi_m \vdash \varphi_\eta \ \bigwedge_{i=1}^{m} RM(\psi_i) \geq \alpha$$

Take n s.t. $2^n > m$. φ_ν, $\eta \subseteq \nu \in 2^{<\omega}$, $|\nu| = |\eta| + n$

contradicts $(*)$. ∎

<u>Def.</u> <u>Multiplicity</u> of $\varphi(\bar{x}) \in L(\mathcal{M})$ s.t. $\underbrace{RM(\varphi(\bar{x}))}_{\alpha \in Ord} < \infty$:

$Mlt(\varphi) =$ the largest $m \in \mathbb{N}$ s.t.

$$\exists \psi_1 \ldots \psi_m \bigwedge_{i=1}^{m} RM(\psi_i) \geq \alpha$$

<u>Properties</u> • $RM(\varphi_1) = RM(\varphi_2)$ and $\varphi_1(\mathcal{M}) \cap \varphi_2(\mathcal{M}) = \emptyset$, then $Mlt(\varphi_1 \vee \varphi_2) = Mlt(\varphi_1) + Mlt(\varphi_2)$

• If $RM(\varphi_1) < RM(\varphi_2) < \infty$, then $Mlt(\varphi_1 \vee \varphi_2) = Mlt(\varphi_2)$

<u>Example</u> If $\underbrace{\varphi(\bar{x}) \text{ is algebraic}}_{\Updownarrow \; RM(\varphi)=0}$, then $Mlt(\varphi) = |\varphi(\mathcal{M})|$

<u>Def.</u> Assume $p(\bar{x})$: a type with $RM(p(\bar{x})) < \infty$.

$$Mlt(p(\bar{x})) = \min\{ Mlt(\varphi(\bar{x})) : p \vdash \varphi \text{ and } RM(\varphi) = RM(p)\}$$

<u>Def.</u> $p(\bar{x})$ is <u>stationary</u>, if $Mlt(p(\bar{x})) = 1$

<u>Remark</u> Assume $p(\bar{x})$: a type over A. Then $\exists q \in S(A)$, $p(\bar{x}) \subseteq q(\bar{x})$ s.t. $RM(p) = RM(q)$

<u>Pf.</u> Let $q_0 = \{\varphi(\bar{x}) \in L(A) : RM(p \cup \{\neg\varphi\}) < RM(p)\}$

- $q_0 \supseteq p$, if $\varphi \in p$, then $RM(p \cup \{\neg\varphi\}) = -1 < RM(p)$ ← consistent

$RM(p)$

- q_0: a consistent type: $\varphi_1, \dots, \varphi_n \in q_0$, $RM(p \cup \{\neg\varphi_i\}) < \overset{!!}{\alpha}$

Choose ψ with $p \vdash \psi$ and $RM(\psi) = \alpha$;

ψ_i with $p \vdash \psi_i$ and $RM(\psi_i \wedge \neg\varphi) < \alpha$.

Wlog $\psi = \bigwedge_{i=1}^{n} \psi_i$

⋮

Let $q_0 \subseteq q \in S(A)$, then $RM(q) = \alpha$. Clearly, $RM(q) \leq \alpha$. If $RM(q) < \alpha \Rightarrow q \vdash \varphi$ with some $RM(\varphi) < \alpha$. By compactness $\exists\, q' \subseteq q$ finite $q' \vdash \varphi$

$\Leftrightarrow \underbrace{\bigwedge q'}_{\chi(\bar{x}) \in q(\bar{x})} \vdash \varphi$, $\chi(\bar{x}) \vdash \varphi(\bar{x})$ so $RM(\chi(\bar{x})) < \alpha$,

then $RM(p \cup \{\chi(\bar{x})\}) < \alpha$

$$\Downarrow$$

$$\begin{cases} \neg\chi(\bar{x}) \in q_0(\bar{x}) \subseteq q(\bar{x}) \\ \chi(\bar{x}) \in q(\bar{x}) \end{cases} \quad \lightning$$

$\overbrace{\text{Example } RM(p)=0}^{\Leftrightarrow p:\text{ algebraic}}$, then $Mlt(p) = |p(\mathcal{M})|$

Example $T = ACF_p$, $\underset{\text{subfield}}{K \subseteq \mathcal{M}}$, $p \in S_1(K)$, $p = tp(a/K)$, $a \in \mathcal{M}$

a) p algebraic $\Leftrightarrow$ $RM(p)=0$,

$\underset{K[x]}{\overset{\cap}{(W)}} = I(a/K) \neq \{0\}$, "$W(x)=0$" $\in p(x)$,

$Mlt(p) = Mlt(W(x)=0) = \#\{\text{roots of } W \text{ in } \mathcal{M}\}$.

b) p: transcendental, then $I(a/K) = \{0\}$,

$RM(p) = 1 = RM(x=x)$

$Mlt(p) = Mlt(x=x) = 1$

So p: stationary.

c) $p = tp(\bar{a}/K)$, $\bar{a} = \langle a_1, \dots, a_n \rangle \subseteq \mathcal{M}$. By q.e. it is determined by $I(\bar{a}/K) \trianglelefteq K[X_1, \dots, X_n]$

$$\| $$

$$\{W(\bar{x}) \in K[\bar{x}] : W(\bar{a}) = 0\}.$$

Let $V_p = V(I(\bar{a}/K)) = \bigcap_{W \in I(\bar{a}/K)} Z(W) =$

$Z(W) = \{\bar{b} \in \mathcal{M}^n : W(\bar{b}) = 0\}$

$= \bigcap_{i=1}^{l} Z(W_i)$ $\leftarrow$ definable on $\mathcal{M}$ over K,

$I(\bar{a}/K) = (W_1, \dots, W_l)$

$\bar{a} \subseteq V_p \Rightarrow p(\bar{x}) \vdash$ "$\bar{x} \in V_p$"

$RM(p) = RM(V_p) = \dim V_p$

$Mlt(p) = Mlt(V_p) =$ The number of irreducible components of V_p of $\dim = \dim V_p$

In $T = Th(ACF_p)$: $RM(\bar{x}=\bar{x}) < \omega$

(Order) indiscernible sets:

Let $(I, \leq)$: a linear ordered set of indicies.

<u>Def.</u> $\{\bar{a}_i : i \in I\} \subseteq \mathcal{M}$: <u>order indiscernible over $A \subseteq \mathcal{M}$</u>, if $\forall k\ \forall i_1 < \ldots < i_k \in I$, $j_1 < \ldots < j_k \in I$ $tp(\bar{a}_{i_1} \ldots \bar{a}_{i_k} / A) = tp(\bar{a}_{j_1} \ldots \bar{a}_{j_k} / A)$

<u>Recall</u>: 1) Assume $p(\bar{x})$: a non-algebraic type over $A \subseteq \mathcal{M}$. Then $\exists \{\bar{a}_n : n < \omega\} \subseteq p(\mathcal{M})$ infinite order indiscernible

2) (stretching) Assume $\{a_i : i \in I\}$ order indiscernible $/A$, I: infinite, $(J, \leq)$: a linear ordering. Then $\exists \{b_j : j \in J\}$: order ind. $/A$ s.t. $\forall k\ \forall i_1 < \ldots < i_k \in I$ $\forall j_1 < \ldots < j_k \in I$

$tp(a_{i_1} \ldots a_{i_k} / A) = tp(b_{j_1} \ldots b_{j_k} / A)$

Pf. (1) by Ramsey thm.

(2) Let $b_j, j \in J$: new constant symbols.

$$T^* = T(A) \cup \{\varphi(b_{j_1}, \dots, b_{j_k}) : \varphi(\bar{x}) \in L_k(A),\ j_1 < \dots < j_k \in J$$
$$\text{and } \forall i_1 < \dots < i_k \in I \models \varphi(a_{i_1}, \dots, a_{i_k})\}.$$

T^* : consistent. $b_{j_1}, \dots, b_{j_k} \xrightarrow{\text{interpret as}} a_{i_1}, \dots, a_{i_k}$

$\Downarrow$

has a model M for any $i_1 < \dots < i_k \in I$

$$M = (\underbrace{M, a^M}_{\models T(A)}, b_j^M)_{\substack{a \in A \\ j \in J}} \models T^*$$

$$T(A) \Rightarrow \exists f : (M, a^M)_{a \in A} \xrightarrow{\equiv} (\mathcal{M}, a)_{a \in A}$$

$$a^M \overset{f}{\longmapsto} a$$

Let $b_j = f(b_j^M)$. $\{b_j : j \in J\} \subseteq \mathcal{M}$ is good.

Example (1) $M = (\mathbb{Q}, \leq) \leftarrow \mathbb{Q}$ is order indisc (indexed by itself)

(2) $T = ACF_p$, $\mathcal{M} \models T$, $\{a_i : i \in I\} \subseteq \mathcal{M}$, acl-indepen. over $K \subseteq \mathcal{M}$ subfield,

then it is indisc$/K$ in $\mathcal{M}$

$\Updownarrow$

$I(a_{i_1} \dots a_{i_k}/K) = \{0\}$

Def. $\{\bar{a}_i : i \in I\} \subseteq \mathcal{M}$ is <u>indiscernible over $A \subseteq \mathcal{M}$</u>

if $\forall k \; \forall i_1, \dots, i_k \subseteq I$ $\; tp(a_{i_1} \dots a_{i_k}/A) = tp(b_{i_1} \dots b_{i_k}/A)$
$\qquad j_1, \dots, j_k \subseteq I$

(3) $T = Th(V, +, 0, k)_{k \in K}$: infinite vector space.

$V \supseteq \{a_i : i \in I\}$ indiscernible $/\emptyset \iff$ linear independ.

Thm. T : stable, $\{a_i\}_{i \in I}$ order indisc., then $\{a_i\}_{i \in I}$ indiscernible.
infinite

Pf. (A.a.) T : $\varkappa$-stable. Wlog $(I, \leq)$ is (by stretching) a dense linear ordering with $J \subseteq_{\text{dense}} I$, $|J| = \varkappa$ s.t. every nonempty interval in I has power $> \varkappa$.

$\{a_i\}_{i \in I}$: ord-indisc. but not indisc.:

$\exists k \; \exists i_1 < \dots < i_k \in I \; \exists j_1, \dots, j_k \in I \;\; tp(a_{i_1} \dots a_{i_k}) \neq tp(a_{j_1} \dots a_{j_k})$

$\Rightarrow$ for some φ: $\models \varphi(a_{i_1} \dots a_{i_k}) \wedge \neg \varphi(a_{i_{\sigma(1)}} \dots a_{i_{\sigma(k)}})$
for some $\sigma \in Sym(k)$