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CONSTRUCTION OF SPECIAL MODELS: $N, M \models T$

Def. (1) M is <u>atomic</u> if $\forall \underset{\text{finite}}{\bar{a}} \subseteq M$ $tp(\bar{a}/\emptyset) = tp(\bar{a})$ is isolated.

(2) M is <u>prime</u> if $\forall N \models T$ $\exists f : M \overset{\equiv}{\Longrightarrow} N$

Example $T = ACF_p$, $\mathbb{F}_p$: prime field of char p

(a) $\hat{\mathbb{F}}_p$: atomic (exercise)

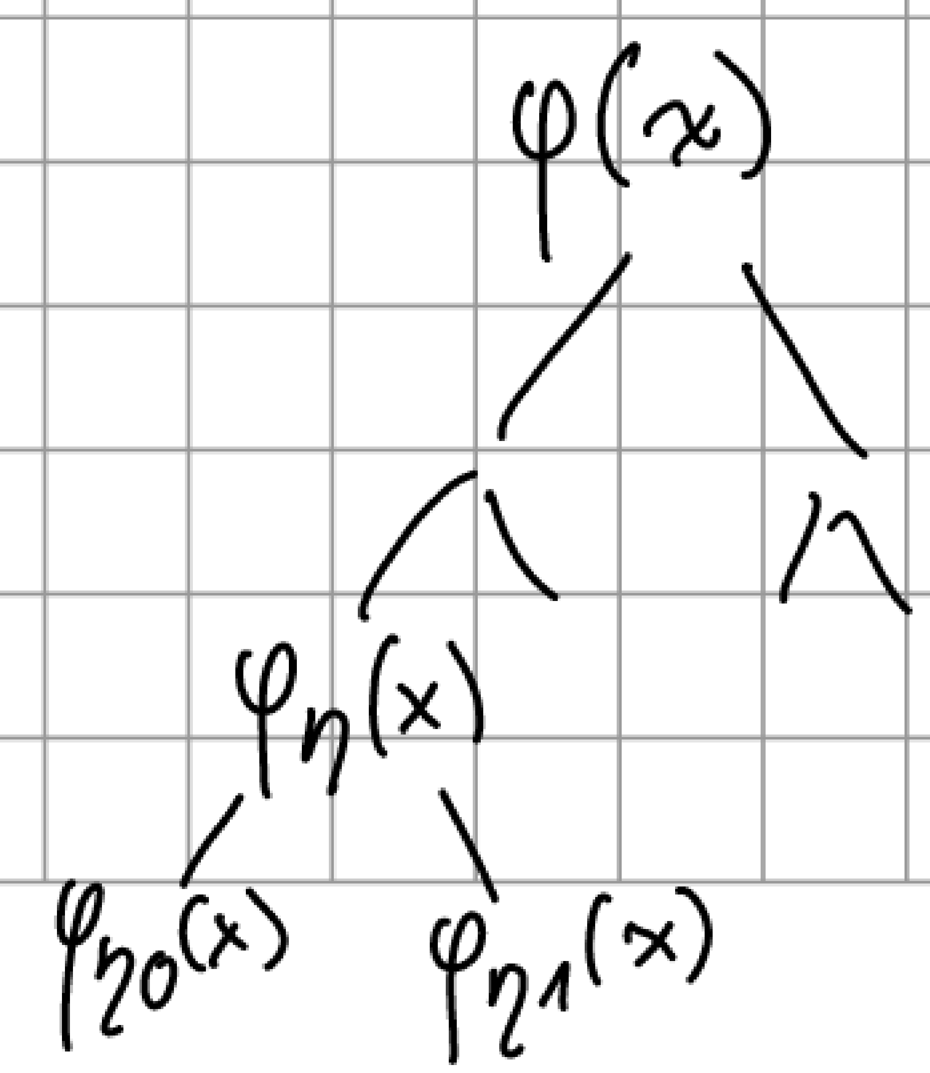
(b) $\hat{\mathbb{F}}_p$: prime (exercise)

Thm. T : $\aleph_0$-stable $\Rightarrow$ T has a prime model.

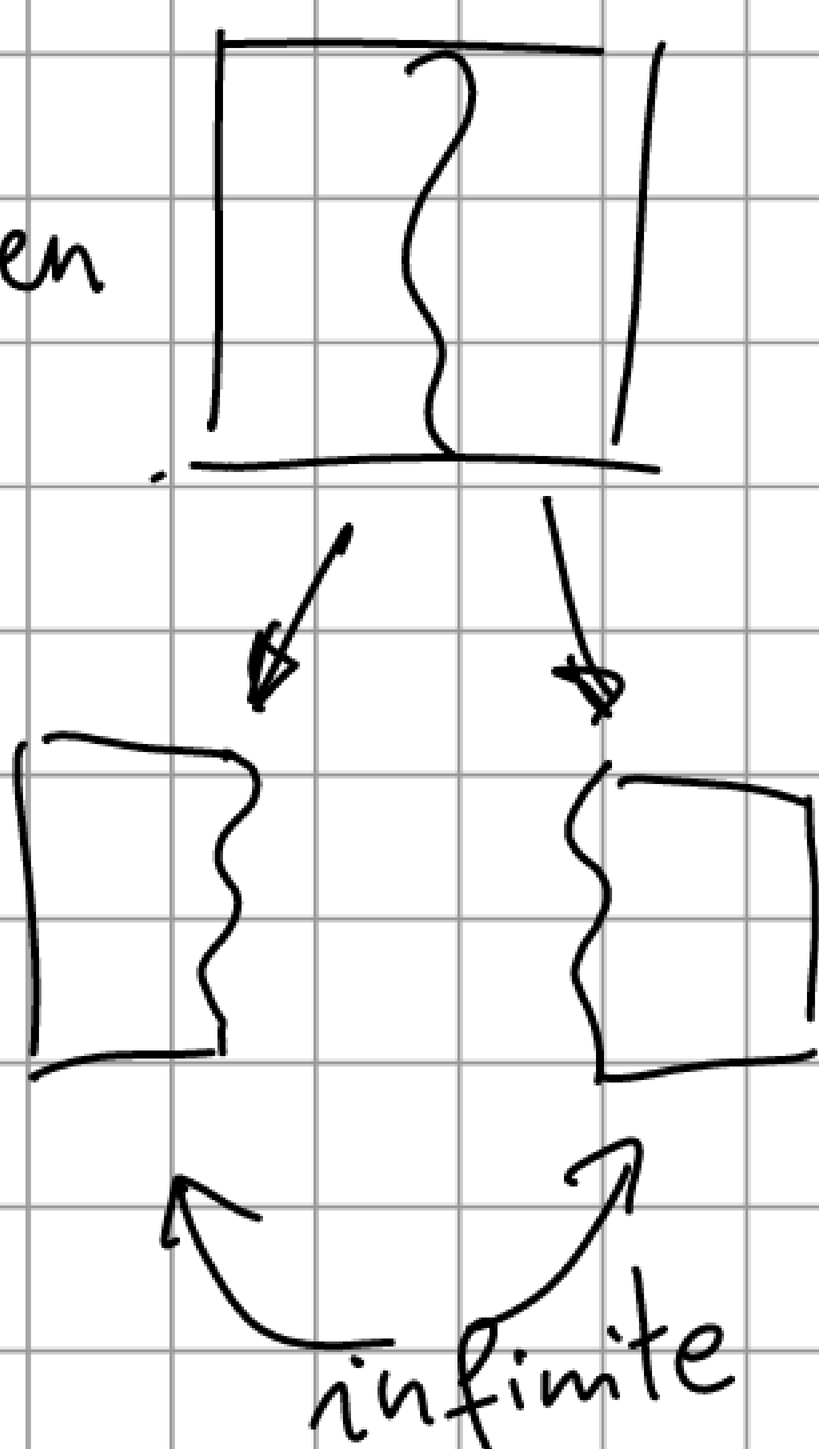
Lemma 1 T : $\aleph_0$-stable $\Rightarrow$ $\forall A \subseteq \mathcal{M}$ $\{\text{isolated types}\} \underset{\text{dense}}{\subseteq} S_1(A)$

Pf. Suppose $\underset{\text{(consistent with } T)}{\varphi(x)} \in L_1(A)$ s.t. in $S_1(A) \cap [\varphi]$

there is no isolated types $\Rightarrow$ a tree of formulas $\varphi_\eta(x) \in L_1(A)$, $\eta \in 2^{<\omega}$



$\rightsquigarrow$ contradicts $\aleph_0$-stability of T

clopen  $\subseteq S_1(A) \cap [\varphi]$

$\square$

<u>Lemma 2</u> $(a, b \in \mathcal{M})$ $tp(a)$ isolated and $tp(b/a)$ isolated $\iff$ $tp(ab)$ isolated

<u>Pf</u> „$\Rightarrow$": $\varphi(x) \vdash tp(a)$, $\psi(a,y) \vdash \underbrace{tp(b/a)}_{p_a(y) \in S_y(a)}(y)$

Then: $\varphi(x) \wedge \psi(x,y) \vdash tp(ab)$

Let $a', b' \in \mathcal{M}$ satisfy $\varphi(x) \wedge \psi(x,y)$

$\models \varphi(a') \Rightarrow tp(a') = tp(a) \Rightarrow \psi(a',y) \vdash p_{a'}(y)$
$p_{a'}(y) \in S_y(a')$

$\models \psi(a',b') \Rightarrow \models p_{a'}(b')$

$\Downarrow$

$$ab \xrightarrow{\equiv} a'b'$$

and $tp(ab) = tp(a'b')$

"$\Leftarrow$": $\theta(x,y) \vdash tp(a,b)$.

(a) "$\exists y\, \theta(x,y)$" $\vdash tp(a)$.

$\underset{tp(a)}{\uparrow}$

<u>because</u> Let $a' \in \mathcal{M}$ satisfy "$\exists y\, \theta(x,y)$".

So there is b' s.t. $\models \theta(a', b')$

$\Rightarrow tp(ab) = tp(a', b') \Rightarrow tp(a) = tp(a')$.

(b) $\theta(a,y) \vdash tp(b/a)(y)$

<u>because</u>: similar to (a) $\blacksquare$

<u>Proof of the thm.</u> Construction of a prime model of T:

$A = \{a_n : n < \omega\} \subseteq \mathcal{M}$ so that:

1) A satisfies TV-test

2) $\forall n$ $tp(a_n / a_{<n})$ is isolated

$[a_{<n} = \{a_k : k < n\}]$

At step n choose a_n:

Let $\varphi(x) \in L_1(a_{<n})$ consistent.

Let $a_n \in \varphi(\mathcal{M})$ s.t. $tp(a_n / a_{<n})$ is isolated (lemma 1)

Suitable choice of φ's ensures (1).

$M \models T$ is prime. Let $N \models T$ arbitrary. We find $f(a_n) \in N$ for $n < \omega$ s.t. $\exists f : M \xrightarrow{\equiv} N$.

At step n $f[a_{<n}] \subseteq N$ with $f : a_{<n} \xrightarrow{\equiv} N$

Let $p(x) = tp(a_n / a_{<n})$ (isolated)

$$\Downarrow$$

$$f^*(p)(x) \in S(f(a_{<n}))$$

isolated too, hence realised by $f(a_n)$

$$\Downarrow$$

$$f : a_{\leq n} \xrightarrow{\equiv} N$$

$\blacksquare$

<u>Remark</u> (1) A prime model $M \models T$ is atomic.

(2) If $M \models T$ ctble atomic, then M prime

<u>Corrolary</u> $\hat{F}_p$ is atomic.

Proof of remark

(1) Let $p(\bar{x}) \in S_n(\emptyset)$ non-isolated. Will show $p(M) = \emptyset$. Let $N \models T$ be ommiting p.

$\exists f : M \xrightarrow{\equiv} N \Rightarrow p(M) = \emptyset$.

(2) Let $\underset{\models T}{M} = \{a_n : n < \omega\}$ atomic.

Then $\forall n \; tp(a_{\leq n})$ is isolated

$\Downarrow$ lemma 2

$\forall n \; tp(a_n / a_{<n})$ is isolated

$\Downarrow$ pf of thm

M prime.

$\blacksquare$

Corollary A prime model of T is unique (up to $\cong$)

Proof Let $M, N \models T$ both prime $\underset{\text{remark}}{\Rightarrow}$ M, N are ctble and atomic, so we have embeddings in both direction, using back-and-forth we get the iso.

Def $M \models T$ is minimal if $\neg \exists N \precneqq M$

Example $\widehat{\mathbb{F}_p}$ is minimal.

Fact T has a prime model $\Leftrightarrow$ $\forall n$ $\{$isolated types$\} \subseteq_{\text{dense}} S_n(\emptyset)$

Proof "$\Rightarrow$": Let $\underset{\text{prime}}{M} \models T \Rightarrow M$: atomic

what we need

$N \models T$, then $\forall n$ $\{p \in S_n(\emptyset) : p(N) \neq \emptyset\}$ is dense in $S_n(\emptyset)$
any

$\boxed{\text{exercise}}$

"$\Leftarrow$": **Claim** Assume $\underset{\text{finite}}{\bar{a}} \subseteq \mathcal{M}$ and $tp(\bar{a})$ is isolated.

Then $\{$isolated types$\} \subseteq_{\text{dense}} S_1(\bar{a})$.

Proof of claim Let $n = |a|$, $\varphi(\bar{x}) \vdash tp(\bar{a})$.

Let $\psi(\bar{x}, y) \in L_{n+1}(\emptyset)$ s.t. $\psi(\bar{a}, y)$ is consistent.

We seek $q(y) \in S_1(A) \cap [\psi(\bar{a}, y)]$ isolated.

Let $\chi(\bar{x}, y) = \varphi(\bar{x}) \wedge \psi(\bar{x}, y)$.

By assumptions of $\Leftarrow$ $\underset{\text{isolated}}{\exists p(\bar{x}, y)} \in S_{\bar{x},y}(\emptyset) \cap [\chi(\bar{x}, y)]$.

Let $\bar{a}', b' \models p(\bar{x}, y)$. Then $\bar{a}' \models \underbrace{p(\bar{x}, y) \restriction_{\bar{x}}}_{\ni \varphi(\bar{x})} = tp(\bar{a})$

Let $f \in Aut(\mathcal{M}) : f(\bar{a}') = \bar{a}$

$b = f(b')$

Then $\bar{a}'b' \xrightarrow[f]{\equiv} \bar{a}b \Rightarrow \bar{a}b \models p(\bar{x}, y)$

so $\underset{\underset{p}{\shortparallel}}{tp(\bar{a}b)}$ is isolated $\underset{\text{lemma 2}}{\Rightarrow}$ $\underset{\underset{\psi(\bar{a}, y)}{\psi}}{tp(b/\bar{a})}$ isolated

So $q(y) = tp(b/\bar{a})$

Claim $\rightsquigarrow$ we construct a model $M = \{a_n : n < \omega\} \prec \mathcal{M}$

s.t. $\forall n$ $tp(a_n / a_{<n})$ is isolated

$\Downarrow$ lemma 2

M atomic ctble $\Rightarrow$ M prime.

<u>Corollary</u> If $\forall n$ $|S_n(\emptyset)| \leq \aleph_0$, then T has a prime model.

<u>Corollary</u> A prime model (of a ctble T) is homogeneous (exercise).

The number of countable models of T: $I(T, \aleph_0)$, $n(T)$.

Remark $1 \leq n(T) \leq 2^{\aleph_0}$

$$\underset{\text{ctble}}{M} \models T \Rightarrow M \cong \underbrace{(\mathbb{N}, \ldots)}_{\leq 2^{\aleph_0}}$$ L-structures like that

Recall $n(T) = 1 \iff \forall n \ |S_n(\emptyset)| < \aleph_0$
(T: $\aleph_0$-categorical)

Vaught conjecture (1961)

$$n(T) > \aleph_0 \Rightarrow n(T) = 2^{\aleph_0}$$

Thm (M. Morley, 1971) $\aleph_0 < n(T) < 2^{\aleph_0} \Rightarrow n(T) = \aleph_1$

Thm (Vaught, 1961) $n(T) \neq 2$

Proof (A.a) suppose $n(T) = 2$.

$$n(T) < 2^{\aleph_0} \Rightarrow T \text{ small (i.e. } \forall n \ |S_n(\emptyset)| \leq \aleph_0)$$

⋮