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$p \in S_\delta(A)$ $[p \in S_\Delta(A)]$ (Δ possibly infinite)

- $CB_\delta(p) =$ CB in $S_\delta(A)$ (CB-delta rank, CB-local rank)

- $\varphi \in L_\delta(A) \rightsquigarrow CB_{\delta,A}(\varphi) = CB(S_\delta(A) \cap [\varphi])$

- $CB_\delta \rightsquigarrow \underset{\substack{\text{local } \delta\text{-rank} \\ \text{local } \Delta\text{-rank}}}{R_\delta} : L(\mathcal{M}) \longrightarrow \{-1\} \cup \mathrm{Ord} \cup \{\infty\}$

The smallest function s.t. $\forall \alpha \in \mathrm{Ord} \cup \{-1\}$

$$R_\delta(\varphi) \geq \alpha + 1 \iff \forall n < \omega \; \exists \underset{\substack{\delta\text{-formulas} \\ \text{pairwise contr.}}}{\psi_1, \dots, \psi_n} \; \bigwedge_i R_\delta(\varphi \wedge \psi_i) \geq \alpha$$

[Alternatively]

$$R_\delta(\varphi) = CB_{\delta,\mathcal{M}}(\{p \in S_\delta(\mathcal{M}) : \underset{\text{consistent}}{p \cup \{\varphi\}}\})$$

$$R_\delta(p) = \min\{R_\delta(\varphi) : p \vdash \varphi\}$$

[Alternatively]

$$R_\delta(p) = CB_{\delta,\mathcal{M}}(\{q(x) \in S_\delta(\mathcal{M}) : p \cup q \text{ consistent}\})$$

<u>Remark</u> (a) $\underset{\delta = \delta(x,y)}{R_\delta} : L_x(\mathcal{M}) \xrightarrow{\text{onto}} (\{-1\} \cup$ proper initial segment of ordinals $\cup \underset{\text{possibly}}{\{\infty\}})$

(b) $R_\delta(\varphi) \geq \alpha + 1 \iff \exists \underset{\substack{\text{pairwise contr.} \\ \delta\text{-formulas}}}{\psi_1, \psi_2, \dots} \; \bigwedge_i R(\varphi \wedge \psi_i) \geq \alpha$

(c) $R_\delta(\varphi \vee \psi) = \max\{R_\delta(\varphi), R_\delta(\psi)\}$

(d) $p(x)$: a type $/A \implies \exists \underset{p \subseteq}{q}(x)$: a complete type $/A$ $R_\delta(p) = R_\delta(q)$

<u>Example</u>

$$R_\delta = R_{\underbrace{\{\delta, \neg\delta\}}_{\Delta}} \underset{\substack{\uparrow \\ \text{Shelah} \\ \text{trick}}}{=} R_{\delta'} \text{ (for some } \delta')$$

$\rightsquigarrow$ We assume that "δ is closed under negation": i.e. $\forall \bar{a} \; \exists \bar{b} \models \delta(x, \bar{a}) \leftrightarrow \neg\delta(x, \bar{b})$

(1) $R_\delta(\varphi(x, \bar{a})) \geq \overset{\omega}{\underset{}{k}} + 1 \iff \forall n < \omega \; \exists \underset{\substack{\delta\text{-formulas} \\ \text{pairwise contr.}}}{\varphi_1, \dots, \varphi_n} \; \bigwedge_i R_\delta(\varphi(x, \bar{a}) \wedge \varphi_i) \geq k$

$\Leftrightarrow \forall n \, \exists \underset{\text{distinct}}{p_1, \dots, p_n} \in S_\delta(\mathcal{M}) \bigwedge_i R_\delta(p_i \cup \{\varphi\}) \geq k$

$$\left[\begin{array}{l} p_i \neq p_j \quad \text{for } i \neq j \\ \exists c_{ij} \; \delta(x, c_{ij}) \quad \neg\delta(x, c_{ij}) \end{array}\right]$$

$\Leftrightarrow \forall n \, \exists \langle c_{ij} : 1 \leq i \neq j \leq n \rangle \bigwedge_i \Big[\overbrace{\underbrace{\bigwedge_{j \neq i} \delta(x, c_{ij}) \wedge \neg\delta(x, c_{ji})}_{\psi_i(x, \bar{c}) : \text{ a } \delta\text{-formula}} \in p_i}^{\text{only a def. of } \psi_i} \Big]$ and $\overbrace{R_\delta(\varphi(x, \bar{a}) \wedge \psi_i(x, \bar{c})) \geq k}^{\text{enough to have this in the condition}}$

[Notice: $\psi_i(x, \bar{c})$, $i = 1, \dots, n$ are explicitely pairwise $\not\downarrow$]

Lemma For every $\varphi(x, \bar{y})$, $k < \omega$ there is a type $\Phi_{\varphi,k}(\bar{y})$ s.t.

$$\forall \bar{a} \subseteq \mathcal{M} \; R_\delta(\varphi(x, \bar{a})) \geq k \Leftrightarrow \models \Phi_{\varphi,k}(\bar{a})$$

Pf Induction on k.

$\underline{k=0}$ $R_\delta(\varphi(x, \bar{a})) \geq 0 \Leftrightarrow \models \exists x \, \varphi(x, \bar{a})$ so $\Phi_{\varphi,0}(\bar{y}) = \{\exists x \, \varphi(x, \bar{y})\}$

$\underline{k \to k+1}$ $R_\delta(\varphi(x, \bar{a})) \geq k+1 \Leftrightarrow_{\models} \underbrace{\forall n \, \exists \bar{c} \bigwedge_{1 \leq i \leq n} \Phi_{\varphi \wedge \psi_i, k}(\bar{a}, \bar{c})}_{\models \Psi_n(\bar{a}) \text{ for a type } \Psi_n(\bar{y})}$

$$\Phi_{\varphi,k+1}(\bar{y}) = \bigcup_n \Psi_n(\bar{y})$$

□

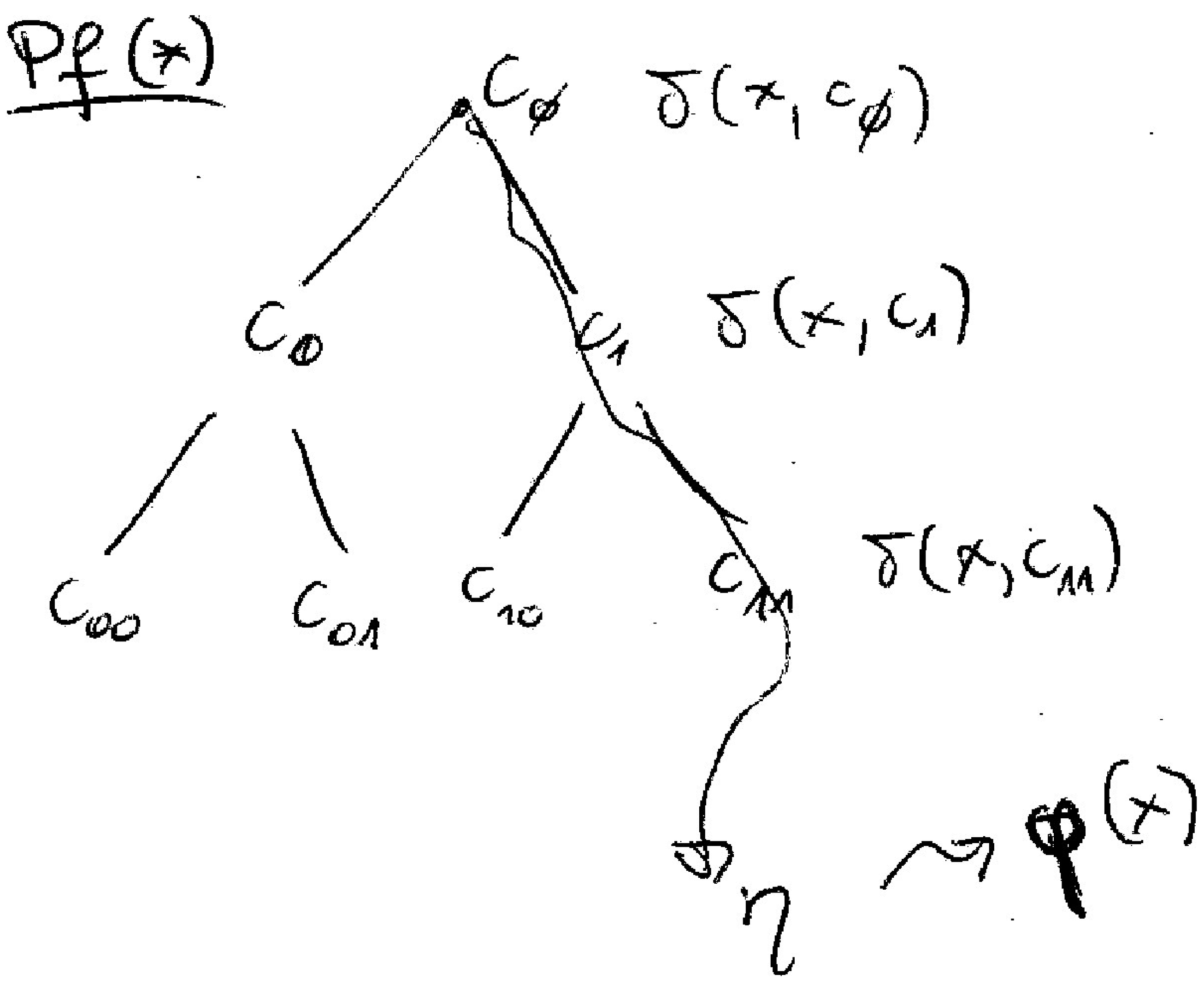
Corollary (1) $R_\delta(\varphi) \geq \omega \Leftrightarrow R_\delta(\varphi) = \infty$

(2) δ is stable $\Leftrightarrow R_\delta(x = x) < \omega$

Proof (1) $\Leftrightarrow$ ($\ast$): $\exists (c_\eta, \eta \in 2^{<\omega}) \forall \eta \in 2^\omega$ $p_\eta(x) = \{\varphi\} \cup \{\delta(x, c_{\eta|n})^{\eta(n)} : n < \omega\}$ is consistent

here $\begin{cases} \chi^0 = \chi \\ \chi^1 = \neg\chi \end{cases}$

Pf ($\ast$)



$\eta \rightsquigarrow \varphi(x) \cup \{\neg\delta(x, c_\emptyset), \neg\delta(x, c_1), \neg\delta(x, c_{11}) \dots\}$ consistent

good

Enough to show $\forall n \, \exists c_\eta, \eta \in 2^{<n} \, \forall \eta \in 2^n \; p_\eta(x)$ consistent. But this follows from $R_\delta(\varphi) \geq n+1$

φ $\qquad\qquad\qquad R_\delta \geq n+1$

$\varphi \wedge \delta(x, c_\varphi) \qquad \varphi \wedge \neg\delta(x, c_\varphi) \qquad\qquad R_\delta \geq n$

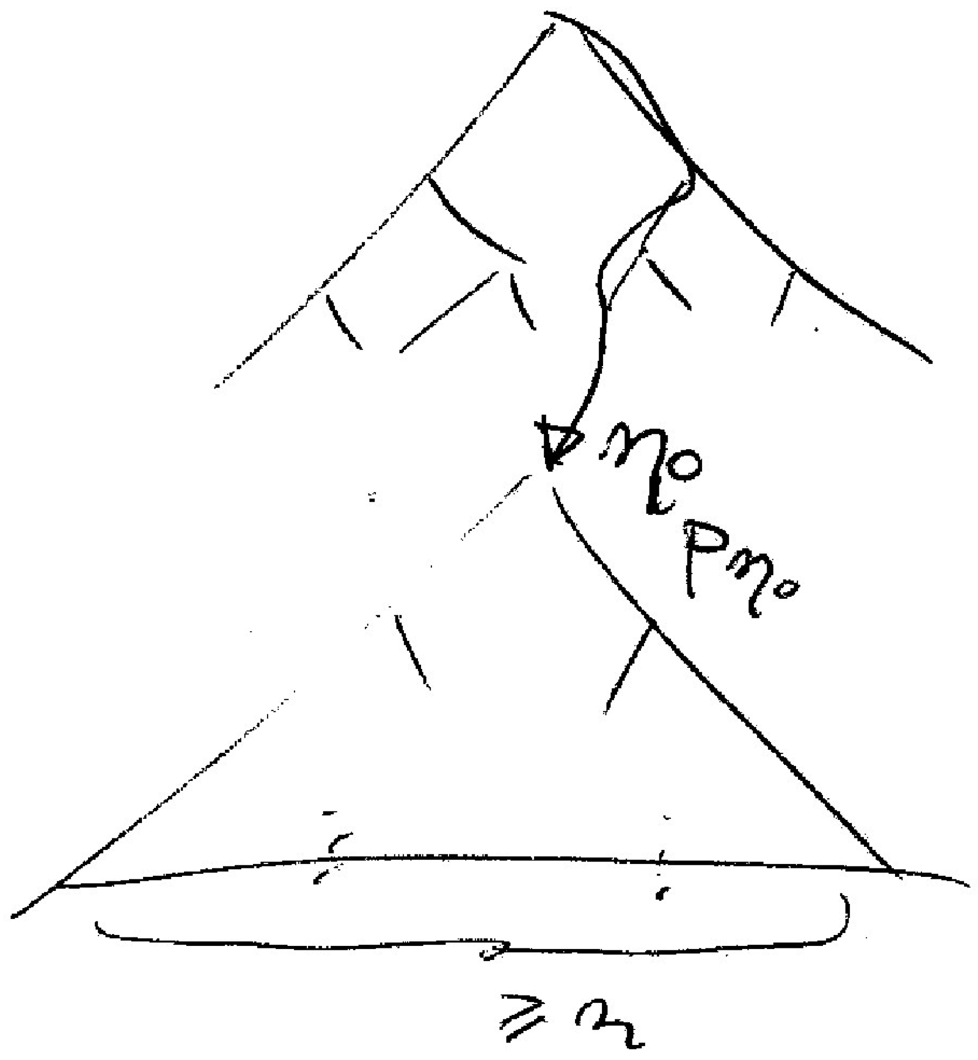
$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots$

$\uparrow$ n many times $\downarrow$

$(*)\square$

Proof of $\Rightarrow$ in (1): (A.c.) Suppose $R_\delta(\varphi) < \infty$. For $\eta \in 2^{<\omega}$ $R_\delta(p_\eta(x)) < \infty$.

Let $\eta_0 \in 2^{<\omega}$ s.t. $\alpha := R_\delta(p_{\eta_0})$ is minimal.



So $\forall n\ \exists \varphi_1, \dots, \varphi_n$ $\bigwedge_i R_\delta(p_{\eta_0} \wedge \varphi_i) \geq \alpha$ $\quad$ ↯

δ-formulas
pairwise ↯

Pf (2) ~~δ not stable $\Leftrightarrow R_\delta$~~ $(\Rightarrow)$(A.c.) $R_\delta(x=x) \geq \omega \overset{(1)}{\Rightarrow} R_\delta(x=x) = \infty$

$\overset{\text{the proof of (1)}}{\Longrightarrow}$ get a tree of instances of $\delta \Rightarrow |S_\delta(\{c_\eta : \eta \in 2^{<\omega}\})| = 2^{\aleph_0}$

$\Rightarrow \delta$ unstable

$(\Leftarrow)$ $R_\delta(x=x) < \omega$, A: a set, $|A| \leq \kappa$, shall prove $|S_\delta(A)| \leq \kappa$

(a) $p \in S_\delta(A) \rightsquigarrow \exists \varphi \in p$ $R_\delta(p) = R_\delta(\varphi) \leq k < \omega$ (for some)

There are $\leq \kappa$-many such $\varphi \in L(A)$

(b) For $\varphi \in L_\delta(A)$ $|\{p \in S_\delta(A) : \varphi \in p \text{ and } R_\delta(p) = R_\delta(\varphi)\}| < \aleph_0$

$\square$

<u>Corollary</u> $R_\delta(p) \geq k+1 \iff \forall n\, \exists \varphi_1, \dots, \varphi_n\ R_\delta(p \cup \{\varphi_i\}) \geq k$

any type $\quad$ δ-formulas pairwise $\perp$

BINARY RANKS : $R_{\delta,2}$

$$R_{\delta,2}(\varphi) \geq \alpha+1 \iff \exists c \left[R_{\delta,2}(\varphi \wedge \delta(x,c)) \geq \alpha \text{ and } R_{\delta,2}(\varphi \wedge \neg\delta(x,c)) \right]$$

<u>Lemma</u> (1) $R_{\delta,2}(\varphi(x,\bar{a})) \geq k \iff \models \Theta_{\varphi,k}(\bar{a})$ for a formula $\Theta_{\varphi,k}(\bar{y})$

(2) $R_{\delta,2}(\varphi(x,\bar{a})) = k \iff \models \Theta_{\varphi,k}(\bar{a}) \wedge \neg \Theta_{\varphi,k+1}(\bar{a})$

(3) $R_{\delta,2}(\varphi) \geq \omega \iff R_{\delta,2}(\varphi) = \infty$

(4) $R_{\delta,2}(x=x) < \infty \iff \delta$ is stable

<u>Comments</u> (a) Usually $R_{\delta,2}(\varphi \vee \psi) \neq \max\{R_{\delta,2}(\varphi), R_{\delta,2}(\psi)\}$.

(b) So usually it is not true that $\forall p\, \forall A\, \exists_{p \subseteq q}\, q \in S(A)\ R_{\delta,2}(p) = R_{\delta,2}(q)$

Definability again

<u>Lemma (Shelah)</u> Let $\delta(x,y)$: stable, $p(x) \in S_\delta(A)$. Then

$$\exists \chi \in L(A)\, \forall \bar{a} \subseteq A \models \chi(\bar{a}) \iff \delta(x,\bar{a}) \in p(x)$$

a δ-definition

<u>Proof</u> $R_{\delta,2}(p) = k < \omega$. Choose $\psi \in p$ with $R_{\delta,2}(\psi) = k$. Then for $\bar{a} \subseteq A$

$$\delta(x,\bar{a}) \in p \iff R_{\delta,2}(\psi \wedge \delta(x,\bar{a})) \geq k \iff \underbrace{\Theta_{\psi\wedge\delta,k}(\bar{a})}_{\chi(\bar{a})}$$

$\blacksquare$

<u>Corollary</u> $\left[\delta\text{: stable}, a \in \mathcal{M}, A \subseteq \mathcal{M}\right]$ $\delta(a,A) = \{b \in A : \models \delta(a,b)\} = \{b \in A : \models \chi(b)\}$ for some $\chi(y) \in L(A)$

<u>Proof</u> $p = tp_\delta(a/A) \rightsquigarrow \chi$ from the lemma

Many-sorted language: L

- $\{S_i : i \in I\}$ a set of sorts
- each sort S_i has a specified set of variables $x_n^{S_i}$, $n<\omega$
- each predicate symbol $P(\underset{\uparrow}{x_1},\dots,\underset{\uparrow}{x_n})$
  prescribed sorts
- each function symbol $\underset{\text{value}}{f}(\overset{\downarrow}{x_1},\dots,\overset{\downarrow}{x_n})$
- likewise constants

A structure for L: $|M| = M = \dot{\bigcup}_{i\in I} \underbrace{M_{S_i}}_{\neq \emptyset}$. Predicates, functions and constants are interpreted respecting sorts
[many-sorted structure]

the S_i-sort of M

Formulas of L:

- $t(\bar{x}) = t(\bar{x})$
  terms of the same sorts
- $P(\vec{t}(x))$
- complex formulas as before
  (but quantify over some fixed sort)

KRL as before

Example: "Plain" $= (P, L; \in)$
points, lines, incidence relation