14.03.2022

Def T is <u>quantifier eliminable</u> if $\forall \varphi \in L\ \exists \underset{\substack{\text{open} \\ \equiv \text{q.f.}}}{\psi} \in L$

$T \vdash \varphi \leftrightarrow \psi$

Def. For $p(\bar{x}) \in S_n(\emptyset)$ let $p_0(\bar{x}) = \{\varphi(\bar{x}) \in p(\bar{x}) : \varphi \text{ open}\}$.

Remark T is q.e. $\Leftrightarrow \forall n\, \forall p \in S_n(\emptyset)\ \ p_0 \vdash p$

Proof „$\Rightarrow$" Obvious. „$\Leftarrow$" Let $\varphi(\bar{x}) \in L$.

- $\forall p \in [\varphi] \cap S_n(\emptyset)\ \exists \underset{\text{open}}{\psi} \in p \quad p \in [\psi] \subseteq [\varphi]$

Why? $p_0 \vdash p$

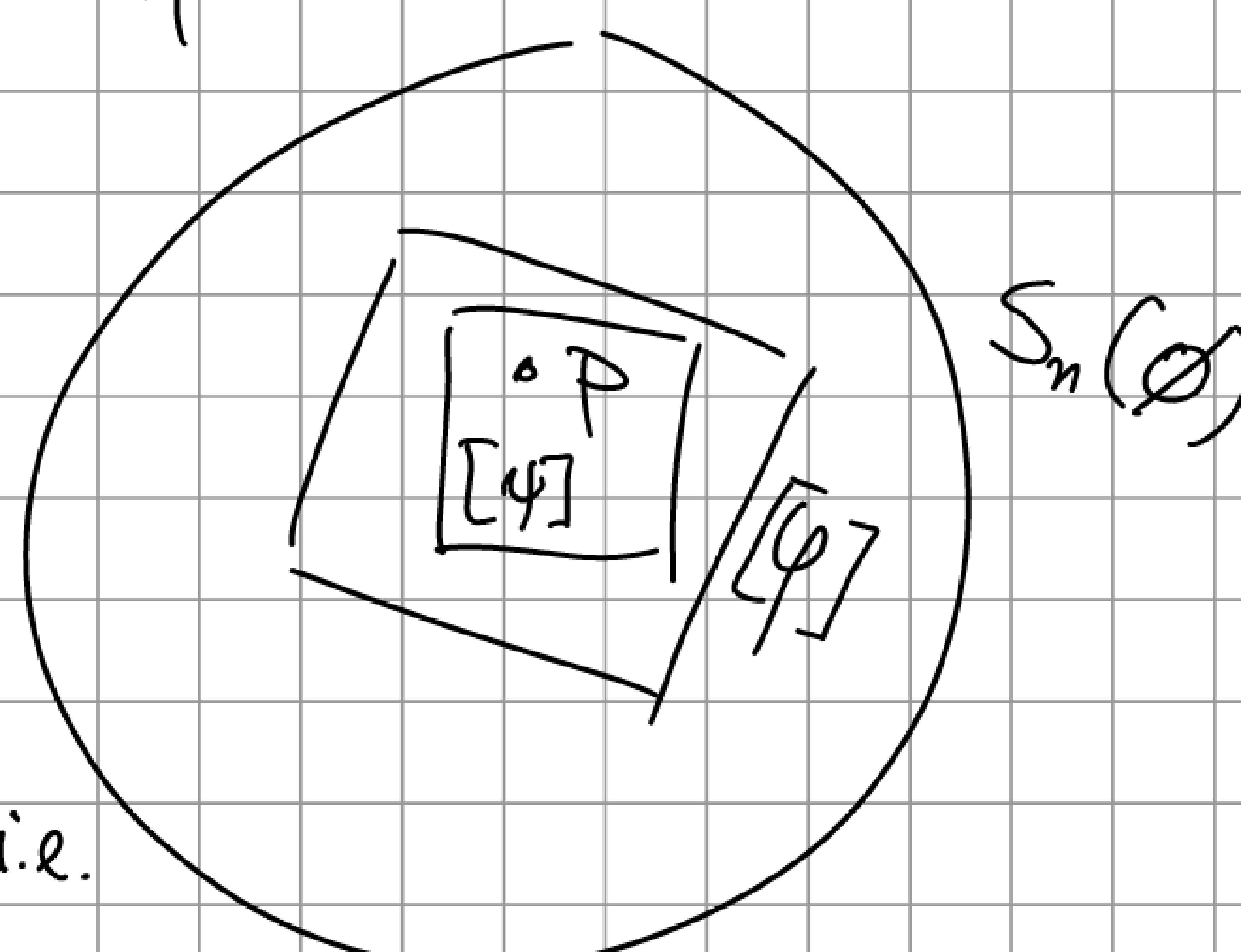
$\quad\quad\ \ \Downarrow$

$p_0 \vdash \varphi$

by compactness

$\exists$ finite $p_0' \subseteq p_0$ s.t.

$p_0' \vdash \varphi$, i.e.



$p_0'(\mathcal{M}) \subseteq \varphi(\mathcal{M})$

$\psi(\mathcal{M}) = \left(\bigwedge_{\psi' \in p_0'} \psi'\right)(\mathcal{M}) \subseteq \varphi(\mathcal{M}) \rightsquigarrow \mathcal{M} \models \psi(\bar{x}) \to \varphi(\bar{x})$

$\Updownarrow$

$\psi \vdash \varphi \Rightarrow [\psi] \cap S_n(\emptyset) \subseteq [\varphi] \cap S_n(\emptyset)$

$\blacksquare$

Application $L = \{+, \cdot, 0, 1\}$: the language of rings.

ACF_p : the theory of algebraically closed fields of char p, in L.

Axioms :

1) field axioms

2) char $= p \neq 0$: $\underbrace{1 + \ldots + 1}_{p} = 0$

2') $p = 0$: $\underbrace{1 + \ldots + 1}_{n} \neq 0$ for $n \geq 1$

3) Every polynomial of deg n has a root:
$0 < n$

$$\forall y_{n-1}, y_{n-2}, \ldots, y_0 \; \exists x \;\; x^n + y_{n-1} x^{n-1} + \ldots + y_0 = 0.$$

Fact ACF_p is complete.

Proof Let $M, N \models ACF_p$. Enough to show that $M \equiv N$. Let $\varkappa > \|M\|, \|N\|$ and let $M' \succ M$, $N' \succ N$.

power $\varkappa$.

M', N': uncountable acl fields of the same power and char

$$\Downarrow \text{algebra}$$

$$M' \cong N' \Rightarrow M' \equiv N'$$

$$\begin{array}{ccc} \rotatebox{90}{$\equiv$} & \Downarrow & \rotatebox{90}{$\equiv$} \\ M & \equiv & N \end{array}$$

<u>Fact</u> ACF_p is q.e. (Chevalley, Tarski)

<u>Proof</u> (in $\mathcal{M}$) We will show that $\forall p \in S_n(\emptyset)$ $p_0 \vdash p$ ($\Leftrightarrow p_0(\mathcal{M}) \subseteq p(\mathcal{M})$). Let $\bar{a} \models p_0$, $\bar{b} \models p$, $\bar{a}, \bar{b} \subseteq \mathcal{M}$. It's enough to prove that $\exists f \in Aut(\mathcal{M})$ $f(\bar{a}) = \bar{b}$.

$\boxed{\begin{array}{l} \bar{a} = (a_1, \dots, a_n) \\ \bar{b} = (b_1, \dots, b_n) \end{array}}$

Let $\langle \bar{a} \rangle, \langle \bar{b} \rangle$: the subrings with 1 of $\mathcal{M}$ generated by $\bar{a}, \bar{b}$. $\bar{a}, \bar{b} \models p_0 \Rightarrow \langle \bar{a} \rangle \cong \langle \bar{b} \rangle$

$a_i \overset{f}{\mapsto} b_i$

Some algebraic magic.

$$\langle \bar{a} \rangle \cong \langle \bar{b} \rangle$$

$$\cap \quad \Downarrow \text{unique} \quad \cap$$

$$\mathcal{M} \supseteq \langle \bar{a} \rangle_0 \cong \langle \bar{b} \rangle_0 \subseteq \mathcal{M}$$

(fraction field)

(not unique)

$$\mathcal{M} \supseteq \underbrace{\langle \bar{a} \rangle_0}_{F_a}{}^{alg} \cong \underbrace{\langle \bar{b} \rangle_0}_{F_b}{}^{alg} \subseteq \mathcal{M}$$

$$\operatorname{trdeg}(\mathcal{M}/F_a) = \|\mathcal{M}\| = \operatorname{trdeg}(\mathcal{M}/F_b)$$

$$\Downarrow$$

$$\mathcal{M} \cong F_a(X_\alpha, \alpha < \lambda)^{alg}$$

$$f \ \cong\!\downarrow \qquad\qquad \text{\rotatebox{90}{$\cong$}}$$

$$\mathcal{M} \cong F_b(X_\beta, \beta < \lambda)^{alg}$$

$$f \in \operatorname{Aut}(\mathcal{M}),\ f(\bar{a}) = \bar{b}.$$

$\blacksquare$

Types in $T = ACF_p$

Let $\mathcal{M} \models T$: a monster model.

$\mathcal{M} \supseteq$ subfield (small) K. We will describe $S_n(K)$.

Let $\bar{a} \subseteq \mathcal{M}$, $|\bar{a}| = n$.

$$K[\bar{x}] \rhd I(\bar{a}/K) = \{ f \in K[\bar{x}] : f(\bar{a}) = 0 \}$$

<u>Remark</u> 1) $tp(\bar{a}/K) = tp(\bar{a}'/K) \iff I(\bar{a}/K) = I(\bar{a}'/K)$

2) $\forall I \triangleleft K[\bar{x}]$ (prime) $\exists \bar{a} \subseteq \mathcal{M}$ $I(\bar{a}/K) = I$.

<u>Proof</u> 1) "$\Rightarrow$" $tp(\bar{a}/K) = tp(\bar{a}'/K) \Rightarrow \exists f \in \operatorname{Aut}(\mathcal{M}/K)$

$I(\bar{a}/K) = I(\bar{a}'/K) \Leftarrow f(\bar{a}) = \bar{a}'$

[alternatively: $f \in I(\bar{a}/K) \iff$ "$f(\bar{x}) = 0$" $\in tp(\bar{a}/K)$]

"$\Leftarrow$" Assume $I(\bar{a}/K) = I(\bar{a}'/K) = I$.

$$\Downarrow$$

$$K[\bar{a}] \underset{K}{\cong} K[\bar{x}]/I \cong K[\bar{a}']$$

$$\Downarrow$$

$$\exists f \in Aut(\mathcal{M}/K)\ f(\bar{a}) = \bar{a}'$$

$$\Downarrow$$

$$tp(\bar{a}/K) = tp(\bar{a}'/K).$$

2) $K \subseteq K[\bar{x}]/I = K[\bar{a}] \ni \bar{a} = \bar{x}/I$ and $I(\bar{a}/K) = I$.

$\vdots$

$\square$

$S_1(K) = \{ tp(a/K) : a \in \mathcal{M} \}$: a top. space.

$\Downarrow$

$p(x) = tp(a/K)$, $I_p = I(a/K) \triangleleft K[x]$

a) $I_p \neq \{0\}$, i.e. a is algebraic $/K$, so

$0 \neq f \in I_p$ "$f(x) = 0$" $\in p(x)$. In fact,

irreducible over K

"$f(x) = 0$" $\vdash p(x)$ (isolates)

Let $a' \in \mathcal{M}$ s.t. $f(a') = 0 \Rightarrow I(a'/K) \ni f \Rightarrow$ f generates $I(a'/K)$

$p = tp(a/K) = tp(a'/K) \Leftarrow I(a/K) = I(a'/K) \Leftarrow$ f generates $I(a'/K)$

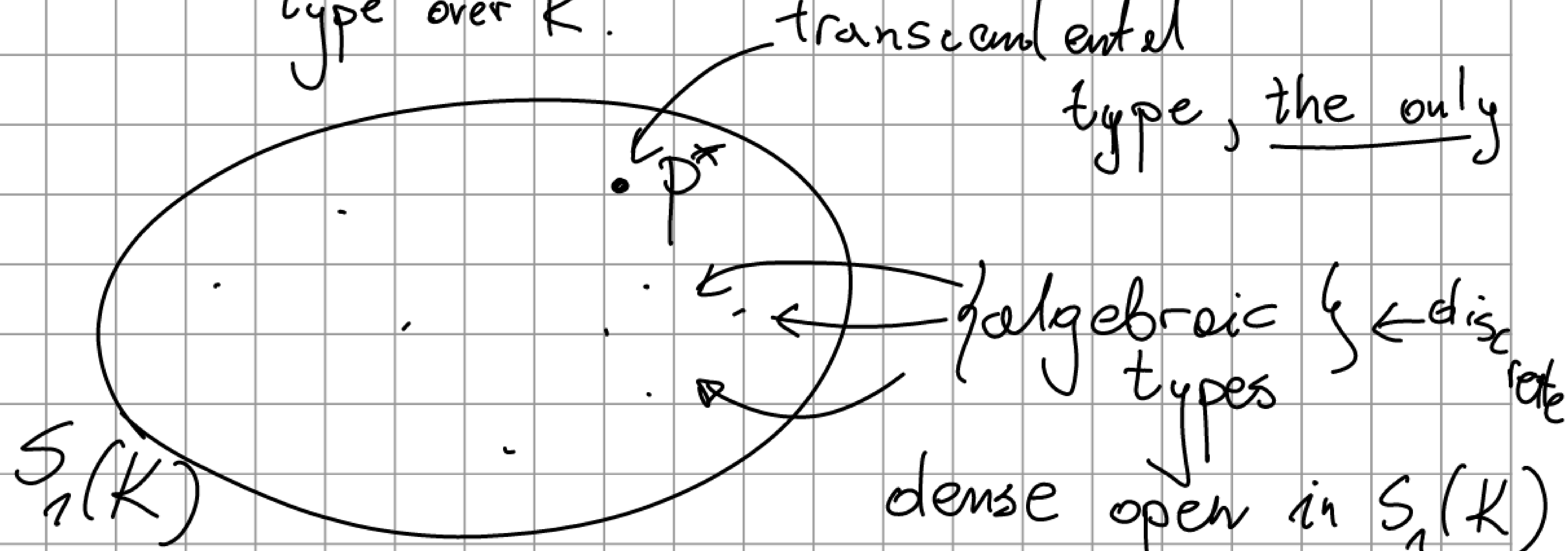
$p(x)$ here is called <u>algebraic</u>.

More generally (T: arbitrary)

<u>**Def**</u> $\varphi(\bar{x}) \in L(A)$ is called <u>algebraic</u>, if $0 < |\varphi(\mathcal{M})| < \aleph_0$, similarly for q: a type.

b) $I_p = \{0\}$: $p = tp(a/K)$ s.t. $a \in \mathcal{M}$ transcendental over K

$\uparrow$ the transcendental type over K.



If K ctble then $S_1(K)$ ctble

Corollary ACF_p is $\aleph_0$-stable [recall: T is $\varkappa$-stable $\iff \forall A \subseteq \mathcal{M}, |A| \leq \varkappa$ $|S_1(A)| \leq \varkappa$]

Proof Let $\underset{\text{ctble}}{A} \subseteq \mathcal{M}$.

$$A \subseteq \underset{\substack{\uparrow \\ \text{ctble} \\ \text{subfield}}}{K} \subseteq \mathcal{M} \quad |S_1(A)| \leq |S_1(K)| = \aleph_0$$

Remark T is totally transcendental $\iff T$: $\aleph_0$-stable

Proof „$\Rightarrow$" from def, „$\Leftarrow$": (A.a.) Let $\varkappa > \aleph_0$.

Suppose $|A| \leq \varkappa < |S_1(A)|$ for some $A \subseteq \mathcal{M}$.

Shall find $\underset{\text{ctble}}{A_0} \subseteq A$ with $|S_1(A_0)| \geq 2^{\aleph_0}$.

(def) $\varphi(x) \in L(A)$ is large iff $|S_1(A) \cap [\varphi]| > \varkappa$

otherwise: $\varphi(x)$ small.

(a) "$x = x$" is large

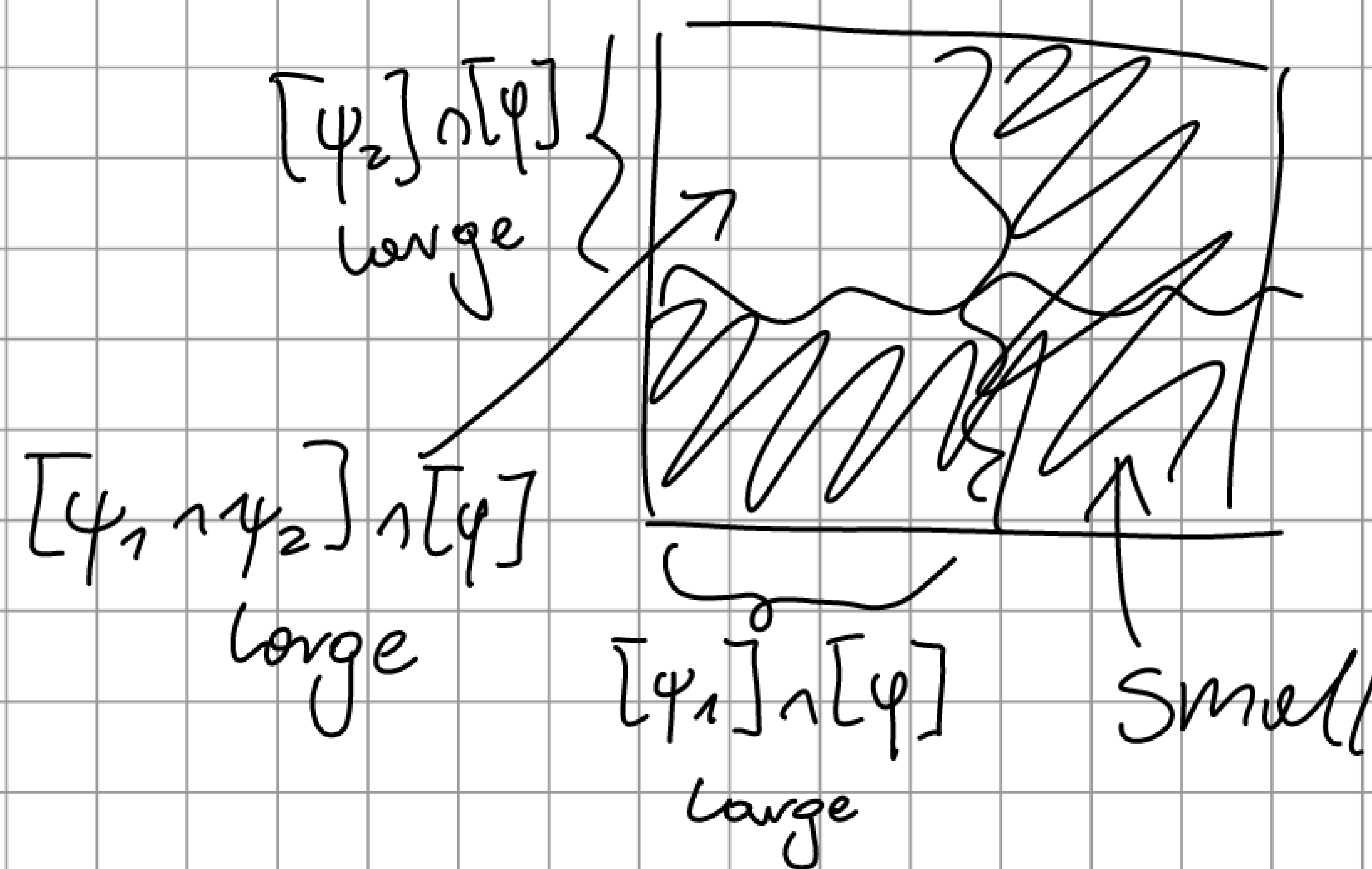
(b) if $\varphi(x)$: large, then $\exists \psi_1, \psi_2 \in L(A)$ large

s.t. $\varphi(\mathcal{M}) = \psi_1(\mathcal{M}) \,\dot\cup\, \psi_1(\mathcal{M})$

Pf. (b) if not then $p^* = \{\psi \in L_1(A) : \psi \wedge \varphi$ is large$\}$

is a complete type is $S_1(A) \cap [\varphi]$

1° p^*: consistent: $\psi_1, \psi_2 \in p^* \Longrightarrow \psi_1 \wedge \psi_2 \in p^*$



2° p^*: complete <u>ok</u>

$p^* \in S_1(A)\cap[\varphi]$: the only large type.

$$\underbrace{S_1(A)\cap[\varphi]}_{>\varkappa} = \underbrace{\{p^*\}\cup \bigcup_{\substack{\psi\in L_1(A)\\ \text{small}\\ \underbrace{\psi\vdash\varphi}_{\le\varkappa}}} \underbrace{(S_1(A)\cap[\psi])}_{\le\varkappa}}_{\le\varkappa}$$

$\lightning$

c) a tree of large formulas in $L_1(A)$ $\varphi_\eta(x)$, $\eta \in 2^{<\omega}$ s.t. $\varphi_\eta(\mathcal{M}) = \varphi_{\eta 0}(\mathcal{M}) \dot\cup \varphi_{\eta 1}(\mathcal{M})$

by (b)

Let $A_0 \subseteq A$: the set of all params of $\varphi_\eta, \eta \in 2^{<\omega}$

Then $|A_0| \leq \aleph_0$.

For $\eta = 2^{\omega}$: $p^0_\eta = \{\varphi_{\eta|n}(x) : x < \omega\}$: a consistent 1-type over A_0

$\cap\!\!|$

When $\nu \neq \eta$ then $p_\eta \neq p_\nu$ $\quad p_\eta \in S_1(A_0)$

$\Downarrow$

$|S_1(A_0)| \geqslant 2^{\aleph_0} > \aleph_0$ $\quad \lightning$