10.06.2022

(1)

<u>Properties</u> (1) dividing $\Rightarrow$ forking

(2) If $\varphi \vdash \psi$, then $\begin{cases} \psi \text{ divides } /A \Rightarrow \varphi \text{ divides } /A \\ \psi \text{ forks } /A \Rightarrow \varphi \text{ forks } /A \end{cases}$

(3) Assume p: a type. Then

p divides $/A$ / forks $/A$ $\Leftrightarrow$ ~~φ divides~~ $\exists \psi_1, \dots, \psi_n \in p$ $\bigwedge_i \psi_i$ divides $/A$ / $\bigwedge_i \psi_i$ forks $/A$

(4) $p \in S(B)$, then

p divides $/A$ / forks $/A$ $\Leftrightarrow$ $\exists \varphi \in p$ divides $/p$ / forks $/p$

(5) p: a type$/B$ and p "does not fork" dnf $/A \Rightarrow \exists q \in S(B)$, $q \supseteq p$, q dnf$/A$

<u>Pf (5)</u> Let $\Gamma = \{\neg\psi(x,\bar{b}) : \psi \in L, \bar{b} \subseteq B \;\&\; \psi(x,\bar{b}) \text{ forks } /A\}$

- $p \cup \Gamma$ is consistent type over B

  If not then $p \cup \Gamma_0$ is inconsistent, thus $p \vdash \bigvee_{i=0}^{k} \psi_i(x, \bar{b}_i)$
  $\Gamma_0 = \{\neg\psi_0, \dots, \neg\psi_m\}$

  But each of ψ_i forks$/A$, thus $\bigvee_{i=0}^{k} \psi_i(x, \bar{b}_i)$ forks $/A$,

  thus p forks $/A$. $\lightning$

- Let $q(x) \in S(B)$. Then q does not fork $/A$.
  $p \cup \Gamma \subseteq q$

From now on: assume T: stable.

FORKING AND RANKS

<u>Remark</u> Assume $\varphi(x,\bar{b})$ forks $/A$. Then:

(1) $\exists \underset{\text{finite}}{\Delta_0 \subseteq L}\ \forall \Delta_0 \subseteq \underset{\text{finite}}{\Delta \subseteq L}\ \forall p \in S(A)\quad R_\Delta(p \cup \{\varphi(x,\bar{b})\}) < R_\Delta(p)$

(2) $RM(p)<\infty$, then $RM(p\cup\{\varphi(x,b)\})<RM(p)$

Proof (1),(2) $\varphi\vdash\bigvee_{i=0}^{k}\underbrace{\varphi_i(x,b_i)}_{\text{divides}/A}$. Let $\Delta_0=\{\varphi_i(x,y_i): i\le k\}$ ← good.

Let $\Delta_0\subseteq\Delta\subseteq L$ and assume $R_\Delta(p)<\infty$. Will show that

$R_\Delta(p\cup\{\varphi(x,b)\})<R_\Delta(p)$. (a.c.) Suppose $R_\Delta(p\cup\{\varphi(x,b)\})=R_\Delta(p)$

Then there's $\overset{\cap}{q}\in S(A\cup\{b,b_i\}_{i\le k})$ $R_\Delta(q)=R_\Delta(p)$. Then

$q\ni\varphi(x,b)\vdash\bigvee_i\varphi_i(x,b_i)$, thus (wlog) $\varphi_0(x,b_0)\in q$,

but φ_0 divides $/A$. By the def. of dividing we get $\{\varphi_0(x,b_0^i): i<\omega\}$: l-contradictory

and $tp(b_0^i/A)=tp(b_0/A)$.

$R_\Delta(\underbrace{p\cup\{\varphi_0(x,b_0^i)\}}) = R_\Delta(p)$ for $i<\omega$

extend to $q_i\in S(C)$, $R_\Delta(q_i)=R_\Delta(p)$

where $C\supseteq A\cup\{b,b_0^i\}_{i<\omega}$

$q_i|_\Delta\in S_\Delta(C)$, $\{q_i|_\Delta : i<\omega\}$ is infinite (by l-contradictionality ~~of φ~~)

and $R_\Delta(p\cup(q_i|_\Delta))=R_\Delta(p)$ for all i ↯

(because there can be only finitely many such types) ▨

Corollary If $p\in S(A)$, then p dnf $/A$.

~~Assume $\varphi(x,y)$ is~~

Recall ~~Lemma~~ Assume $\varphi(x,y)\in L$. Then $\exists n_\varphi<\omega\ \forall I$: infinite indiscernible $\forall a\ (|\varphi(I,a)|<n_\varphi$ or $|\neg\varphi(I,a)|<n_\varphi)$

Def. Let $I\subseteq M$ infinite indiscernible ← permanent assumption, $A\subseteq M$.

$Av(I/A)=\{\varphi(x)\in L(A): \varphi(I)$ infinite $\}\in S(A)$
the average type of I over A — infinite $=$ cofinite in I

Lemma If $\varphi(x)$: a formula over $\mathcal{M}$, $a \in acl(A)$ and $\models \varphi(a)$, then φ dnf $/A$

Def. I is based on A iff $\forall B$ $Av(I/B)$ dnf $/A$.
$(\iff Av(I/\mathcal{M})$ dnf $/A)$

Remark For $\alpha < \gamma \in Ord$ let $a_\alpha \models Av(I/AIa_{<\alpha})$. Then:

(1) $I \cup \{a_\alpha : \alpha < \gamma\}$ is indiscernible

(2) $\{a_\alpha : \alpha < \gamma\}$: indiscernible $/A$

Proof hint: (a) If $f \in Aut(\mathcal{M}/I)$, $A \subseteq \mathcal{M}$, then $f(Av(I/A)) = Av(I/f(A))$

(b) If $a \notin I$, then $I \cup \{a\}$ is indiscernible $\iff a \models Av(I/I)$

Lemma If $a \notin acl(A)$, then $\exists I$: indiscernible $/A$ based on A, $a \in I$

Proof Choose $M \supseteq Aa$, where $\varkappa = \max(|A|, \aleph_0)$.
M is $\varkappa^+$-saturated

Let $p = tp(a/A) \subseteq_{nf} q \in S(M)$. Then $\forall \Delta \subseteq_{finite} L$ $Mlt_\Delta(q) = 1$.
(non forking) dnf $/A$

Then $\exists B \subseteq M$, $A \subseteq B$, $|B \setminus A| \le \omega$

$\forall \Delta \subseteq_{finite} L$ $\begin{cases} Mlt_\Delta(q) = Mlt_\Delta(q|B) = 1 \\ R_\Delta(q) = R_\Delta(q|B) \end{cases}$

Let $M \ni a_n \models q|_{Ba_{<n}}$ for $n = 0, 1, \ldots < \omega$

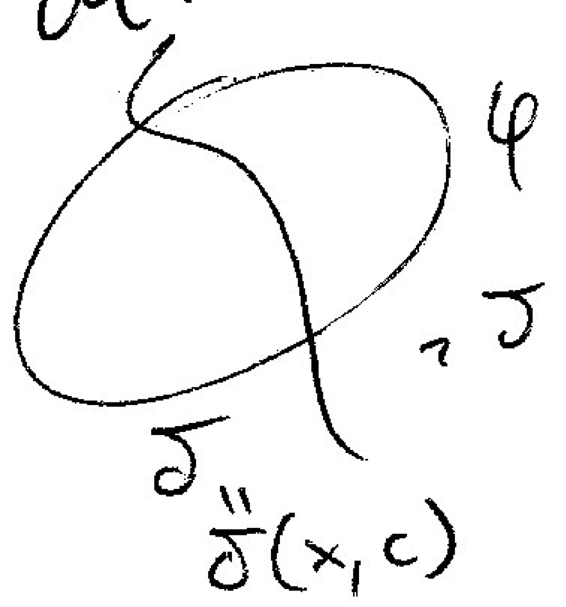
Claim $I = \{a_n : n < \omega\}$: indiscernible $/B$

[hint: as with Morley sequence]
crucial point: If $C \xrightarrow[g]{\cong} D$, $g|B = id_B$

then $g(q|C) = q|D$]

(if >1: choose $\varphi(x) \in q$ with $\omega > R_\Delta(q) = R_\Delta(\varphi)$ and $Mlt_\Delta(\varphi) = Mlt_\Delta(q) > 1$

in $\mathcal{M}$:



δ: Δ-formula
$R_\Delta(\varphi \wedge \delta) = R_\Delta(\varphi \wedge \neg\delta) = R_\Delta(\varphi)$

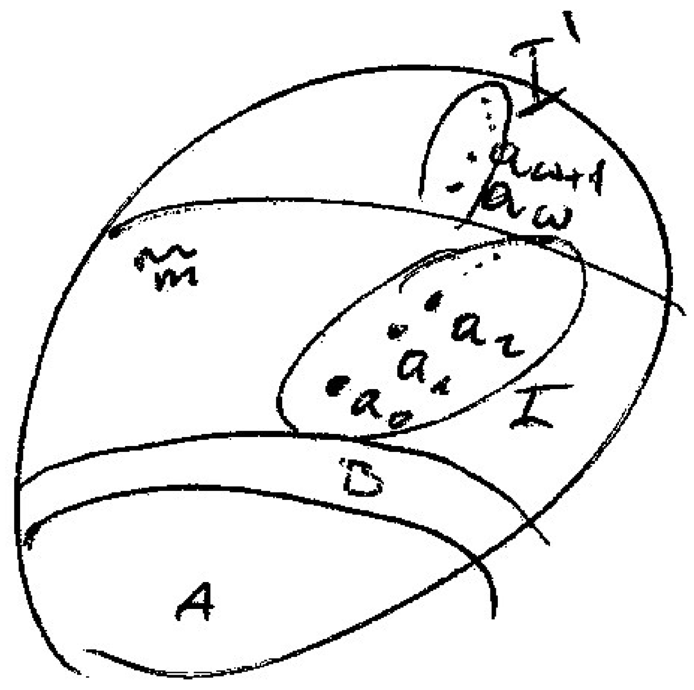
choose $c' \in M$ with $tp(c'/\bar{a}) = tp(c/\bar{a})$

Then $R_\Delta(\varphi \wedge \delta(x, c')) = R_\Delta(\varphi \wedge \neg\delta(x, c')) = R_\Delta(\varphi)$

q or q ↯

Claim $q = Av(I/M)$. ~~Let $C \subseteq \mathcal{M}$ and $\varphi(x,c) \in Av(I/C)$~~

Pf. Let $\varphi(x,m) \in q$. Let $a_{\omega+k} \models q|Ba_{<\omega+k}m$ for all $k=0,1,\ldots<\omega$



$J = I \cup I'$: also indiscernible $/B$

But $I' \subseteq \varphi(J, m)$, thus infinite in J, thus cofinite in J, thus cofinite in I, thus $\varphi(x,m) \in Av(I/M)$ $\square$

Claim I is based on A.

Pf Let $C \subseteq \mathcal{M}$ any and $\varphi(x,c) \in Av(I/C)$. Want: $\varphi(x,c)$ dnf $/A$.

Let $c' \in M$ with $tp(c'/AI) = tp(c/AI)$. So $\varphi(x,c)$ dnf $/A$ $\Leftrightarrow \varphi(x,c')$ dnf $/A$;

$\varkappa^+$-saturated

$\varphi(x,c') \in Av(I/c') \subseteq Av(I/M) = q$ ← nf ext. of p

So $\varphi(x,c')$ dnf $/A$ $\square$

Choose $f \in Aut(\mathcal{M}/A)$ with $f(a_0) = a$. So $a \in f[I]$ ← an A-indisc. set based on A.

Lemma $\square$

Lemma Assume $|B| > \varkappa := (|A|+2)^{\aleph_0}$. Then $\exists I \subseteq B$ infinite, indiscernible $/A$

Proof For a type $p(x)$ let $\vec{R}(p) = \langle (R_\Delta(p), Mlt_\Delta(p)) : \Delta \subseteq_{\text{finite}} L \rangle$

$\forall p \in S(A)\ \exists A_0 \subseteq_{\text{ctble}} A\ \ \vec{R}(p) = \vec{R}(p|A_0)$.

~~Let $B_0 \subseteq_{\text{ctble}} B$~~ Let $p \in S(A \cup B_0)$ where $B_0 \subseteq_{\text{ctble}} B$ s.t.

$p(B) = \{b \in B : b \models p\}$ has power $> \varkappa$ and

$\vec{R}(p)$ is minimal possible (lexicographically) (over all B_0, p)

$\forall B_1 \subseteq_{\text{ctble}} B\ \exists!\, p' \in S(AB_1) \left(\vec{R}(p') = \vec{R}(p)\ \&\ |p'(B)| > \varkappa\right)$ $\Big]\ \otimes$

$B_1 \supseteq B_0$, $p' \supseteq p$

~~If not~~ Let $P = \{p' \in S(AB_1) : p \subseteq p'\}$.

$\overset{\cup\!\!|}{p'} \Rightarrow \vec{R}(p') \leqslant \vec{R}(p)$

$\exists p' \in P \ |p'(B)| > \varkappa$ (as $p(B) = \bigcup_{\substack{p' \in P \\ |P| \leqslant \varkappa}} p'(B)$).

Then $\vec{R}(p') = \vec{R}(p)$

Uniqueness of p': $\mathrm{Mlt}_\Delta(p') = \mathrm{Mlt}_\Delta(p)$ for every $\Delta \subset L$ finite

If there's other p'' like that then

$\underset{\varphi}{\overset{}{p'}} \neq \underset{\neg\varphi}{p''}$, $\Delta \overset{L}{\underset{\cup\!\!|}{}} \varphi \Rightarrow \mathrm{Mlt}_\Delta(p') < \mathrm{Mlt}_\Delta(p)$

φ is a Δ-formula

$\circledast$ ↯

Let $I = \{b_n : n < \omega\} \subseteq B$ st. $\forall n \ p_n = tp(b_n / AB_0 b_{<n})$ is as in $(\ast)$.

clearly I is good.

■

<u>Lemma</u> Assume $I = \{a_n : n < \omega\}$ indiscernible $/A$, based on A.

Assume $\varphi(x,y) \in L$ and for $n < \omega$ $\varphi'(x, \underbrace{\bar{y}_{2n}}_{\langle y_0, \dots, y_{2n-1}\rangle}) = \bigvee_{\substack{W \subseteq 2n \\ |W| \geqslant n}} \bigwedge_{i \in W} \varphi(x, y_i)$

Then for $n \geqslant n_\varphi$ $\varphi'(x, \bar{a}_{<2n})$ is almost over A.

[that is, it is over $acl^{eq}(A)$]

$\Updownarrow$ Recall

$\{f(\varphi'(x, \bar{a}_{<2n})) : f \in Aut(\mathcal{M}/A)\}$ is finite

<u>Theorem</u> (Shelah, finite equivalence relation thm)

Assume $p \neq q \in S(\mathcal{M})$ dnf $/A$. Then $\exists E \in FE(A)$ $p(x) \cup q(y) \vdash \neg E(x,y)$