4.04.22

Def. (1) $\varphi(\bar{x}, \bar{a}) \in L_n(\bar{a})$ is <u>algebraic</u> if $0 < |\varphi(\mathcal{M})| < \aleph_0$

(2) a type $p(\bar{x})$ (over $\mathcal{M}$) is <u>algebraic</u> if $0 < |p(\mathcal{M})| < \aleph_0$

(3) $a \in \underset{\text{algebraic closure}}{acl}(A)$ if $tp(a/A)$ is algebraic

(4) $a \in \underset{\text{definable closure}}{dcl}(A)$ if $a \in \mathcal{M}$ is the only realisation of $tp(a/A)$

Remark (1) $p(\bar{x})$: an algebraic type $\iff p(\bar{x}) \vdash \varphi(\bar{x})$ for some algebraic formula $\varphi(\bar{x})$

(2) $|p(\mathcal{M})| = 1 \iff \exists \varphi \, (p \vdash \varphi$ and $|\varphi(\mathcal{M})| = 1)$

Proof. 1 $(\Leftarrow)$ $p(\mathcal{M}) \subseteq \varphi(\mathcal{M})$

$(\Rightarrow)$ (a.a.) let $n \in \mathbb{N}$ arbitrary. Will show: $|p(\mathcal{M})| \geq n$.

Let $\bar{x}_1, \ldots, \bar{x}_n$: disjoint tuples of variables, $|\bar{x}_i| = |\bar{x}|$.

$\{\varphi(\bar{x}_i) : \varphi(\bar{x}) \in p, i = 1, \ldots, n\} \cup \{``\bar{x}_i \neq \bar{x}_j" : 1 \leq i < j \leq n\}$: a consistent type.

It is realised in $\mathcal{M}$ so it has $\geq n$ realisations. ↯

<u>Fact</u> $acl(A) = \bigcup\{\varphi(\mathcal{M}) : \varphi(x) \in L_1(A) \text{ algebraic}\}$

$dcl(A) = \bigcup\{\varphi(\mathcal{M}) : \varphi(x) \in L_1(A) \wedge |\varphi(\mathcal{M})| = 1\}$

<u>Remark</u> Let $\varphi(\bar{x}) \in L_n(M)$. Then $\varphi(\bar{x})$ algebraic $\iff 0 < |\varphi(M)| < \aleph_0$

<u>Proof</u> $M \prec \mathcal{M}$, $|\varphi(M)| = k \iff M \models (\exists !^k \bar{x})\varphi(\bar{x})$

$\iff \mathcal{M} \models (\exists !^k \bar{x})\varphi(\bar{x}) \iff |\varphi(\mathcal{M})| = k$

<u>Remark</u> Let $\underset{a, b \in \mathcal{M}}{A \subseteq \mathcal{M}}$, then: $tp(ab/A)$ is algebraic

$\iff tp(a/A)$ is algebraic and $tp(b/Aa)$ is algebraic

<u>Pf.</u> $(\Rightarrow)$ $p(x, y) = tp(ab/A)$. Let $q(x) = p\restriction_x$
$= tp(a/A)$. Let $f : \mathcal{M}^2 \to \mathcal{M}$ projection to the first coord. Then $f : p(\mathcal{M}) \twoheadrightarrow q(\mathcal{M})$

Why?

Take $a' \in q$, choose $g \in Aut(\mathcal{M}/A)$, $g(a) = a'$, then $b' = g(b) \Rightarrow tp(ab/A) = tp(a'b'/A)$.

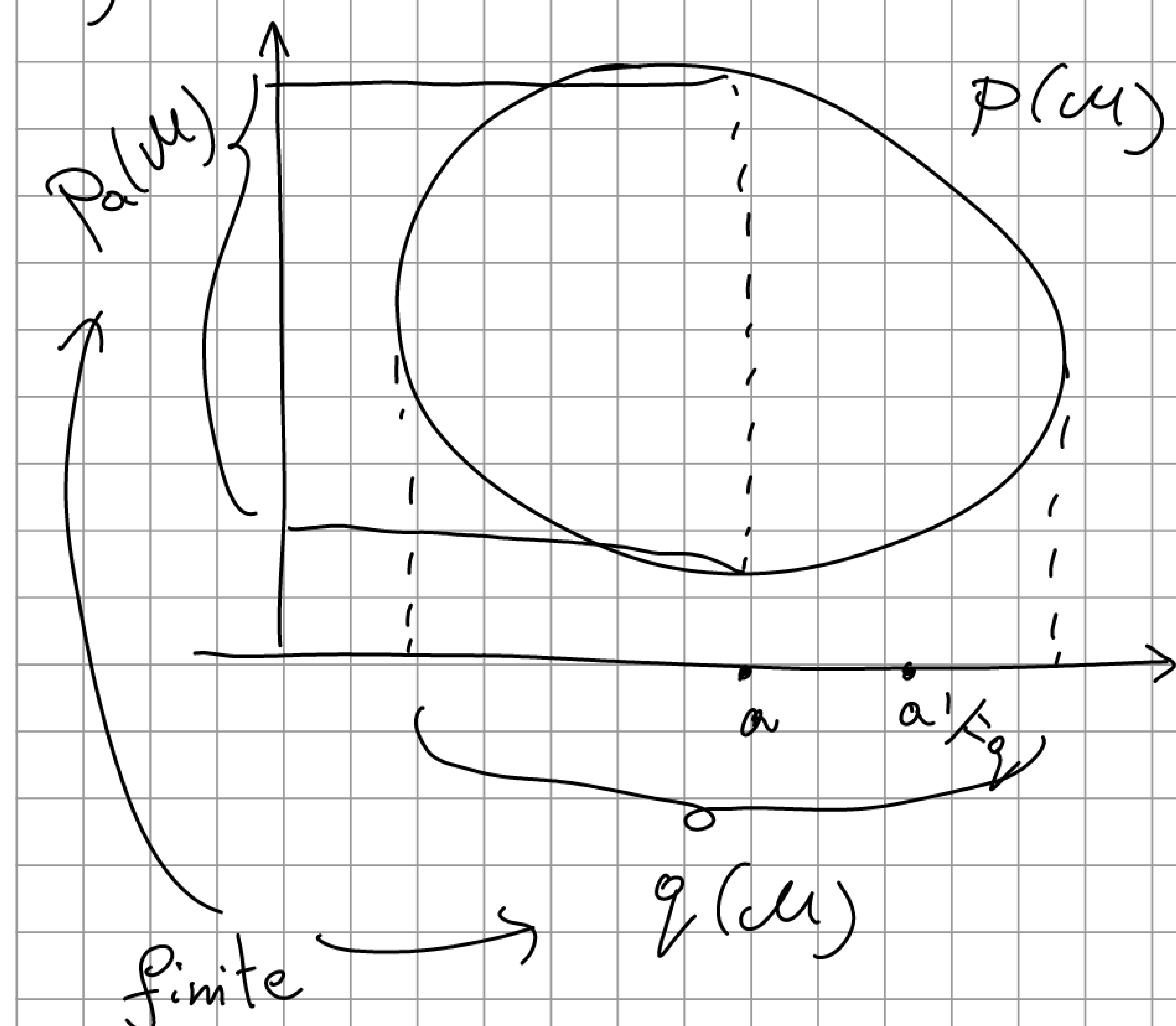
Then $|p(\mathcal{M})| < \aleph_0 \Rightarrow |q(\mathcal{M})| < \aleph_0$ and $|p_a(\mathcal{M})| < \aleph_0$

Let $p_a(y) = tp(b/Aa) = \{\varphi(a,y) : \varphi(x,y) \in p\}$

Then $p_a(\mathcal{M}) = p(\mathcal{M})_a$ ← "verticalization"

this is finite

($\Leftarrow$)



$p(\mathcal{M})_{a'} = g(p(\mathcal{M})_a)$ for any $g \in Aut(\mathcal{M}/A)$
with $g(a) = a'$.

$p(\mathcal{M}) = \bigcup_{a' \models q} (\{a'\} \times p_{a'}(\mathcal{M}))$: finite $\square$

Remark (1) (i) $A \subseteq acl(A)$, (ii) $acl(acl(A)) = acl(A)$,
(iii) $A \subseteq B \Rightarrow$ (iv) $acl(A) \subseteq acl(B)$ (a closure operator)
$acl(A) = \bigcup_{\substack{A_0 \subseteq A \\ \text{finite}}} acl(A_0)$ (finite character)

(2) The same for dcl

Pf. 1 • $a \in acl(A) \Leftrightarrow a \in \varphi(\mathcal{M})$, $\varphi(x) \in L_1(A)$,
so $\varphi \in L_1(A_0)$ for some $\underset{\text{finite}}{\overset{\text{finite}}{A_0 \subseteq A}}$, so (iv).

(ii): $a \in acl(acl(A)) \Rightarrow a \in acl(A \cup \bar{b})$,
for some $\bar{b} \subseteq acl(A) \Rightarrow tp(\bar{b}/A)$ algebraic
and $tp(a/A\bar{b})$ algebraic $\overset{\text{remark}}{\Longrightarrow} tp(a\bar{b}/A)$ is algebraic

$\Rightarrow tp(a/A)$ algebraic $\blacksquare$

Example $T = ACF_p$, $A \subseteq \mathcal{M} \models T$. $acl(A) =$
the algebraic closure (in $\mathcal{M}$) of the field
generated by A.

Measuring definable sets and types.

The Cantor–Bendixson rank.

Def. Let X: compact T_2 space.

$$\underbrace{X'}_{\text{CB-derivative}} = X \setminus \underbrace{\{\text{isolated points}\}}_{\text{open in } X} \Rightarrow \underbrace{X' \subseteq X}_{\text{closed}}$$

Iteration: $X^{(\alpha+1)} = \left(X^{(\alpha)}\right)'$, $X^{(\delta)} = \bigcap_{\alpha<\delta} X^{(\alpha)}$ where $\delta \in Lim$,

$$X^{(\infty)} = \bigcap_{\alpha \in Ord} X^{(\alpha)} = X^{(\beta)} \text{ for some } \beta < |w(X)|^+$$

the perfect core of X

minimal cardinality of basis of X

Def $\underline{CB}: X \to Ord \cup \{\infty\}$

$$\underset{\text{(CB-rank)}}{CB(p)} = \begin{cases} \min\{\alpha \in Ord : p \notin X^{(\alpha+1)}\} & \text{if } p \notin X^{(\infty)} \\ \infty & p \in X^{(\infty)} \end{cases}$$

Now: $X = S(A)$: 0-dimensional (extremly disconnected).

Let: $Clopen(X) = \{V \subseteq X : V \text{ clopen}\}$.

Def. $CB: \mathrm{Clopen}(X) \to \{-1\} \cup \mathrm{Ord} \cup \{\infty\}$: the smallest function (value-wise) s.t.: $CB(V) \geq \alpha+1 \iff$ $\forall n < \omega\ \exists V_1, \dots, V_n \subseteq V$ clopen disjoint $CB(V_i) \geq \alpha$.

(also we define "$\geq \delta$")

Then $CB(V) := \min\{\alpha : \neg CB(V) \geq \alpha+1\}$

Properties Let $\mathcal{U}, V \subseteq X$ clopen.

(0) $CB(\mathcal{U}) = -1 \iff \mathcal{U} = \emptyset$

(1) $CB(\mathcal{U}) = 0 \iff 0 < |\mathcal{U}| < \aleph_0$

(2) $\mathcal{U} \subseteq V \implies CB(\mathcal{U}) \leq CB(V)$

very easy

(3) $CB(\mathcal{U} \cup V) = \max\{CB(\mathcal{U}), CB(V)\}$

Pf. 3 Obv. $CB(\mathcal{U} \cup V) \geq \max\{CB(\mathcal{U}), CB(V)\}$.

Now assume $CB(\mathcal{U} \cup V) \geq \alpha \implies \max\{CB(\mathcal{U}), CB(V)\} \geq \alpha$.

Pf by ind on α. Base and limit easy.

Succesor step $\alpha \to \alpha+1$. Assume $CB(\mathcal{U} \cup V) \geq \alpha+1$.

So $\forall n\ \exists W_1, \dots, W_n \subseteq \mathcal{U} \cup V$ clopen disjoint $\bigwedge_{i=1}^{n} CB(W_i) \geq \alpha$.

By ind hyp.: $\max\{CB(W_i \cap U), CB(W_i \cap V)\} \geq \alpha$.

So $\forall n \left(\exists W_1, \dots, W_n \subseteq U \ \bigwedge_{i=1}^{n} CB(W_i) \geq \alpha \right)$

(or the same for V)

$\Downarrow$

$CB(U) \geq \alpha+1$ or $CB(V) \geq \alpha+1$.

(4) $CB(V) \geq \alpha+1 \iff (\exists V_n \underset{\substack{\text{clopen} \\ \text{disjoint}}}{\subseteq} V, n<\omega) \bigwedge_n CB(V_n) \geq \alpha$

(5) For $p \in X$

$$CB(p) = \min\{CB(U) : p \in U \underset{\text{clopen}}{\subseteq} X\}$$

Notation

In model theory: $X = S(A)$.

$CB(a/A) := CB(tp(a/A))$

$CB(a/A) = 0 \iff tp(a/A)$ is isolated

$p \in S(A) \rightsquigarrow CB(p) = CB_A(p)$

$\varphi \in L(A) \rightsquigarrow [\varphi] \underset{\text{clopen}}{\subseteq} S(A) \rightsquigarrow CB_A(\varphi)$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \parallel$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad CB_A([\varphi] \cap S(A))$

Morley rank "CB on $L(\mathcal{M})$, in $S(\mathcal{M})$".

Def. RM : $L(\mathcal{M}) \longrightarrow \{-1\} \cup \mathrm{Ord} \cup \{\infty\}$:

the minimal function s.t. For $\mathcal{U} \subseteq \mathcal{M}^n$ definable

$[\varphi(\bar{x}) \in L_n(\mathcal{M})$ identified with $\mathcal{U} = \varphi(\mathcal{M}) \subseteq \mathcal{M}^n]$

(1) $\mathcal{U} = \emptyset \Rightarrow RM(\mathcal{U}) = -1$

(2) If $\mathcal{U} \neq \emptyset$, then $RM(\mathcal{U}) \geqslant \alpha + 1$

$\Longleftrightarrow \forall n < \omega \ \exists V_1, \dots, V_n \underset{\substack{\text{def.} \\ \text{disjoint}}}{\subseteq} \mathcal{U} \ \bigwedge_{i \leq n} RM(V_i) \geqslant \alpha$

Def. • For $\varphi(\bar{x}) \in L(\mathcal{M})$, $RM(\varphi) = RM(\varphi(\mathcal{M}))$

• For a type $p(\bar{x})$ over $\mathcal{M}$ (not necesserly completed)

$RM(p) = \min\{RM(\varphi) : p \vdash \varphi\}$

$= \min\{RM(\mathcal{U}) : p(\mathcal{M}) \subseteq \mathcal{U}\}$

Remark (1) $RM(\varphi) = CB_{\mathcal{M}}(\varphi)$

(2) For $p \in S(\mathcal{M})$ $RM(p) = CB_{\mathcal{M}}(p)$

Here $\mathcal{M}$ may be replaced with any $\aleph_0$-saturated $M \prec \mathcal{M}$.

Properties (1) $\varphi \vdash \psi \Rightarrow RM(\varphi) \leq RM(\psi)$

(2) $p \vdash q \Rightarrow RM(p) \leq RM(q)$

(3) $RM(\varphi \vee \psi) = \max\{RM(\varphi), RM(\psi)\}$

(4) $RM(\varphi) \geq \alpha+1 \iff (\exists \varphi_n \vdash \varphi, n<\omega) \bigwedge_n RM(\varphi_n) \geq \alpha$
pairwise contradictory

(5) If $\delta \in Lim$, then $RM(\varphi) \geq \delta \iff (\forall \alpha < \delta) RM(\varphi) \geq \alpha$

Thm T is $\aleph_0$-stable $\iff RM(\text{"}x=x\text{"}) < \infty$.

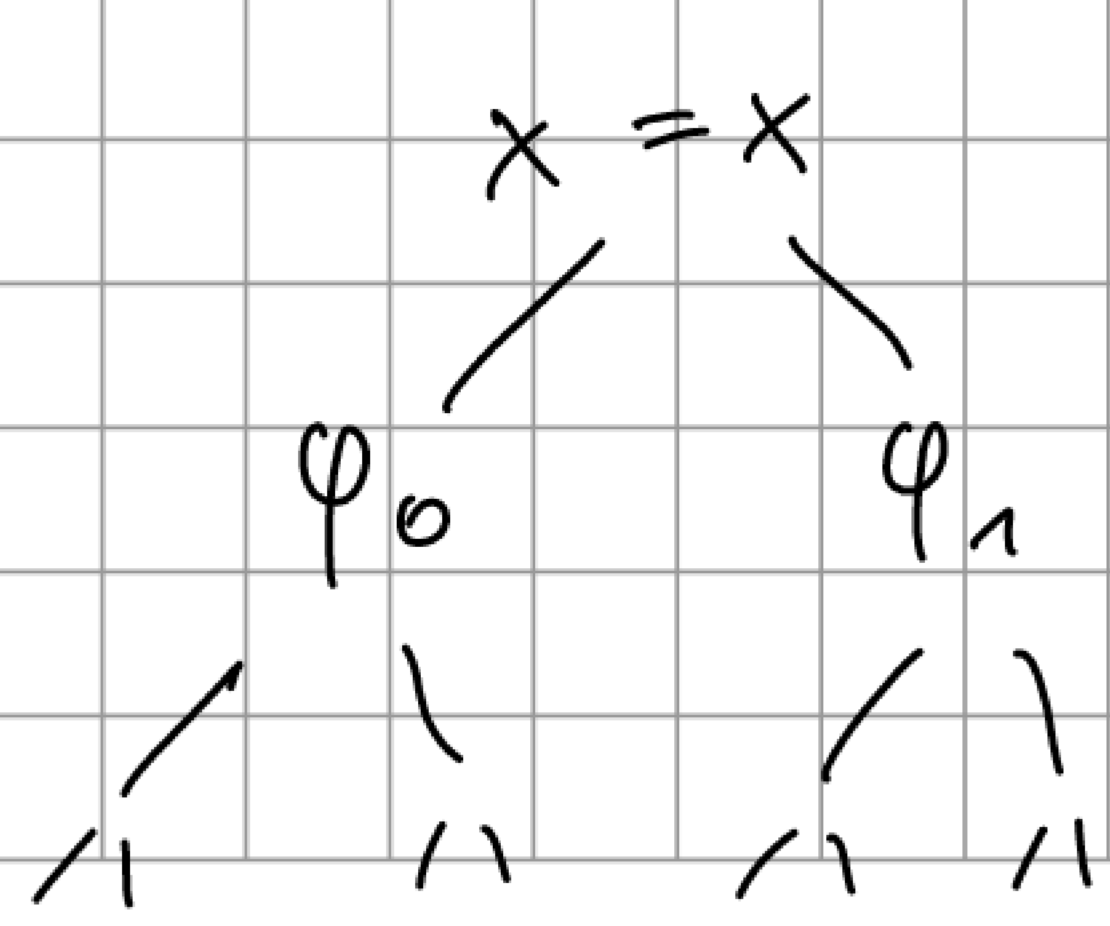
Pf. $(\Rightarrow)$ (a.a.) If $tp(\bar{a}) = tp(\bar{b})$, then

$RM(\varphi(\bar{x},\bar{a})) = RM(\varphi(\bar{x},\bar{b}))$

$\exists f \in Aut(\mathcal{M})\ f(\bar{a}) = \bar{b} \Rightarrow f(\varphi(\mathcal{M},\bar{a})) = \varphi(\mathcal{M},\bar{b})$

So $|Rng(RM)| \leq 2^{|T|} \Rightarrow \exists \alpha \in Ord\ \forall \varphi [RM(\varphi) \geq \alpha \Rightarrow RM(\varphi) = \infty]$.

Suppose $RM(\text{"}x=x\text{"}) = \infty \Rightarrow \geq \alpha+1$



: pairwise contradictory, $RM \geq \alpha \Rightarrow \geq \alpha+1$

proceed like that

We get $2^{\aleph_0}$ many types over ctble set A

$\Rightarrow T$ is not $\aleph_0$-stable.