<u>13.06.2022</u> <u>Lemma</u> $I = \{a_n : n < \omega\}$ indiscernible $/A$, based on A, $\varphi(x,y) \in L$

$\varphi'(x, \bar{y}_{2n}) = \bigvee_{\substack{W \subseteq 2n \\ |W| \geq n}} \bigwedge_{i \in W} \varphi(x, y_i)$. Then for $n \geq n_\varphi$, $\varphi'(x, \bar{a}_{<2n})$ is almost over A

($\bar{y}_{2n} = \langle y_0, \dots, y_{2n-1} \rangle$)

<u>Proof</u> (A-a.) Let $\varkappa = (|A|+2)^{\aleph_0}$. Let $\{\bar{a}^i_{<2n} : i < (2^\varkappa)^+\}$ s.t.

$tp(\bar{a}^i_{<2n}/A) = tp(\bar{a}_{<2n}/A)$ and $\{\varphi'(x, \bar{a}^i_{<2n}) : i < (2^\varkappa)^+\}$ are pairwise non-equivalent.

$$\bar{a}^i_{<2n} \subseteq I^i = \{a^i_n : i < \omega\}$$

$$I \xrightarrow{f_i} I^i, \quad f_i \in Aut(\mathfrak{M}/A)$$

I based $/A$ $\Longrightarrow$ I^i based $/A$

Let $M \supseteq A \cup \bigcup_i I_i$ (sufficiently saturated). Let $p_i = Av(I^i/M)$, $i < (2^\varkappa)^+$

<u>Claim</u> If $i \neq j < (2^\varkappa)^+$, then $p_i \neq p_j$

<u>Pf</u> Let $b \in M$ s.t. $\models \varphi'(b, \bar{a}^i_{<2n}) \wedge \neg\varphi(b, \bar{a}^j_{<2n})$

$$\Downarrow$$

$$|\varphi'(b, I^i)| \geq n \geq n_\varphi$$

$$\Downarrow$$

$\varphi(b, I^i)$ is co-finite in I^i

$$\Downarrow$$

$$\varphi(b, y) \in Av(I^i/M)$$

Likewise $\neg\varphi(b, y) \in Av(I^j/M)$ □

Define $\sim$ on M:

$b \sim_0 b' \Leftrightarrow \exists J$ infinite, indiscernible $/A$, both $\{b\}^\frown J$, $\{b'\}^\frown J$ are indiscernible $/A$

Let $\sim$ be the transitive closure of $\sim_0$.

So: $b \sim b' \Leftrightarrow \exists\, b = b_0, \dots, b_n = b'$ $\bigwedge_i b_i \sim_0 b_{i+1}$

<u>~~Sub~~ Claim</u> $b \sim b'$, $p \in S(M)$ dnf $/A$, $\Rightarrow \varphi(x,b) \in p \Leftrightarrow \varphi(x,b') \in p$
$\varphi(x,y) \in L$

<u>Pf</u> May assume $b \sim_0 b'$ ($\exists$ witness J) (A.c.) Suppose $\varphi(x,b) \in p \wedge \neg\varphi(x,b') \in p$.
Let $c \models p$. $\varphi(c,y) \in Av(J/Ac)$ or $\neg\varphi(c,y) \in Av(J/Ac)$
(consider this: $\varphi(c,y) \in Av(J/Ac)$)

Then there is $b^* \in J$ s.t. $\models \varphi(c,b^*)$.
Look at $J' = \{b'\} \cup J$. $b^* \neq b' \in J'$: infinite indiscernible $/A$
and $\varphi(x,b^*) \wedge \neg\varphi(x,b') \in p$, so p forks $/A$ ↯
(but this forks $/A$ (see problem from list 3))

Cases:
1° $\sim$ has $\leq \varkappa$ classes on M.

Let $B \subseteq M$, $|B| \leq \varkappa$, B contains representatives of all $\sim$-classes, $A \cup$
by previous claim, for $i \neq j$ $p_i|B \neq p_j|B$. So $|S(B)| \geq (2^{\varkappa})^+$,
but $|S(B)| \leq 2^{\varkappa}$ ↯

2° $\sim$ has $> \varkappa$ classes on M.

Let $\{b_\alpha : \alpha < \varkappa^+\} \subseteq M$, pairwise ~~...~~ non-$\sim$-equivalent
By the previous lemma: $\exists J \subseteq \{b_\alpha : \alpha < \varkappa^+\}$ infinite, indiscernible $/A$

$b_\alpha, b_\beta \in J$ for some $\alpha < \beta < \varkappa^+$
$\Downarrow$
$b_\alpha \sim_0 b_\beta \Rightarrow b_\alpha \sim b_\beta$ ↯

<u>Thm</u> (Shelah) $p \neq q$, $\in S(\mathfrak{M})$, both dnf $/A$
$\Downarrow$
$\exists E \in FE(A)$ $p(x) \cup q(y) \vdash \neg E(x,y)$

<u>Proof</u> $\varphi(x,b) \in p$, $\neg\varphi(x,b) \in q$. Let $I \subseteq \mathfrak{M}$ indiscernible $/A$, based $/A$ with $b \in I = \{a_n : n < \omega\}$

This implies (by the claim):

$\forall c \in I \, (\varphi(x,c) \in p \ \& \ \neg\varphi(x,c) \in q)$

[or else $\varphi(x,b) \wedge \neg\varphi(x,c)$, $\neg\varphi(x,b) \wedge \varphi(x,c)$]
← fork for every $c \in I$

Let $n \geq n_\varphi$. $\varphi'(x, \bar{a}_{<n})$ is almost $/A$, $\neg\varphi'(x, \bar{a}_{<n}) \in q$

So $\exists E \in FE(A)$ s.t. $\varphi'(\mathcal{M}, \bar{a}_{<n})$ = a union of E-classes.

$p(x) \cup q(y) \vdash \varphi'(x, \bar{a}_{<n}) \wedge \neg\varphi'(y, \bar{a}_{<n}) \vdash \neg E(x,y)$. □

__Def__ $p \in S(A)$ is __stationary__ if $\forall B \supseteq A \ \exists! q \in S(B)$, $q \supseteq p$ nf

__Corollary__ (1) $p \in S(acl^{eq}(A)) \Rightarrow p$: stationary

(2) $p \in S(M) \Rightarrow p$: stationary

__Proof__ (1) Let $B \supseteq acl^{eq}(A)$. Suppose $p \subseteq q_1, q_2 \in S(B)$ nf.

Then $q_i \subseteq q_i' \in S(\mathcal{M})$ ← still dnf $/acl^{eq}(A)$

$q_1' \neq q_2' \Rightarrow \exists E \in FE(acl^{eq}(A)) \ q_1'(x) \cup q_2'(y) \vdash \neg E(x,y)$ (†)

Let $a_1, \dots, a_k$: names of E-classes. $a_1, \dots, a_k \in acl^{eq}(acl^{eq}(A)) = acl^{eq}(A)$
$\mathcal{M}$

"$x/E = a_i$" $\in L(acl^{eq}(A))$, $i = 1, \dots, k$

for some $1 \leq i, j \leq k$ "$x/E = a_i$" $\in q_1'$, "$x/E = a_j$" $\in q_2'$

By (†): $i \neq j$. But $q_1' \restriction acl^{eq}(A) = p = q_2' \restriction acl^{eq}(A)$, so $i = j$. ↯

(2) $M = acl^{eq}(M)$ □

<u>Corollary</u> (1) $A \subseteq B$, $p \in S(B)$. Then p dnf$/A \iff \forall \Delta \subseteq L$ finite $R_\Delta(p) = R_\Delta(p|_A)$

(2) $p \in S(A)$ is stationary $\iff \forall \Delta \in L$ finite $Mlt_\Delta(p) = 1$

(3) $p \in S(A)$ is stationary $\implies \left(\forall \Delta \in L \text{ finite } \forall B \supseteq A \; R_{\Delta,2}(p|B) = R_{\Delta,2}(p)\right)$ and $p|B$ is definable over A (unique nf extension of p is $S(B)$)

<u>Proof</u> (1) $\Leftarrow$ clear. $\Rightarrow$: $p \subseteq p' \in S(acl^{eq}(B))$ ~~s.t. p' dnf /A~~
$\cup$ $acl^{eq}(A)$

Then $(*)$ $\forall \Delta \subseteq L$ finite $R_\Delta(p') = R_\Delta(p)$ (exercise)

$$R_\Delta(p'|acl^{eq}(A)) = R_\Delta(p|_A)$$

~~By~~ We can assume p' dnf $/A$, hence p' dnf $/acl^{eq}(A)$

Therefore we may assume $A = acl^{eq}(A)$, $B = acl^{eq}(B)$

Shall prove $\forall \Delta \subseteq L$ finite $R_\Delta(p') \overset{!}{=} R_\Delta(p'|acl^{eq}(A))$

Let $q = p|_A$. $A = acl^{eq}(A)$, so q is stationary and $q \underset{nf}{\subseteq} p \in S(B) = S(acl^{eq}(B))$.

Let Δ: finite. Want: $R_\Delta(p) = R_\Delta(q)$ ($\leq$ obvious)

$\geq$: Let $k = R_\Delta(q)$, $k \in \mathbb{N}$, wlog $\Delta = \{\delta\}$, closed under $\neg$.

⋮ $\square$

Comment: similarly one can show that in this situation (i.e. when $A = acl^{eq}(A)$, $B = acl^{eq}(B)$) $Mlt_\Delta(p) = Mlt_\Delta(q)$

<u>Pf (3)</u> As in (1) ("lifting") we show that $R_{\Delta,2}(p|_B) = R_{\Delta,2}(p)$ for every $\Delta \subseteq L$ finite (but easier, binary tree)

By Shelah proof of definability lemma $p|B$ is definable $/A$ $\square$

Pf (2) $\Leftarrow$: Let $B \supseteq A$. Suppose

$$S(B) \ni q \overset{\text{nf}}{\supseteq} p \in S(A), \quad S(B) \ni q' \neq q \text{ nf } \supseteq p\ ?$$

By (1) $R_\Delta(q) = R_\Delta(q') = R_\Delta(p)$ for every $\Delta \subseteq L$ finite

Let $\delta(x,\bar c) \in q$, $\neg\delta(x,\bar c) \in q'$. Let $\Delta = \{\delta(x,\bar y)\} \subseteq L$

$$R_\Delta(q) = R_\Delta(q') = R_\Delta(p) \Rightarrow Mlt_\Delta(p) > 1 \quad ↯$$

$\Rightarrow$: Let $p \in S(A)$: stationary.

$\overset{\text{nf}}{q} \supseteq p$, $q \in S(\mathcal{M})$, so $Mlt_\Delta(q) = 1$

by the comment $Mlt_\Delta(q) = Mlt_\Delta(p)$

<u>Corollary</u> Assume $p \in S(A)$, $RM(p) < \infty$, $p \subseteq q \in S(B)$. Then

$$q \text{ dnf } /A \Leftrightarrow RM(p) = RM(q)$$

<u>Corollary</u> Assume $p \in S(M)$.

(1) $RM(p) < \infty \Rightarrow Mlt_{RM}(p) = 1$

(2) $\forall \Delta \subseteq L$ finite $Mlt_\Delta(p) = 1$

<u>Corollary</u> $p \in S(B)$, $B \supseteq A$. Then p dnf $/A \Leftrightarrow p$ is definable $/ acl^{eq}(A)$

FORKING INDEPENDENCE

$A, B, C \subseteq \mathcal{M}$, $a \in \mathcal{M}$

<u>Def</u> $a \perp_C B$ [a is independent from B over C] if $tp(a/B \cup C)$ dnf $/C$

<u>Def</u> $A \perp_C B$ if $\forall \bar a \subseteq A$ $\bar a \perp_C B$

$\perp$: has properties