Constructions of models:

• saturated, $\Longrightarrow$ • (strongly) homogeneous $\not\Longleftarrow$

Thm. $\underbrace{\kappa = 2^{<\kappa},\ \kappa \in Reg}_{\kappa^{<\kappa} = \kappa}$, $\kappa > \aleph_0 \Longrightarrow \exists M \models T$ saturated, of power κ.

<u>Proof</u>

$(*)$ $|S_1(A)| \leq 2^{|A| + \aleph_0}$, because: $|L_1(A)| = |A| + \aleph_0$

Here: $|A| < \kappa \Rightarrow |S_1(A)| \leq \kappa$.

<u>Lemma</u> $N \models T$, $\|N\| \leq \kappa \Rightarrow X_N := \bigcup \{S_1(A) : A \subseteq N \ \& \ |A| < \kappa\}$ the set has power $\leq \kappa$.

<u>Pf</u> • $|\{A \subset N : |A| < \kappa\}| \leq \kappa^{<\kappa} = \kappa$.

• $|S_1(A)| \leq \kappa$ for such A.

Proof of the thm.

M_α, $\alpha < \kappa$: elementary chain of models of T of power κ.

• M_0 : whatever

• $M_\delta = \bigcup_{\alpha<\delta}^{\uparrow} M_\alpha$, when $\delta < \kappa$ limit.

• $M_{\alpha+1} \succ M_\alpha$ such that $\forall p \in X_{M_\alpha}$ $p(M_{\alpha+1}) \neq \emptyset$:

$T' = Th(M_\alpha, m)_{m \in M_\alpha} \cup \bigcup_{\beta<\kappa} \{\varphi(c_\beta) : \varphi(x) \in p_\beta\}$,

where $X_{M_\alpha} = \{p_\beta : \beta < \kappa\}$ — c_β: new constant symbols,

and T' in language $L(M_\alpha) \cup \{c_\beta : \beta < \kappa\}$.

T': consistent, has model of power $\varkappa$: $M_{\alpha+1}$ such that $M_\alpha \prec M_{\alpha+1}$.

$M = \bigcup\limits_{\alpha<\varkappa}^{\nearrow} M_\alpha$: of power $\varkappa$, saturated:

Let $A \subseteq M$, $|A| < \varkappa$ and $p \in S_1^M(A)$

$\varkappa \in Reg \Rightarrow A \subseteq M_\alpha$ for some $\alpha < \varkappa$.

proof: $A = \{a_\beta : \beta < \mu\}$ for some $\overset{\text{CN}}{\mu} < \varkappa$.

- $\forall \beta < \mu \quad \exists\, \alpha_\beta < \varkappa \quad a_\beta \in M_{\alpha_\beta}$
- $\{\alpha_\beta : \beta < \mu\} \subseteq \varkappa$, $\mu < cf(\varkappa) = \varkappa$

$\Rightarrow \exists\, \alpha < \varkappa \quad \forall \beta < \mu \quad \alpha_\beta < \alpha$

$A \subseteq M_\alpha$.

~~Let~~ $M_\alpha \prec M \Rightarrow p \in S_1^{M_\alpha}(A) = S_1^M(A)$

p realized in $M_{\alpha+1}$ by some $a \in M_{\alpha+1}$

$a \models p$ in $M_{\alpha+1} \underset{M_{\alpha+1} \prec M}{\Rightarrow} a \models p$ in M.

Monster model:

Let $\bar{\varkappa}$: a large cardinal number.

"Ideal model" $\mathcal{M} \models T$: saturated of power $\bar{\varkappa}$

because: $\forall M \models T \ (\|M\| < \bar{\varkappa} \Rightarrow \exists M' \prec \mathcal{M} \quad M \cong M')$.

Advantages of saturated model M:

(i) universality

(ii) strong homogeneity

~~More~~ More Weakly (a bit):

(1) $\overline{\kappa}$ - universality

(2) strong $\overline{\kappa}$ - homogeneity

$\mathrm{Aut}(M)$: the group of automorphisms of M

$\mathrm{Aut}(M/A) = \{f \in \mathrm{Aut}(M) : f\restriction_A = id_A\}$: automorphisms of M over A.
$A \subseteq M$

Lemma Assume M is strongly κ-homogeneous, κ-saturated, $A \subseteq M$, $|A| < \kappa$. Then:

(1) For $a, b \in M$ ($tp(a/A) = tp(b/A) \Leftrightarrow a, b$ are in the same orbit of $\mathrm{Aut}(M/A)$ on M.

(2) $[\text{orbits } \mathrm{Aut}(M/A) \text{ on } M^n] \xleftrightarrow[\text{onto}]{1\text{-}1} S_n(A)$

Proof (1) $\Leftarrow$: $f \in \mathrm{Aut}(M/A)$, $f(a) = b$

$$\Downarrow$$

$$tp(a/A) = tp(b/A).$$

$\Rightarrow$: $tp(a/A) = tp(b/A) \Rightarrow f: Aa \xrightarrow{\equiv} Ab$

$f\restriction_A = id_A$, $f(a) = b$

strong κ-homogeneity, $|A| < \kappa$ $\Rightarrow f \subseteq g \in \mathrm{Aut}(M)$, $g \in \mathrm{Aut}(M/A)$

$g(a) = b$: a, b in the same orbit of $\mathrm{Aut}(M/A)$

(2) $M^n \supseteq \mathcal{O} \xrightarrow[\varphi]{(1)} p_{\mathcal{O}} \in S_n(A)$

$\mathcal{O}$: orbit of $\mathrm{Aut}(M/A)$

$p_{\mathcal{O}}$ = common type $tp(a/A)$ for $a \in \mathcal{O}$.

Φ: {orbits of $\mathrm{Aut}(M/A)$ on M^n} $\xrightarrow{\varphi}$ $S_n(A)$

$\mathcal{O}_1 \neq \mathcal{O}_2 \overset{(1)}{\Rightarrow} p_{\mathcal{O}_1} \neq p_{\mathcal{O}_2}$ $\boxed{\text{so } \varphi : 1\text{-}1}$

[if $p_{\mathcal{O}_1} = p_{\mathcal{O}_2}$ then let $a \in \mathcal{O}_1, b \in \mathcal{O}_2 \Rightarrow \exists g \in \mathrm{Aut}(M/A)$ $g(a) = b$ ↯.]

M: κ-saturated $\Rightarrow \varphi$: "onto".

<u>Def</u> Let $\bar{\kappa}$: a (large) cardinal number,
$\mathfrak{M} \models T$ <u>monster model</u> (w.r. to $\bar{\kappa}$), if $\mathfrak{M}$: $\bar{\kappa}$-saturated, strongly $\bar{\kappa}$-homogeneous

Thm. Assume $\aleph_0 \leq \kappa \in CN$. Then
$\exists M$: κ-saturated & strongly $\bar{\kappa}$-saturated.

<u>Proof</u> $M = \bigcup^{\uparrow}_{\alpha < \kappa^+} M_\alpha$: union of elementary chain s.t.:

(1) $M_0 \models T$ any

(2) $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$ if $\delta \in Lim$,

(3) $M_{\alpha+1} \succ M_\alpha$ s.t.:

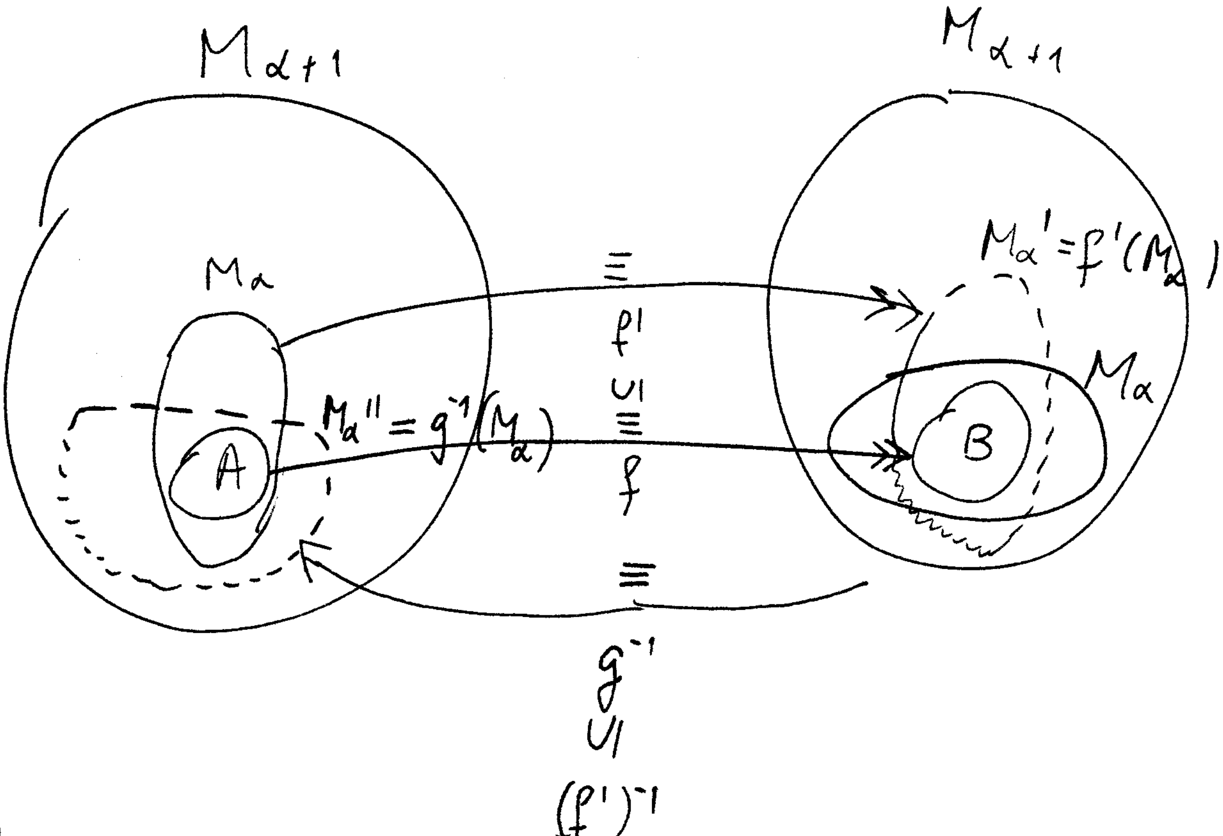
(a) $\forall p \in S_1(M_\alpha)$ p realized in $M_{\alpha+1}$

(b) $(\forall f: \underset{\substack{\cap\\ M_\alpha}}{A} \xrightarrow{\equiv} \underset{\substack{\cap\\ M_\alpha}}{B})(\exists\, g \supseteq f)(g: \underset{\substack{\cup\\ M_\alpha}}{A'} \xrightarrow{\equiv} \underset{\substack{\cup\\ M_\alpha}}{B'} \text{ in } M_{\alpha+1})$

It is enough that $M_{\alpha+1}$ is $\|M_\alpha\|^+$-saturated, $M_{\alpha+1} \succ M_\alpha$

To satisfy (a), (b).

Proof of (b) for such $M_{\alpha+1}$:



I. M: κ-saturated: <u>clear</u>

II M strongly κ-homogeneous:

Assume $A \subset M$, $|A| < \kappa$. Then $A \subseteq M_\alpha$, $B \subseteq M_\alpha$ for some $\alpha < \kappa$.

$f: A \xrightarrow{\equiv} M$

$B = f[A]$

- we construct

a sequence f_β, $\alpha \le \beta < \varkappa^+$:-

- increasing $f_\beta : M \xrightarrow{\overset{\equiv}{0}} M$ partial elementary
- ~~$\exists$~~ $f \subseteq f_\alpha$

(*) $M_\beta \subseteq \operatorname{dom} f_\beta \cap \operatorname{rng} f_\beta$.

- f_α constructed according to 3)(6)

$$f_\alpha : M_{\alpha+1} \xrightarrow{\overset{\equiv}{0}} M_{\alpha+1}$$

- $f_\beta : M_{\beta+1} \xrightarrow{\overset{\equiv}{0}} M_{\beta+1}$, as in 3.(6) when β : successor.
- $f_\delta = \bigcup_{\beta<\delta} f_\beta$ when δ limit. , still $f_\delta : M_\delta \xrightarrow{\overset{\equiv}{0}} M_\delta$.

$$f_\infty = \bigcup^{\nearrow}_{\alpha \le \beta < \varkappa^+} f_\beta \ , \quad f_\beta \in \operatorname{Aut}(M), \ f \subseteq f_\beta .$$

<u>Assumptions</u> Let $\bar{\varkappa}$: a cardinal number <u>large enough</u> so that:

(1) We consider only small models of T
$\quad\quad$ ($\shortparallel$ of power $< \bar{\varkappa}$, or even $\ll \bar{\varkappa}$)

(2) We work within a monster model $\mathcal{M} \models T$ (w.r.to $\bar{\varkappa}$)

(3) We consider only $\underbrace{\text{small}}$ models $M \prec \mathcal{M}$
$\quad\quad$ ($\shortparallel$ of power $< \bar{\varkappa}$, or even $\ll \bar{\varkappa}$,)

Consequences:

(1) For $M, N \prec \mathcal{M}$, $M \subseteq N \iff M \prec N$

(2) Convention: For $\bar{a} \subseteq \mathcal{M}$

$\models \varphi(\bar{a})$ means $\mathcal{M} \models \varphi(\bar{a})$

(3) For $A \subseteq M \prec \mathcal{M}$:

$S_n^M(A) = S_n^{\mathcal{M}}(A) =: S_n(A)$

<u>Notation</u> Assume $p(\bar{x})$, $q(\bar{x})$ types (small, over $\mathcal{M}$)

- $p(\bar{x}) \vdash q(\bar{x}) \iff p(\mathcal{M}) \subseteq q(\mathcal{M})$
  "p implies q"
- $p(\bar{x}) \underset{\uparrow}{\equiv} q(\bar{x}) \iff p \vdash q \;\&\; q \vdash p$
  equivalent

Special case: $p(\bar{x}) = \{\varphi(\bar{x})\}$.

$\varphi(\bar{x}) \vdash q(\bar{x})$: "φ isolates q".

<u>Remark</u>: Syntactically:

$p(\bar{x}) \vdash q(\bar{x}) \iff \forall \varphi(\bar{x}) \in q \; \exists \, p_0(\bar{x}) \subseteq p(\bar{x})$ finite

types over A

$T(A) \vdash \bigwedge p_0(x) \to \varphi(x)$

Remark (exercise)

$p(\bar{x}) \vdash q(\bar{x}) \iff \forall M \models T$ $|A|^+$-saturated $p(M) \subseteq q(M)$.

Def. (reminder)

Let $p(\bar{x})$: a type over A.

p is isolated over $A \iff \exists \underbrace{\varphi(\bar{x}) \in L(A)}_{\text{consistent (with } T)}$ $\varphi \vdash p$.

Thm (omitting types, Ehrenfeucht)

Assume $p_n(\bar{x}_n)$, $n<\omega$: ~~non~~ a family of non-isolated types over $\emptyset$. Then $\exists M \models T$ $\forall n$ $\underbrace{p_n(M)=\emptyset}_{M \text{ omits } p_n}$.

<u>Lemma</u>

Assume A is ctble, $p_n(\bar{x}_n)$, $n<\omega$: a family of non-isolated types over A, $\varphi(x)\in L_1(A)$, $\underbrace{\varphi(\mathcal{M})\neq\emptyset}_{\text{i.e. } \varphi \text{: consistent}}$.

Then $\exists c \in \varphi(\mathcal{M})$ $\forall n$ p_n non-isolated over $A\cup\{c\}$.

Proof [Lemma $\Rightarrow$ Thm]

By the Lemma: $\exists \underbrace{\{a_n : n<\omega\}}_{= A} \subseteq \mathcal{M}$ s.t.

(1) A satisfies the TV-test

(2) p_n, $n<\omega$: non-isolated over A.

Construction of a_n, $n<\omega$: recursion on n.

Let $\{\varphi_n(x,\bar{y}) : n<\omega\}$: all formulas of L of this form.

Suppose $n<\omega$ and $\{a_i : i<n\} = a_{<n}$ already constructed so that all p_k, $k<\omega$ still non-isolated over $a_{<n}$.

We consider a consistent formula $\varphi_n(x) \in L_1(a_{<n})$

By the Lemma we find $\underset{a_n}{\overset{\shortparallel}{c}} \in \varphi_n(\mathcal{M})$ so that

all p_k, $k<\omega$, still non-isolated over $a_{\leq n}$.

- the formulas $\varphi_n(x)$, $n<\omega$ may be chosen so that after ω steps:

  $\forall \underset{\text{consistent}}{\varphi(x)} \in L(A)\ \exists n\ \varphi = \varphi_n$.

Then $A = \{a_n : n<\omega\}$ satisfies TV-test

$A = M \prec \mathcal{M}$, every p_k still non-isolated over A.

$p_k(M) = \emptyset$ [if not, some $\underset{M}{\overset{\cap}{\overline{m}}} \models p_k$. then $(\overline{x}_k = \overline{m}) \vdash p_k(\overline{x}_k)$ ↯)

<u>Proof of the Lemma.</u>

Let $p(\overline{x})$: one of the types $p_n(\overline{x}_n)$.

Let $h(\overline{x}, y, \overline{a}) \in L(A$

$h(\overline{x}, c, \overline{a}) \vdash p(\overline{x}) \iff h(\mathcal{M}, c, \overline{a}) \subseteq p(\mathcal{M})$.

$\iff \forall \psi(\overline{x}) \in p(\overline{x})\ h(\mathcal{M}, c, \overline{a}) \subseteq \psi(\mathcal{M})$

$\iff \forall \psi \in p\ \ \mathcal{M} \models \forall \overline{x}\,(h(\overline{x}, c, \overline{a}) \rightarrow \psi(\overline{x}))$

$\Longleftrightarrow \forall \psi \in p \quad \psi_h(y) \in tp(c/A)(y)$

where $\psi_h(y) = \forall \bar{x} \, (h(\bar{x}, y, \bar{a}) \to \psi(\bar{x}))$

<u>hence</u>:

$tp(c/A) = tp(c'/A) \Rightarrow [\, h(\bar{x}, c, \bar{a}) \vdash p \Leftrightarrow h(\bar{x}, c', \bar{a}) \vdash p \,]$

$h(\bar{x}, c, \bar{a})$ consistent $\Longleftrightarrow (\exists \bar{x} \, h(\bar{x}, y, \bar{a})) \in tp(c/A)(y)$.

Let $X_{h,p} = \{ q \in S_1(A) : \text{For } c \models q \;\; h(\bar{x}, c, \bar{a}) \vdash p(\bar{x}) \}$
and $h(\bar{x}, c, \bar{a})$ consistent.

$\uparrow$
"bad types"

Let $q \in S_1(A)$ then $\qquad$ For $c \models q \;\; h(\bar{x}, c, \bar{a})$ consistent

$$q \in X_{h,p} \Longleftrightarrow q(y) \in S_1(A) \cap \underbrace{[\exists \bar{x}\, h(\bar{x}, y, \bar{a})]}_{\Updownarrow} \cap \underbrace{\bigcap_{\psi \in p} [\psi_h(y)]}_{\text{For } c \models q \;\; h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})}$$

(*) $X_{h,p}$: nowhere dense in $S_1(A)$.

Proof of (*) : (a.a.)

Suppose $\theta(y) \in L_1(A)$ and $\emptyset \neq S_1(A) \cap [\theta] \subseteq X_{h,p}$.

Let $\alpha(\bar{x}) = \exists y (h(\bar{x}, y, \bar{a}) \wedge \theta(y))$
$\in L(A)$

- $\alpha(\bar{x})$: consistent :

Let $c \in \theta(\mathcal{M})$

$\Downarrow$ $[\theta] \subseteq X_{h,p}$)

$\mathcal{M} \models \exists \bar{x}\ h(\bar{x}, c, \bar{a})$.

Let $\bar{d} \subseteq \mathcal{M}$ s.t. $\mathcal{M} \models h(\bar{d}, c, \bar{a})$.

$\bar{d}$ satisfies in $\mathcal{M}$: $\underbrace{\exists y\, h(\bar{x}, y, \bar{a})}_{= \alpha(\bar{x})}$.

- $\alpha(\bar{x}) \vdash p(\bar{x})$, i.e. $\alpha(\mathcal{M}) \subseteq p(\mathcal{M})$.

Let $\bar{d} \in \alpha(\mathcal{M})$. So there is $c \in \mathcal{M}$ st

$\models h(\bar{d}, c, \bar{a}) \wedge \theta(c)$

$\Downarrow$ $[\theta] \subseteq X_{h,d}$

$\forall \psi \in p \quad \models \psi_h(c)$

$\Downarrow$

$\forall \psi \in p \quad h(\bar{x}, c, \bar{a}) \vdash \psi(\bar{x}) \implies h(\bar{x}, c, \bar{a}) \vdash p(\bar{x})$

$\bar{d} \implies \bar{d}$ ↯.

as : $p(\bar{x})$ non-isolated

Let $X = \bigcup_{h, p_n} X_{h,p_n} \subseteq S_1(A)$.

↑ meager. Let $q \in S_1(A) \cap [\varphi] \setminus X$

$c \models q$ good.

Pf (a.a) Suppose $p = p_n$ isolated over $A \cup \{c\}$.

$\exists\ h(\bar{x}, c, \bar{a}) \quad h(\bar{x}, c, \bar{a}) \vdash p(x) \implies q \in X_{h,p_n}$ ↯.

Consistent $\qquad q = tp(\bar{c}/A)$