How to extend elementary mappings?

~~Bat.~~ BAlg : Category of Boolean algebras

Comp$_0$: —"— of compact Hausdorff 0-dimensional spaces

$F : BAlg \longrightarrow Comp_0$ $\quad$ $G : Comp_0 \longrightarrow BAlg$

$F(A) = S(A)$ $\quad$ $G(X) = Clopen(X)$.

F, G : contravariant functors "inverse" to each other

~~(F, G) is a duality of categories... (look it up)~~

Categories BAlg and Comp$_0$ are dually equivalent.

A, B : Boolean algebras

$f : A \longrightarrow B$ homomorphism $\Rightarrow \underbrace{F(f)}_{f^*} : S(B) \longrightarrow S(A)$

$F(f)(p) = f^{-1}[p]$

continuous.

Assume $f : A \xrightarrow{\equiv} B$, $\quad T \models M \supseteq A$, $\quad N \models T$, $N \supseteq B$. Then $\hat{f} : L_n(A) \longrightarrow L_n(B)$

$\hat{f}(\varphi(\bar{x}, \bar{a})) = \varphi(\bar{x}, f(\bar{a}))$

homomorphism of Boolean algebras.

even: monomorphism.

We skip ^ in $\hat{f}$, so:

$f : L_n(A) \longrightarrow L_n(B)$ monomorphism

$\Downarrow$

$f^* : S_n(B) \longrightarrow S_n(A)$ epimorphism in Comp$_0$

i.e.: continuous onto

Lemma (on extensions of elementary mappings)

Assume $M, N \models T$, $A \subseteq M$, $B \subseteq N$, $f : A \xrightarrow{\equiv} B$ "onto".

Assume $\overset{\rotatebox{90}{$\in$}}{a}$, $\overset{\rotatebox{90}{$\in$}}{b}$, $p = tp(a/A)$, $q = tp(b/B)$.

Then $f \cup \{\langle a, b\rangle\}$ is elementary $\iff f^*(q) = p$.

[here $f^* : S(B) \xrightarrow{\approx} S(A)$]
homeomorphism

Proof. exercise.

Def. M is $(<\aleph_0)$-universal $\iff \forall n\, \forall p \in S_n(\emptyset)\ p(M) \neq \emptyset$.

Remark M : $\varkappa$-universal $\Rightarrow$ M : $(<\aleph_0)$-universal.

Proof Let $p \in S_n(\emptyset)$.

Choose a countable $N \models T$ with $p(N) \neq \emptyset$.

M : $\varkappa$-universal $\Rightarrow$ $\exists f : N \xrightarrow{\equiv} M$

$$\underset{\bar a \models p}{\rotatebox{90}{$\in$}} \overset{f}{\leadsto} \underset{f(a) \models p}{\rotatebox{90}{$\in$}}.$$

Thm. (1) M : $\varkappa$-saturated $\Rightarrow$ M : $\varkappa$-homogeneous and $\varkappa$-universal.

(2) M : $\varkappa$-~~universal~~ homogeneous and $(<\aleph_0)$-universal $\Rightarrow$ M : $\varkappa$-saturated.

Proof. (1) $\varkappa$-homogeneity of M:

Assume $f : A \xrightarrow{\equiv} M$, $A \subseteq M$, $|A| < \varkappa$, $a \in M$.

We seek $b \in M$ s.t. $g = f \cup \{\langle a, b \rangle\}$ elementary

$$\Updownarrow \text{ Lemma}$$

$$f^*(tp(b/B)) = tp(a/A).$$

Let $\underset{\substack{\uparrow \\ S_1(A)}}{p} = tp(a/A)$, $q = (f^*)^{-1}(p) \in S_1(B)$

Let $b \in M$ (exists by $\varkappa$-saturation) of M

$\uparrow$ good.

- ~~$\varkappa$-universality~~ of M:

Assume $N \equiv M$, $\|N\| \leq \varkappa$.

We seek $f : N \xrightarrow{\equiv} M$.

Let $\{a_\alpha : \alpha < \mu\}$: an enumeration of N, $\underset{\varkappa}{\overset{}{}}\mu = \|N\|$, $\mu \leq \varkappa$.

We define $f(a_\alpha)$ by induction on $\alpha < \mu$:

- Suppose $f(a_\beta)$ defined for all $\beta < \alpha$ so that

  ~~and~~ $f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} M$

  Want to find $f(a_\alpha)$ so that

  $$f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} M.$$

~~By the Lemma it is enough that t~~

Let $p = tp(a_\alpha / \{a_\beta : \beta < \alpha\})$.

By the lemma it is enough to find $f(a_\alpha) \in M$

so that $f^*(tp(f(a_\alpha)/\{f(a_\beta) : \beta < \alpha\})) = p$.

So let $q = (f^*)^{-1}(p) \in S_1(\underbrace{\{f(a_\beta) : \beta < \alpha\}}_{\text{power} < \kappa})$

M κ-saturated $\Rightarrow$ q realized in M.

Let $f(a_\alpha) \in M$ s.t. $f(a_\alpha) \models q$.

(2) Assume M is κ-homogeneous & $(<\aleph_0)$-~~saturated~~ universal.

Want: M: κ-saturated.

So: Let $A \subseteq M$, $|A| < \kappa$, $p \in S_1(A)$. Show: $p(M) \neq \emptyset$.

Induction on $|A|$.

Case (a): $|A| < \aleph_0$.

$\exists N \succ M \quad p(N) \neq \emptyset$. So let $b \models p$.

Let $A^* = \underset{\{a_1,\dots,a_k\}}{\overset{\|}{A}} \cup \{b\}$

Let $q = tp^N(a_1,\dots,a_k,b) \in S_{k+1}(\emptyset)$

q is realized in M (by $(<\aleph_0)$-universality),

by $\langle \underbrace{a_1',\dots,a_k'}_{A'}, b' \rangle$

Let $g : A \cup \{b\} \longrightarrow A' \cup \{b'\}$, $g(a_i) = a_i'$, $g(b) = b'$.

g: elementary.

$\Rightarrow \; g\restriction_A : A \xrightarrow{\equiv} A'$

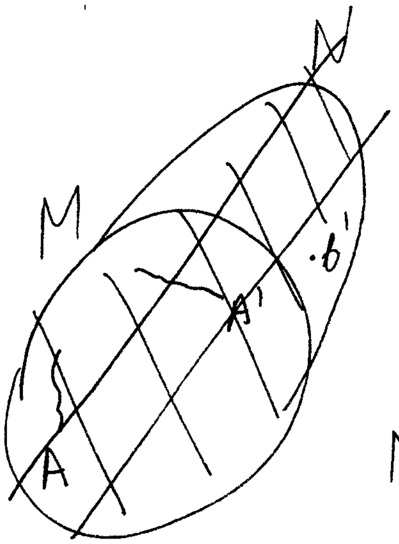
$\Downarrow$

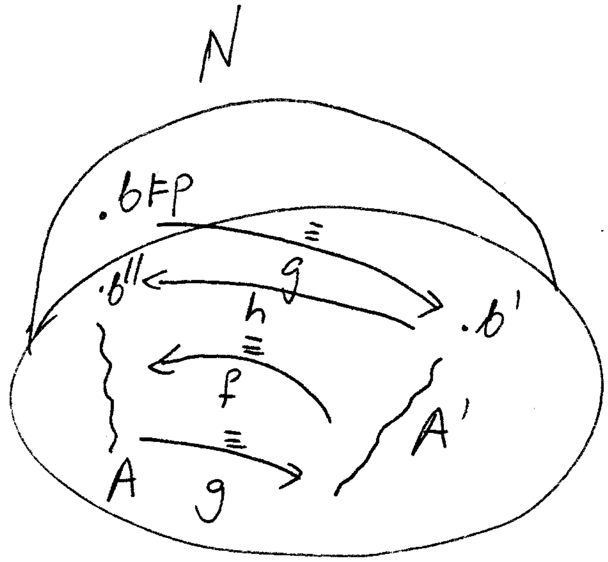
$f := (g\restriction_A)^{-1} : A' \xrightarrow{\equiv} A$

$\cap|$

M: $\varkappa$-homogeneous $\Rightarrow \; \exists \; h : A' \cup \{b'\} \xrightarrow{\equiv} A \cup \{b''\}$

for some $b'' \in M$.





$$\begin{tikzcd} Ab \arrow[rr, "\equiv", "h\circ g"'] \arrow[dr, "g"', "\equiv"] & & Ab'' \\ & A'b' \arrow[ur, "\equiv", "h"'] & \end{tikzcd}$$

Let $s = h \circ g$

$s\restriction A = \underbrace{(h\restriction_{A'})}_{= (g\restriction_A)^{-1}} \circ (g\restriction_A) = id_A$.

$\Downarrow$

$\cancel{s^*(tp(b''/A)) = tp(b}$

$s\restriction_A = id_A \Rightarrow \underbrace{s^*}_{= id_{S(A)}} : S(A) \xrightarrow{\simeq} S(A)$

hence: $p = tp(b/A) \underset{\substack{\uparrow \\ \text{Lemma} \\ \text{and } b'' \models p}}{=} s^*(tp(b''/A)) \underset{\substack{\uparrow \\ s^* = id_{S(A)}}}{=} tp(b''/A)$

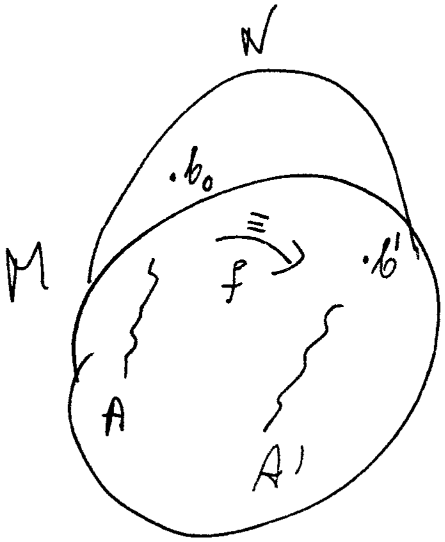
Case (b) $|A| = \mu$, $\aleph_0 \le \mu < \kappa$.

$A = \{a_\alpha : \alpha < \mu\}$, $p \in S_1(A)$.

$p \restriction \emptyset \in S_1(\emptyset) \Longrightarrow \exists b' \in M \ b' \models p \restriction \emptyset$.

M: $(<\aleph_0)$-universal

$\exists N \succ M \ \exists b_0 \in N$, $b_0 \models p$



Will find $A' = \{a'_\alpha : \alpha < \mu\} \subseteq M$

s.t. $f : Ab_0 \longrightarrow A'b'$

given by $f(a_\alpha) = a'_\alpha$

$f(b_0) = b'$

is <u>elementary</u>!

We find a'_α, $\alpha < \mu$ by induction on $\alpha < \mu$.

So suppose $\alpha < \mu$ and a'_β already defined for all $\beta < \alpha$

so that $\underbrace{f\restriction_{\{a_\beta : \beta<\alpha\} b_0}}_{=: f_0} : \{a_\beta : \beta<\alpha\} b_0 \xrightarrow{\equiv} \{a'_\beta : \beta < \alpha\} b'$

We look for a'_α.

Let $q = tp(a_\alpha / \{a_\beta : \beta < \alpha\} \cup \{b_0\})$

Then $(f_0^*)^{-1}(q) \in S(\underbrace{\{a'_\beta : \beta < \alpha\} \cup \{b'\}}_{\text{power } < \mu \le |A|})$

By the lemma it is enough that $a'_\alpha \models_M (f_0^*)^{-1}(q)$.

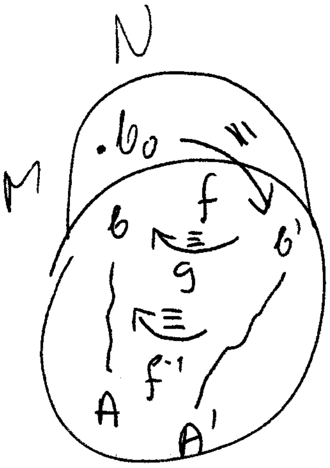
~~But M: κ-~~

~~Let A~~ By the inductive assumption on A:

$(f_0^*)^{-1}(q)$ is realized in M, so we are done with constructing A'.

Now: $f^{-1} : A' \xrightarrow{\equiv} A$ in $M \leftarrow \varkappa$-homogeneous, so

$\exists\ \hat{g}' : A'b' \xrightarrow{\equiv} Ab$ for some $b \in M$.



~~Let~~ Let $s = g \circ f$

$s : Ab_0 \xrightarrow{\equiv} Ab$

$s\restriction_A = \underset{\substack{\| \\ (f\restriction_A)^{-1}}}{(g\restriction_{A'})} \circ (f\restriction_A) = id_A$,

so $p = tp(b_0/A) = tp(b/A)$

and p is realized in M.

<u>Corollary</u>

M is $\varkappa$-saturated $\iff$ M is $\varkappa$-homogeneous and ~~$\varkappa$-universal~~ and $\varkappa$-universal.

<u>Proof</u> $\Rightarrow$ by Thm (1).

$\Leftarrow$ $\varkappa$-homogeneous + $\varkappa$-universal $\Rightarrow$

$\varkappa$-homogeneous + $(<\aleph_0)$-universal $\underset{\text{Thm (2)}}{\Rightarrow}$ $\varkappa$-saturated

Properties of saturated models.

<u>Thm</u>. Assume $M, N \models T$ saturated models of the same power. Then $M \cong N$.

<u>Proof</u> $M = \{m_\alpha : \alpha < \varkappa\}$, $N = \{n_\alpha : \alpha < \varkappa\}$,

$\varkappa = \|M\| = \|N\|$. We find $f : M \xrightarrow{\cong} N$

back-and-forth method:

$$f = \bigcup_{\alpha<\varkappa}^{\nearrow} f_\alpha \qquad f_\alpha : M \xrightarrow[\circ]{\equiv} N \qquad \text{s.t.:}$$

partial, elementary

(1) $m_\alpha \in \mathrm{Dom}\, f_{\alpha+1}$

$n_\alpha \in \mathrm{Rng}\, f_{\alpha+1}$, $|f_\alpha| \le 2\cdot|\alpha|$

(2) $f_0 = \emptyset$

(3) For $\delta \in \mathrm{Lim}$, $f_\delta = \bigcup_{\alpha<\delta}^{\nearrow} f_\alpha$.

(4) $f_{\alpha+1} = f_\alpha \cup \{\langle m_\alpha, \underset{\substack{\uparrow \\ N}}{n}\rangle, \langle \underset{\substack{\uparrow \\ M}}{m}, n_\alpha\rangle\}$

<u>Inductive step:</u>

Suppose we have f_α. Want: $f_{\alpha+1}$.

Let $A_\alpha = \mathrm{Dom}\, f_\alpha \subseteq M$, $B_\alpha = \mathrm{Rng}\, f_\alpha \subseteq N$.

$$f_\alpha : A_\alpha \xrightarrow{\equiv} B_\alpha$$
$$\Downarrow$$
$$f_\alpha^{*} : S(B_\alpha) \xrightarrow{\cong} S(A_\alpha).$$

"forth": Find $n \in N$ st. $f_\alpha \cup \{\langle m_\alpha, n\rangle\}$ elementary

$$\Updownarrow$$

$$(f_\alpha^*)^{-1}(tp(m_\alpha/A_\alpha)) = tp(n/B_\alpha).$$

Let $p = tp(m_\alpha/A_\alpha)$.

So $(f_\alpha^*)^{-1}(p) \in S(B_\alpha)$ is realized in N by some n.

"~~forth~~ back": similarly.

<u>Thm</u> Assume $M, N \models T$ are homogeneous, of the same power and $\forall n<\omega\ \forall p \in S_n(\emptyset)\ (p(M) \neq \emptyset \Leftrightarrow p(N) \neq \emptyset)$.
Then $M \cong N$.

<u>Lemma</u> Under the assumptions of the Thm,

$$\forall A \subseteq M\ \exists f : A \xrightarrow{\equiv} N.$$

<u>Proof</u>. Induction on $|A|$.

Case (a) $|A| < \aleph_0$. $A = \{a_1, \dots, a_n\}$.

Let $p = tp(\langle a_1, \dots, a_n\rangle) \in S_n(\emptyset)$. realized in M

$\Downarrow$

realized in N

by some $\langle b_1, \dots, b_n\rangle \subseteq N$.

$f(a_i) = b_i$ is <u>good</u>.

Case (b) $|A| = \mu \geq \aleph_0$, $A = \{a_\alpha : \alpha < \mu\}$

We find $f(a_\alpha)$ by induction on $\alpha < \mu$.

Inductive step.

Suppose $\alpha < \mu$ and for every $\beta < \alpha$ we have $f(a_\beta)$

s.t. $f : \{a_\beta : \beta < \alpha\} \xrightarrow{\equiv} N$.

We shall find $f(a_\alpha) \in N$ s.t. $f : \{a_\beta : \beta \leq \alpha\} \xrightarrow{\equiv} N$.

Let $a_{<\alpha} := \{a_\beta : \beta < \alpha\}$. Likewise $a_{\leq\alpha}$.

$|a_{\leq\alpha}| < \mu = |A|$

By inductive assumption: $\exists g : a_{\leq\alpha} \xrightarrow{\equiv} N$.

Then $f \circ g^{-1} : \underset{\substack{\cap \\ N}}{g(a_{<\alpha})} \xrightarrow{\equiv} \underset{\substack{\cap \\ N}}{f(a_{<\alpha})}$

By homogeneity of N: $\exists f(a_\alpha) \in N$ s.t.

$$f \circ g^{-1} : \underbrace{g(a_{<\alpha})\, g(a_\alpha)}_{= g(a_{\leq\alpha})} \xrightarrow{\equiv} f(a_{<\alpha}) f(a_\alpha) = f(a_{\leq\alpha})$$

Then $f = (f \circ g^{-1}) \circ g : a_{\leq\alpha} \xrightarrow{\equiv} N$.

Proof of the theorem $\qquad \kappa := \|M\| = \|N\|$

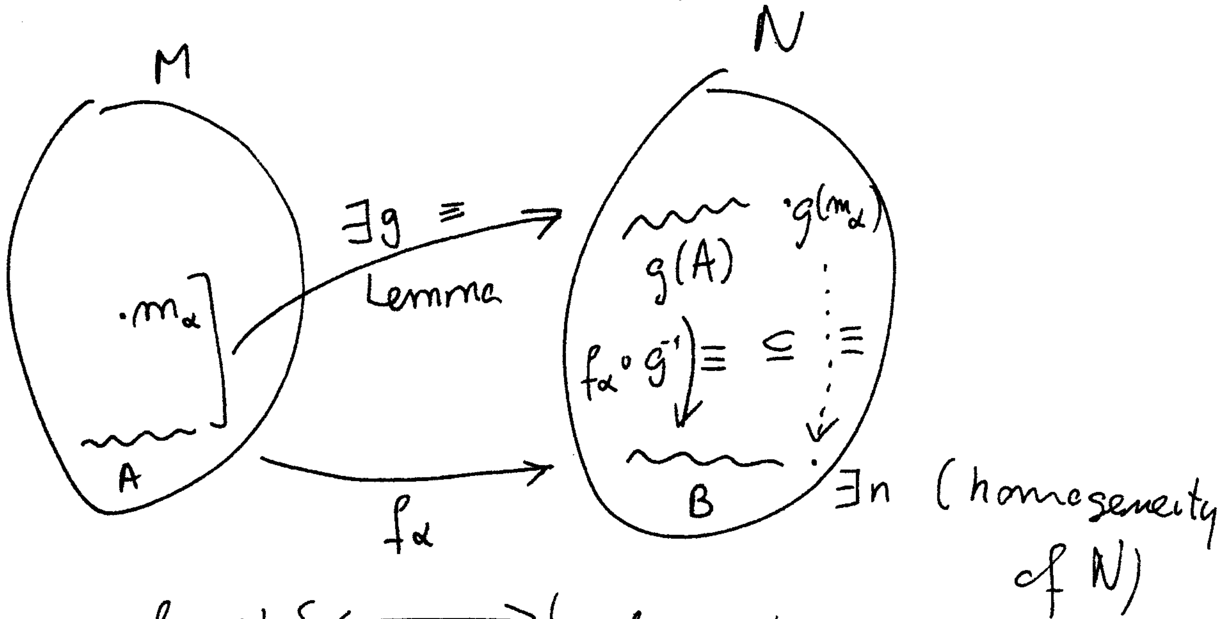
$f : M \xrightarrow{\cong} N$ constructed by back-and-forth method

$f = \bigcup\limits_{\alpha<\kappa}^{\uparrow} f_\alpha$, $f_\alpha : M \xrightarrow[\circ]{\equiv} N$ (partial elementary), $\alpha < \kappa$

$|f_\alpha| \leq 2 \cdot |\alpha|$ + the same conditions as in the previous thm.

Inductive step $f_\alpha \longmapsto f_{\alpha+1}$

$A = \mathrm{Dom}\, f_\alpha \qquad B = \mathrm{Rng}\, f_\alpha$ $\qquad$ $\boxed{\text{"forth":}}$



$\cancel{f_\alpha \cup \{\langle m_\alpha, n\rangle\} \text{ elementary}}$

$$h = (f_\alpha \circ g^{-1})\restriction_{g(A)} \cup \{\langle g(m_\alpha), n\rangle\} \text{ elementary}$$

$$\begin{array}{l} h \circ g : A m_\alpha \xrightarrow{\equiv} B n \subseteq N \\ \;\;\cup\!\mid \\ \;\;f_\alpha \end{array}$$

"back": similarly.