M for L, $T = Th(M)$, $ER(T) = \{E(\bar{x},\bar{y}) \in L : M \models$ "E is an equivalence relation"$\}$

$\overset{\in}{E} \rightsquigarrow S_E$: new sort symbol.

$L \rightsquigarrow L^{eq}$: many sorted language.

L^{eq}:

- the standard sort $S_=$
- S_E, $E \in ER(T)$ (those are imaginary sort symbols)
- symbols of L refer to $S_=$ ~~(the standard sort)~~
- new function symbols of L^{eq}:
  - $\underset{E(\bar{x},\bar{y})}{\underset{\shortparallel}{E}} \in ER(T) \rightsquigarrow F_E(\bar{x})$, F_E of sort S_E, $\bar{x}$ of $S_=$

$M \models T \rightsquigarrow \underset{\text{for } L^{eq}}{M^{eq}} = \dot{\bigcup}_{E \in ER(T)} S_E^{M^{eq}}$, where $S_E^{M^{eq}} = \{\bar{a}/E : \bar{a} \in M^n\}$

$S_=^{M^{eq}} = M$

the ~~home~~ standard sort of M^{eq}

$M_E := S_E^{M^{eq}}$, elements of M_E: imaginary elements, imaginaries where $E \neq x=y$

$F_E^{M^{eq}}(\bar{a}) = \bar{a}/E$. The symbols of L are naturally interpreted in M.

~~$T^{eq} := Th(M^{eq})$~~

Fact For $\varphi(\bar{x}) \in L$ and $\bar{a} \in M^n$ $M \models \varphi(\bar{a}) \iff M^{eq} \models \varphi(\bar{a})$

Def $T^{eq} := Cn\Big(T \cup \{$"$F_E : S_= \xrightarrow{\text{onto}} S_E$"$\}_{E \in ER(T)} \cup \{F_E(\bar{x}) = F_E(\bar{y}) \leftrightarrow E(\bar{x},\bar{y})\}_{E \in ER(T)}\Big)$

Fact (1) $M \models T \iff M^{eq} \models T^{eq}$

(2) $\forall M^* \models T^{eq}\ \exists M \models T\ \big(M^* \underset{\text{essentially}}{=} M^{eq}$ and $M^*_= = M\big)$

i.e. $\exists f : M^* \xrightarrow{\cong} M^{eq}$

$f\restriction_M = id_M$

(3) T^{eq}: complete

(4) If $E(1), \dots, E(k) \in ER(T)$, $\varphi(x_1, \dots, x_k) \in L^{eq}$, $x_i \in E(i)$
then there's $\psi(y_1, \dots, y_k) \in L$ s.t. $T^{eq} \vdash (\forall y_1, \dots, y_k)\Big(\psi(y_1, \dots, y_k) \leftrightarrow$
of sort $S_=$
$\varphi(F_{E(1)}(y_1), \dots, F_{E(k)}(y_k))\Big)$

Corollary (1) $\forall M \prec \mathcal{M}$ $M^{eq} \prec \mathcal{M}^{eq}$

(2) $\mathcal{M}^{eq} = dcl^{eq}(\mathcal{M}_=)$, $M^{eq} = dcl^{eq}(M_=)$

(3) $A \subseteq \mathcal{M} = \mathcal{M}_=$ is definable [with parameters] in $\mathcal{M}^{eq}$
$\iff$ A is definable [with parameters] in $\mathcal{M}$

(4) $\mathcal{M}^{eq}$ is $\varkappa$-saturated and strongly $\varkappa$-homogeneous

Proof of fact (1), (2): obvious

(3) Let $M^*, N^* \models T^{eq}$ $\overset{(2)}{\Longrightarrow}$ $M^* = M^{eq}$, $N^* = N^{eq}$ for some $M, N \models T$
$\underset{\Longrightarrow}{T \text{ complete}}$ $M \equiv N$ $\overset{(4)}{\Longrightarrow}$ $M^{eq} \equiv N^{eq}$

(4) Induction on φ:

- φ: atomic
- the connectives

] easy

- φ: $\exists x_0 \, \varphi_0(x_0, x_1, \dots, x_k)$. By the ind. hyp. φ_0 is "equivalent" in T^{eq} to $\psi_0(\bar{y}_0, \dots, \bar{y}_k)$
of sort S_{E_0} So: $T^{eq} \vdash \exists x_0 \, \varphi_0(x_0, F_{E(1)}(\bar{y}_1), \dots, F_{E(k)}(\bar{y}_k))$

$\Vert$ (by axioms of T^{eq})

$\exists \bar{y}_0 \, \varphi_0(F_{E(0)}(\bar{y}_0), \dots, F_{E(k)}(\bar{y}_k))$

$\Vert$

$\exists \bar{y}_0 \, \psi(\bar{y}_0, \dots, \bar{y}_k)$

$\blacksquare$

Proof of corollary (1) Goal: $M^{eq} \prec \mathcal{M}^{eq}$. Take $\bar{a}^{eq} \subseteq M^{eq} \models \varphi(\bar{a}^{eq})$, "$\bar{a} = \bar{b}/E$"
by fact (4) $M^{eq} \models \varphi(\bar{a}^{eq}) \iff M^{eq} \models \psi(\bar{b}) \iff M \models \psi(\bar{b}) \iff \mathcal{M} \models \psi(\bar{b})$
$\iff \mathcal{M}^{eq} \models \psi(\bar{b}) \iff \mathcal{M}^{eq} \models \varphi(\bar{a})$

Fact Assume $E \in ER(T^{eq})$. Then there is $\psi \in ER(T)$ on $\mathcal{M}^n$ and

$$\exists f : \mathcal{M}^n \xrightarrow{\text{onto}} \mathrm{dom}(E) \text{ s.t. } \forall \bar{a}, \bar{b} \in \mathcal{M} \; (\mathcal{M}^{eq} \models \psi(\bar{a}, \bar{b}) \leftrightarrow f(\bar{a}) E f(\bar{b}))$$

f: 0-definable in T^{eq}. So: $\exists g : \mathcal{M}_\psi \xrightarrow[\text{onto}]{1-1} \mathcal{M}^{eq}_E$ in $(\mathcal{M}^{eq})^{eq}$, g: 0-definable

Def Assume $X \subseteq \mathcal{M}^{eq}$ definable (with parameters) and $c \in \mathcal{M}^{eq}$. c is a name (code) of X iff

$$\forall f \in Aut(\mathcal{M}^{eq}) \left(f[X] = X \Leftrightarrow f(c) = c \right)$$

Remark (1) $X \subseteq_{def} \mathcal{M}^{eq} \Rightarrow X$ has a name $c \in \mathcal{M}^{eq}$

(2) If c: name of X then $X = \varphi(\mathcal{M}^{eq}, c)$ for some $\varphi \in L^{eq}$

Proof (1) 1° $X \subseteq_{def} \mathcal{M}$, $X = \psi(\mathcal{M}, \bar{a})$, $\bar{a} \in \mathcal{M}$. $E_\psi \in ER(T)$,

$$E_\psi(\bar{y}_0, \bar{y}_1) = \forall x \left(\psi(x, \bar{y}_0) \leftrightarrow \psi(x, \bar{y}_1) \right)$$

So for $\bar{a}_0, \bar{a}_1 \in \mathcal{M}$ $\quad \models E_\psi(\bar{a}_0, \bar{a}_1) \Leftrightarrow \psi(\mathcal{M}, \bar{a}_0) = \psi(\mathcal{M}, \bar{a}_1)$

$$c = \bar{a}/E_\psi = F_{E_\psi}(\bar{a}) \in \mathcal{M}_{E_\psi}$$

good, name for X

2° $X \subseteq \mathcal{M}^{eq} \rightsquigarrow$ a name of X in $(\mathcal{M}^{eq})^{eq} \Rightarrow$ get $c' \in \mathcal{M}^{eq}$ a name of X

$c \in$ $\qquad (\mathcal{M}^{eq})^{eq} = \mathcal{M}^{eq}$

(2) $X \subseteq_{def} \mathcal{M}^{eq}$, c: a name of X in $\mathcal{M}^{eq}$. By def. $\forall f \in Aut(\mathcal{M}^{eq})$
$(f[X] = X \Leftrightarrow f(c) = c)$

hence X is invariant under $Aut(\mathcal{M}^{eq}/c)$

$\Downarrow$

X definable over c

Example $X = \psi(\mathcal{M}, \bar{a})$, $c = \bar{a}/E_\psi$ a name of X

$$x \in X \Leftrightarrow \underbrace{\exists \bar{y} \left(\psi(x, \bar{y}) \wedge F_{E_\psi}(\bar{y}) = c \right)}_{\sigma(x, c) \text{ defines } X}$$

From now on we work in $\underset{\mathcal{M}^*}{\underset{\shortparallel}{\mathcal{M}^{eq}}}$, T^{eq}

Definition Assume $\underset{\text{definable}}{X \subseteq \mathcal{M}}$. X is <u>definable almost over</u> $A \subseteq \mathcal{M}$

$\iff$ the set $\{f[X] : f \in \mathrm{Aut}(\mathcal{M}/A)\}$ is finite

<u>Lemma</u> X is definable almost over $A \iff X$ is definable over $\mathrm{acl}^{eq}(A)$

<u>Proof</u> $(\Leftarrow)$ $X = \varphi(\mathcal{M}, \underset{\mathrm{acl}^{eq}(A)}{\underset{\cap}{\bar{a}}})$ $\quad f[X] = \varphi(\mathcal{M}, \underbrace{f(\bar{a})}_{\text{finitely many possibillities}})$, $f \in \mathrm{Aut}(\mathcal{M}/A)$

$(\Rightarrow)$ $X = \varphi(\mathcal{M}, \bar{b})$, $|\{f[X] : f \in \mathrm{Aut}(\mathcal{M}/A)\}| \underset{k}{\underset{\shortparallel}{\leq}} \aleph_0$. A type over A in $\bar{y}_1, \dots, \bar{y}_{k+1}$

$\{\varphi(\mathcal{M}, \bar{y}_i) \neq \varphi(\mathcal{M}, \bar{y}_j)\}_{1 \leq i \neq j \leq k+1}$ ① $\cup \{\text{"}\bar{y}_i \models tp(\bar{b}/A)\text{"}\}_{1 \leq i \leq k+1}$.

This type has to be inconsistent. So for some $\Theta(\bar{x}) \in \underbrace{tp(\bar{b}/A)}_{p(\bar{x})}$

① $\cup \{\bigwedge_{1 \leq i \leq k+1} \Theta(\bar{y}_i)\}$ is inconsistent with T.

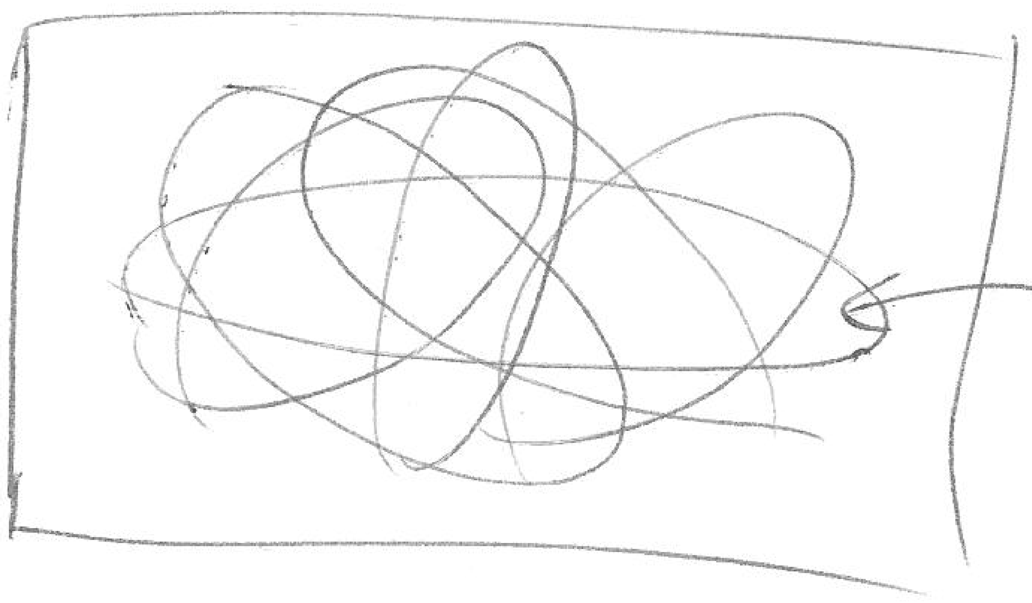
So: $|\underbrace{\{\varphi(\mathcal{M}, \bar{c}) : \bar{c} \in \Theta(\mathcal{M})\}}_{(*)}| \leq k$

$\underset{L(A)}{\underset{\cap}{E}}(x, x') : \forall \bar{c} \in \Theta(\mathcal{M}) \, (\varphi(x, \bar{c}) \leftrightarrow \varphi(x', \bar{c}))$.

E defines in $\mathcal{M}^{eq}$ an equiv. rel. with finitely many classes.

$E \in \underbrace{FE(A)}$

formulas over A defining in $\mathcal{M}$ an equiv. rel. with finitely many classes



$\varphi(\mathcal{M}, \bar{c})$
$\bar{c} \in \Theta(\mathcal{M})$

E-classes are atoms of the algebra of sets generated by $(*)$

X = union of some E-classes $\mathcal{C}_1, \dots, \mathcal{C}_n$

definable in $\mathcal{M}$ $\Longrightarrow$ have names $c_1, \dots, c_n \in \mathcal{M}^{eq}$

$\Longrightarrow c_1, \dots, c_n \in$ ~~dcl~~ $\mathrm{acl}^{eq}(A)$

So X is definable over $\mathrm{acl}^{eq}(A)$

FORKING OF TYPES

Example $p \in S(A)$, $A \subset B$, $RM(p) < \infty$

$$\begin{matrix} p \\ \cap \\ q \in S(B) \end{matrix}$$

1° $RM(p) = RM(q)$: "q is a free extension" of p"

2° $RM(q) < RM(p)$: "q is $\underbrace{\text{a degenerated}}_{\text{forking}}$ extension of p"

Def (1) $\{\varphi(x, a_i) : i < \omega\}$ is <u>k-contradictory</u> ($k < \omega$) if every k-element subset is inconsistent (with T)

(2) $\varphi(x, a)$ <u>divides over A</u> if

$$\exists k < \omega \; \exists \{\varphi(x, a_i) : i < \omega\} : k\text{-contradictory} \bigwedge_{i<\omega} tp(a_i/A) = tp(a/A)$$

(3) A type $p(x)$ (over $\mathcal{M}$) <u>divides over A</u> if $p \vdash \varphi(x, \bar{a})$ for some $\varphi(x, \bar{a})$ that divides over A

Remark In (2) can request $a_0 = a$ and $\langle a_i, i < \omega \rangle$ is order indiscernible

Def (1) $\varphi(x, \bar{a})$ <u>forks over A</u> if $\varphi(x, \bar{a}) \vdash \bigvee_{i \leq n} \underbrace{\varphi_i(x, \bar{b}_i)}_{\text{divides over } A}$ for some $\varphi_i(x, \bar{b}_i)$, $i \leq n$

(2) A type $p(x)$ <u>forks$/A$</u> if $p(x) \vdash \underbrace{\varphi(x, \bar{a})}_{\text{forks}/A}$ for some $\varphi(x, \bar{a})$