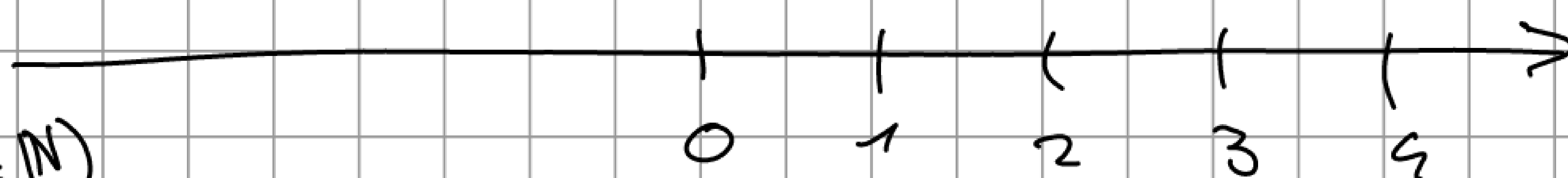
<u>25.03.2022</u>

Example (Andrzej Ehrenfeucht) Theory with exactly 3 ctble theories.

$T_0 = Th(\mathbb{Q}, \leq)$. Look at $T = T_0(\mathbb{N}) = Th(\mathbb{Q}, \leq, n)_{n\in\mathbb{N}}$.

T_0 is q.e. $\Rightarrow$ T is also q.e.

$S_1^T(\emptyset) = S_1^{T_0}(\mathbb{N})$. The types in $S_1^T(\mathbb{N})$:



$(\mathbb{Q}, \leq \mathbb{N})$

realised in, isolated

- $p_i(x) \equiv \{x = i\}$, $i \in \mathbb{N}$
- $r_i(x) \equiv \{i-1 \leq x \leq i\}$, $i \in \mathbb{N}$, $-1 \approx -\infty$
- $s(x) \equiv \{x > i : i \in \mathbb{N}\}$

ommited

We will point 3 ctble models $N \models T$.

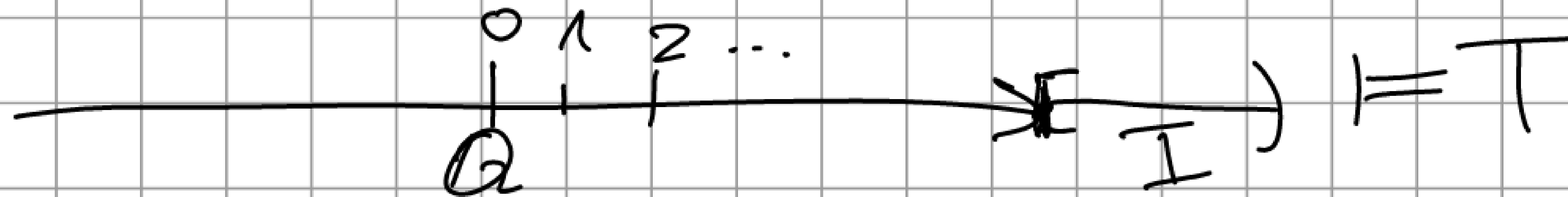
1° $N = M$: prime model of T.

2° $s(N)$ has the minimal element.

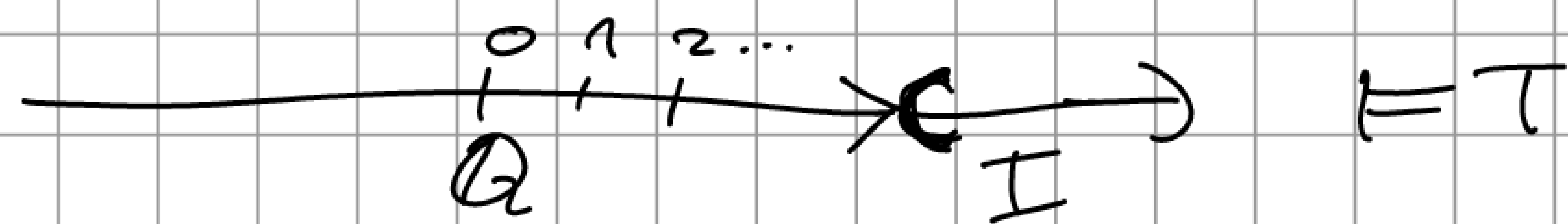
3° $s(N)$ hasn't minimal element.

2°: "$\mathbb{Q} \cup \left([0,1) \cap \mathbb{Q}\right)$" :

N:



3°:



Variants: 3, 4, 5, 6, ...

<u>Problem</u> Does there exist a stable T with

$1 < n(T) < \aleph_0$?

<u>Def</u> ($A \subseteq_{\text{small}} \mathfrak{M}$). $M \prec \mathfrak{M}$ is <u>prime over A</u> if:

(1) $A \subseteq M$

(2) $\forall \underset{A}{\overset{}{N}} \supseteq \prec \mathfrak{M}\ \exists f : M \xrightarrow{\equiv} N : f\restriction_A = id_A$

<u>Equivalently</u> M is a prime model of $T(A)$.

- If A is ctble $\longrightarrow$ full description of prime models of $T(A)$.
- If A is unctble $\longrightarrow$ in general not much can be set

Thm T $\aleph_0$-stable $\Longrightarrow$ $\forall A \exists M \models T(A)$ prime
$A \subseteq \mathcal{M}$

Proof $M = A \cup \{a_\alpha : \alpha < ?\}$

Construction of a_α's s.t.:

(i) $A \cup \{a_\alpha : \alpha < ?\}$ satisfies the TV-test

(ii) $\forall \alpha < ?$ $tp(a_\alpha / A a_{<\alpha})$ is isolated.

At some point it has to terminate (i.e. we cannot add more elements).

Claim Assume $N \prec \mathcal{M}$, $A \subseteq N$. Then $\forall \alpha \exists f : A \cup a_{<\alpha} \xrightarrow{\equiv} N$ s.t. $f\restriction_A = id_A$.

Proof We define $f(a_\beta)$ for all $\beta < \alpha$ by ind. on β so that $f\restriction_A = id_A$ and $f : A a_{\leq \beta} \xrightarrow{\equiv} N$.

Take $\beta < \alpha$ and suppose $\forall \beta' < \beta$ $f(a_{\beta'})\downarrow$ so that the condition holds.

$p(x) = tp(a_\beta / A a_{<\beta})$ is isolated.

$f : A a_{<\beta} \xrightarrow{\equiv} f[A a_{<\beta}] \subseteq N$

$f(p)$ is realised by c

$\underset{\text{isolated}}{p(x)} \in S_1(A a_{<\beta}) \overset{f^*}{\rightsquigarrow} \underset{\text{isolated}}{f(p)} \in S_1(f[A a_{<\beta}])$

Now we put $f(a_\beta) = c$.

Claim $\blacksquare$

By the claim after some time we cannot get any more elements.

Additional property of the construction:

At the step α we consider a consistent formula $\varphi(\alpha) \in L(A a_{<\alpha})$ with no realisation in $A a_{<\alpha}$, choose a_α s.t. $\models \varphi(a_\alpha)$.

<u>Problems</u> Is a prime model over A unique up to isomorphism over A?

Answer: not always. However the prime model M over A constructed by the previous construction is unique up to $\underset{A}{\cong}$ and it's called primary over A.

<u>Thm</u> M, N: primary over $A \Longrightarrow M \underset{A}{\cong} N$.

Proof $M = A \cup \{a_\alpha : \alpha < \gamma\}$: an „isolated construction" of M over A, i.e. $tp(a_\alpha / Aa_{<\alpha})$ is isolated by a formula $\varphi_\alpha(x)$ over Aa_{C_α} s.t. $\underset{\text{finite}}{C_\alpha} \subseteq \gamma$

Def. $X \subseteq \gamma$ is <u>closed</u> if $\forall \alpha \in X\ C_\alpha \subseteq X$

Remark (1) $\alpha \in \gamma \Rightarrow \exists$ minimal $\underset{\text{finite}}{X \subseteq \alpha}$ s.t. $\underbrace{X \cup \{\alpha\}}_{d(\alpha)}$ is closed.

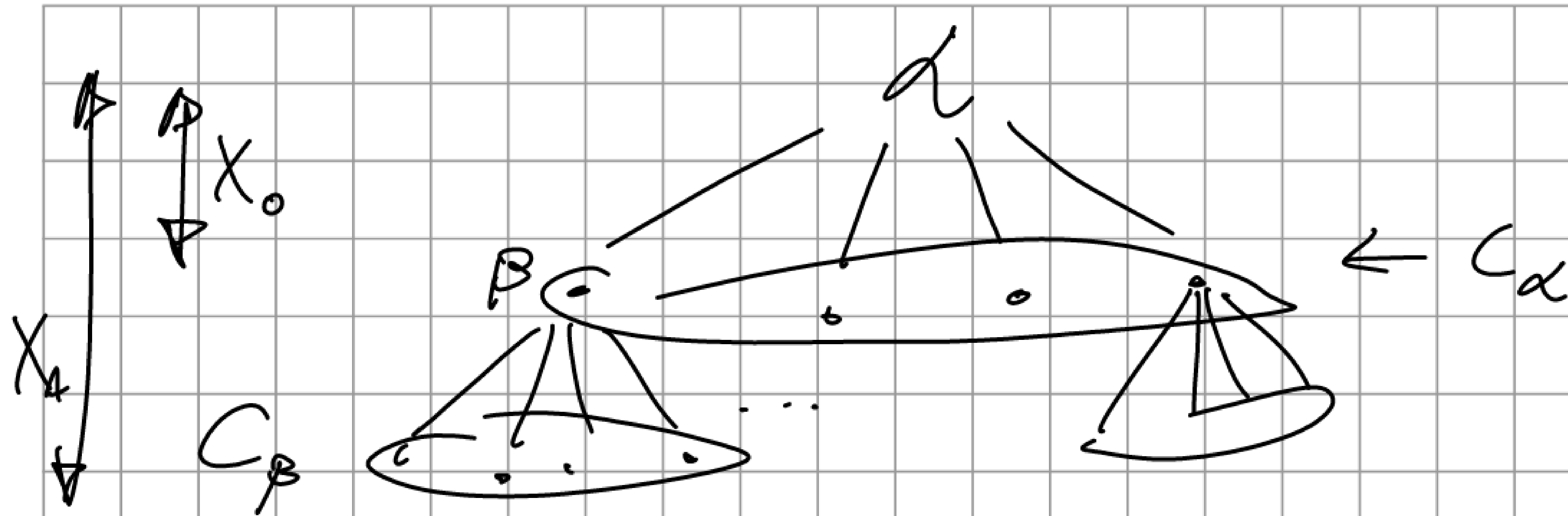
(2) A union of family of closed subsets of γ is closed.

Proof (remark) (2) is obvious.

(1): Take $X_0 = C_\alpha$, then $X_n = X_{n+1} \cup \bigcup_{\beta \in X_n} C_\beta$

finite because C_β finite

Then $X = \bigcup_n^{\nearrow} X_n$. Then $X \cup \{\alpha\}$ is finite and closed!



This tree has no infinite branch (because there are no infinite decreasing sequence of ordinals).

remark

Remark Assume X: closed. Then
$X \subseteq \gamma$

$$A \cup \langle a_\alpha : \alpha \in X\rangle ^\frown \langle a_\alpha : \alpha \in \gamma \setminus X\rangle$$

concatenation

is an isolated construction over A.

Pf. (remark) as $A \cup \langle a_\alpha : \alpha \in X\rangle$ is an i-construction over A (by the fact that X is closed)

(2) Suppose that $\alpha < \gamma$ and $\alpha \notin X$. Will show that $tp(a_\alpha / A a_X a_{X^c \cap (<\alpha)})$ is isolated.

$\varphi_\alpha(a) \vdash tp(a_\alpha / A a_{<\alpha}) \vdash tp(a_\alpha / A a_X a_{X \cap \alpha < \alpha})$
will show

Let $\underset{\text{finite}}{X_0 \subseteq X}$ s.t. $X_0 \cap \alpha \neq \emptyset$.

Enough to show that $tp(a_\alpha / A a_{<\alpha}) \vdash tp(a_\alpha / A a_{<\alpha} a_{X_0})$

Wlog By the remark $X_0 \cup \alpha$ is closed.

So: $A \cup \langle a_\beta : \beta < \alpha \rangle ^\frown \langle a_\beta : \beta \in X_0 \rangle$ is

an i-construction over A, but for $\beta \in X_0$

$tp(a_\beta / A a_{<\alpha} a_{(<\beta) \cap X_0}) \vdash tp(a_\beta / A a_{<\beta})$

Because $\varphi_\beta(x)^{\vdash}$ and $\varphi_\beta^{\in}$.

So it implies also $tp(a_\beta / A a_{\leq \alpha} a_{(<\beta) \cap X_0})$.

$\Rightarrow$ $tp(a_\alpha / A a_{<\alpha} a_{(<\beta) \cap X_0}) \vdash tp(a_\alpha / A a_{<\alpha} a_{(\leq \beta) \cap X})$

the proof of e1.3

(we switch a_α with a_β)

We just continue with induction on β

(start with $\beta = \min X_0$).

romeu

Claim M: primary$/A \Rightarrow$ atomic$/A$

Pf. Let $\bar{m} \subseteq M = A \cup \{a_\alpha : \alpha < \gamma\}$.

$\bar{m} \subseteq A \cup a_X$, $\underset{\text{finite closed}}{X \subseteq \gamma}$

$A \cup a_X$: a partial i-construction.

$\Downarrow$

$tp(a_X/A)$ is isolated

$\Downarrow$

$tp(\bar{m}/A)$ —— 〃 ——

claim $\square$

Pf (of thm) $M = A \cup \{a_\alpha : \alpha < \gamma\}$,

$N = A \cup \{b_\alpha : \alpha < \delta\}$: i-constructions $/A$.

We construct $f : M \xrightarrow{\cong} N$, $f = \bigcup_\alpha f_\alpha$: elementary.

(i) $\mathrm{Dom} f_\alpha \supseteq A$, $\mathrm{Rng} f_\alpha \supseteq A$, $f_\alpha \restriction_A = id_A$

(ii) $|\mathrm{Dom} f_\alpha \setminus A|$, $|\mathrm{Rng} f_\alpha \setminus A| \leq |\alpha| \cdot \aleph_0$

(iii) $\beta \in \mathrm{Lim}$ $f_\beta = \bigcup_{\alpha < \beta} f_\alpha$

(iv) $a_\alpha \in \mathrm{Dom} f_{\alpha+1}$, $b_{\alpha+1} \in \mathrm{Rng} f_{\alpha+1}$.

(v) $\mathrm{Dom} f_\alpha \setminus A = a_X$, $\mathrm{Rng} f_\alpha \setminus A = b_Y$, where $X \subseteq \gamma$, $Y \subseteq \delta$ are closed.

The recursive step from f_α to $f_{\alpha+1}$.

Let $A' = A \cup \mathrm{Dom}\, f_\alpha$: an iconstruction over A,

likewise $A'' = A \cup \mathrm{Rng}\, f_\alpha$: —— " ——

and M is primary over A' (by the remark)

and N is primary over A''.

$p(x) = tp(a_\alpha / A')$ is isolated, so $f_\alpha(p)$ is isolated too, therefore $\exists b \in N$ s.t. $f_\alpha(p) = tp(b/A'')$.