<u>Lemma 2</u> Assume δ is stable, $p \in S(M)$ or $p \in S_\delta(M)$. Then
(1) $p(x)$ has a δ-definition $\psi(y)$, that is a positive δ-formula
(2) If $A \subset M \leftarrow |A|^+$-saturated, then $\exists c_1, c_2, \ldots \in M$ $\bigwedge_i c_i \models p\restriction_{A c_{<i}}$ and ψ from (1) is a positive Boolean combination of formulas $\delta(c_i, x)$, $i<\omega$

<u>Proof</u> Let $N<\omega$ s.t.

(i) $\neg(\exists c_i, a_i, i \leq N)(\bigwedge_{i,j} \models \delta(c_i, a_j) \Leftrightarrow i \leq j)$

(ii) $\neg(\exists c_i, b_i, i \leq N)(\bigwedge_{i,j} \models \neg\delta(c_i, b_j) \Leftrightarrow i<j)$

Let $c^* \models p$. So: $\delta(x,a) \in p \Leftrightarrow \models \delta(c^*, a)$ for $a \in M$.

We shall define $c_i \subseteq M$, $\underbrace{A_i, B_i}_{\text{finite}} \subseteq M$, $i<\omega$ as follows:

- $c_0 \subseteq M$ arbitrary ($c_0 \models p\restriction_A$)
- Suppose we have $c_0, \ldots, c_n, A_0, \ldots, A_n, B_0, \ldots, B_n$. We shall find $c_{n+1}, A_{n+1}, B_{n+1}$.
- $A_{n+1} \subseteq M$ s.t. $\forall W \subseteq \{0, \ldots, n\}$ <u>if</u> $(\exists a \in M)(\bigwedge_{j \in W} \models \delta(c_j, a)$ and $\underbrace{\models \neg\delta(c^*, a)}_{\text{i.e. } \delta(x,a) \notin p})$ <u>then</u> $\exists$ such an $a_W \in A_{n+1}$
- $B_{n+1} \subseteq M$ s.t. $\forall W \subseteq \{0, \ldots, n\}$ <u>if</u> $(\exists b \subseteq M)(\bigwedge_{j \in W} \models \neg\delta(c_j, b)$ and $\models \delta(c^*, b))$ <u>then</u> $\exists$ such an $b_W \in B_{n+1}$.
- for $b \in B_{\leq n+1}$ $\models \delta(c^*, b)$, for $a \in A_{\leq n+1}$ $\models \neg\delta(c^*, a)$
we may assume
- $c_{n+1} \in M$ s.t. $(\forall a \in A_{\leq n+1})(\forall b \in B_{\leq n+1})(\models \delta(c_{n+1}, b)$ and $\models \neg\delta(c_{n+1}, a))$
[for (2): $c_{n+1} \models p|_{A A_{\leq n+1} B_{\leq n+1}}$ is enough]
- So for $a \in A_{\leq n}$, $b \in B_{\leq n}$ $\models \neg\delta(c_n, a) \wedge \delta(c_n, b)$
- <u>if</u> $0 \leq i_0 < \ldots < i_n < \omega$ and $\exists a \in M$
$\models \delta(c_{i_0}, a) \wedge \delta(c_{i_1}, a) \wedge \ldots \wedge \delta(c_{i_n}, a) \wedge \neg\delta(c^*, a)$
<u>then</u> $\exists d_0, \ldots, d_n \in M$ $(\forall 0 \leq j, r \leq n)(\models \delta(c_{i_j}, d_r) \Leftrightarrow j \leq r)$

pf Let $W_r = \{i_0, \dots, i_{r-1}\} \subseteq \{i_0, \dots, i_n\}$. Let $d_r = a_{W_r} \in A_{(i_{r-1})+1}$ s.t.
$\models \sigma(c_{i_0}, d_r) \wedge \dots \wedge \sigma(c_{i_{r-1}}, d_r) \wedge \neg\sigma(c^*, d_r)$, $\models \neg\sigma(c_j, d_r)$ for all $j > i_{r-1}$

- simmetrically: if $0 \le i_0 < \dots < i_n < \omega$ and $\exists b \in M$
  $\models \neg\sigma(c_{i_0}, b) \wedge \dots \wedge \neg\sigma(c_{i_n}, b) \wedge \sigma(c^*, b)$
  then $\exists d_0, \dots, d_m \in M$ $\forall 0 \le j, r \le n$ $(\models \neg\sigma(c_{i_j}, d_r) \Leftrightarrow j < r)$

- Therefore:
  (a) If $a \in M$ and $\exists W \subseteq \{0, \dots, 2N\}$ s.t. $\bigwedge_{i \in W} \models \sigma(c_i, a)$, then $\models \sigma(c^*, a)$
  $|W| = N+1$

  (b) If $a \in M$ and $\exists W \subseteq \{0, \dots, 2N\}$ s.t. $\bigwedge_{i \in W} \models \neg\sigma(c_i, a)$, then $\models \neg\sigma(c^*, a)$
  $|W| = N+1$

- Let $a \in M$. $\sigma(x, a) \in p \Leftrightarrow \models \sigma(c^*, a) \Leftrightarrow \models \underbrace{\bigvee_{\substack{W \subseteq \{0, \dots, 2N\} \\ |W| = N+1}} \bigwedge_{i \in W} \sigma(c_i, a)}_{\text{put } y\,:\,\psi(c_{\le 2N}, y)}$

- So $\forall a \in M$ $(\sigma(x, a) \in p \Leftrightarrow \models \psi(c_{\le 2N}, a))$: a delta definition of p.

Lemma ■

Corollary Let $\sigma \in L$. TFAE:

(1) $\sigma(x, y)$: stable

(2) $\forall M \forall p \in S_\sigma(M)$ p has a σ-definition

(3) $(\forall \lambda \ge \aleph_0)\left(\forall A \subseteq \mathcal{M}, |A| \le \lambda\right)\left(|S_\sigma(A)| \le \lambda\right)$

(4) $(\exists \lambda \ge \aleph_0)$ —— " ——

Proof (1) ⇒ (2): by lemma 2

- (2) ⇒ (3) $A \subseteq M \prec \mathcal{M}$, $\|M\| \le \lambda$; $|S_\sigma(A)| \le |S_\sigma(M)| \le \|M\| \le \lambda$
  there are $\le \|M\|$ σ-definitions$/M$

- (3) ⇒ (4) trivial

(4) $\Rightarrow$ (1) by contradiction: suppose δ has order property. Let $\lambda \geq \aleph_0$.

We shall find A s.t. $|A| \leq \lambda$ and $|S_\delta(A)| > \lambda$

- $(\exists a_i, b_i, i<\omega)(\models \delta(a_i, b_j) \iff i<j)$, choose $(\underset{\text{dense}}{I}, \leq) \models DLO_0$, $|I| > \lambda$

  where $\underset{|J| \leq \lambda}{\exists J \subseteq I}$: J dense. By compactness

  $$(\exists a_i, b_i, i \in I) \bigwedge_{i,j \in I} (\models \delta(a_i, b_j) \iff i<j)$$

- δ defines an order:

  $$0 < \underset{i \in I \supseteq J}{\frac{a_i}{b_i}} < 0 \underset{\delta}{\leq} 0 < \dots$$

  Let $A = \{b_j : j \in J\}$, then $|A| \leq \lambda$ and $|S_\delta(A)| > \lambda$:

  when $i_1 < i_2 \in I$ then $tp_\delta(a_{i_1}/A) \neq tp_\delta(a_{i_2}/A)$,

  just choose $j \in J$ s.t. $i_1 < j < i_2$. $\blacksquare$

<u>Corollary</u> (1) T: stable $\iff (\forall \delta \in L)(\delta$: stable in $T)$

(2) T stable and $\lambda^{|T|} = \lambda$, then T: λ-stable

<u>Proof</u> (1) $(\Rightarrow)$ T: λ-stable. Then $\overbrace{\forall \delta \in L}\ \underset{|A| \leq \lambda}{\forall A \subseteq M}$ $|S_\delta(A)| \leq |S(A)| \leq \lambda$, so by previous corollary δ: stable.

$(\Leftarrow)$ later...

(2) $|T| = \aleph_0$, assume $\lambda = \lambda^{\aleph_0}$. Let $|A| \leq \lambda$, ~~$S(A)$~~

$$S(A) \ni p \xmapsto{1-1} \langle p|_\delta : \delta \in L \rangle \in \prod_{\delta \in L} S_\delta(A)$$

$$\left|\prod_{\delta \in L} S_\delta(A)\right| \leq \lambda^{\aleph_0} = \lambda \rightsquigarrow |S(A)| \leq \lambda$$

$(\Leftarrow)$ Assume $\forall \delta \in L$ δ: stable in T. Then $\underset{|A| \leq \lambda}{\wedge}$ $|S(A)| \leq \left|\prod_{\delta \in L} S_\delta(A)\right| \leq \lambda^{\aleph_0}$

$$\Downarrow$$

T stable in λ if $\lambda^{\aleph_0} = \lambda$ $\blacksquare$

Def (1) $\delta(x,y) \in L$ has <u>strict order property</u> (SOP)

if $(\exists b_i, i<\omega)(\delta(\mathcal{M}, b_0) \subsetneq \delta(\mathcal{M}, b_1) \subsetneq \dots)$

$[\Leftrightarrow \ \delta(b_0, \mathcal{M}) \subsetneq \delta(b_1, \mathcal{M}) \subsetneq \dots]$

alternative definition

T has SOP $\Leftrightarrow (\exists \delta \in L)(\delta$ has SOP$)$

(2) δ has independence property (IP)

if $(\exists b_i, i<\omega)$ $\{\delta(\mathcal{M}, b_i) : i<\omega\}$ is independent

T has IP $\Leftrightarrow (\exists \delta \in L)(\delta$ has IP$)$

<u>Theorem</u> (Shelah)

T has OP $\Leftrightarrow$ T has SOP or T has IP

(is unstable)

<u>Examples</u> • $DLO_0 = Th(\mathbb{Q}, \leq)$ has SOP : $\varphi(x,y) = x \leq y$ has SOP,

but DLO_0 has NIP

• Random graph has IP and NSOP

~~TYPES AND DEFINABL~~

<u>TYPES & TYPE-DEFINABLE SETS.</u>

$p(\bar{x})$: a type $\rightsquigarrow$ $p(\mathcal{M}) = \{a \in \mathcal{M}^n : a \models p\}$: a type-definable set (over A)

$|\bar{x}| = n$ (over A)

$= \bigcap_{\varphi \in p} \varphi(\mathcal{M})$

$= (\bigwedge_{i \in I} \varphi_i)(\mathcal{M})$

$p = \{\varphi_i : i \in I\}$

Let $p(\bar{x}), q(\bar{x})$: types, $(p \wedge q)(\bar{x}) = (p \cup q)(\bar{x})$,

$(p \vee q)(\bar{x}) = \{((\bigwedge p_0) \vee (\bigwedge q_0))(\bar{x}) : p_0 \subseteq p, q_0 \subseteq q$ finite$\}$

<u>Fact</u> (1) $(p \wedge q)(\mathcal{M}) = p(\mathcal{M}) \cap q(\mathcal{M})$

(2) $(p \vee q)(\mathcal{M}) = p(\mathcal{M}) \cup q(\mathcal{M})$

$$(\exists \bar{y}\, p)(\bar{x},\bar{y}) = \exists \bar{y} \bigwedge_i \varphi_i(\bar{x},\bar{y}) \overset{def}{=} \{ \exists \bar{y} \bigwedge p_0(\bar{x},\bar{y}) : p_0 \subseteq_{\text{finite}} p \}$$

where $p = \{\varphi_i : i \in I\}$

(3) $(\exists \bar{y}\, p)(\mathcal{M}) = \pi_{\bar{x}}(\underset{\mathcal{M}^{\bar{x}} \times \mathcal{M}^{\bar{y}}}{\overset{\text{in}}{p(\mathcal{M})}})$

(4) $(\forall \bar{y} p)(\mathcal{M}) = \mathcal{M}^{\bar{x}} \setminus \pi_{\bar{x}}(\mathcal{M}^{\bar{x}\bar{y}} \setminus p(\mathcal{M}))$

<u>Remark</u> $\mathcal{M} \setminus p(\mathcal{M})$ is usually <u>not</u> type-definible

Let $\Delta = \{\varphi_0(\underset{\text{tuples}}{x, y}) \ldots \varphi_n(x,y)\} \subseteq L$

Δ-formula, Δ-type

<u>Shelah trick</u> There's formula $\delta(x,\bar{y})$ s.t. "instances of formulas Δ" = "instances of δ" over any set A of power ≥ 2.

<u>Proof</u> $\delta = \delta(x, y, z_0, \ldots, z_n) =$

$$\begin{cases} \varphi_0(x,y), & \text{when } z_0 = z_1 \\ \varphi_1(x,y), & \text{when } z_0 \neq z_1 \wedge z_0 = z_2 \\ \varphi_2(x,y), & \text{when } z_0 \neq z_1, z_2 \wedge z_0 = z_3 \\ \vdots & \\ \varphi_n(x,y), & \text{when } z_0 \neq z_1, \ldots, z_n \end{cases}$$

- $\delta(\mathcal{M}, a, \bar{c}) = \varphi_i(\mathcal{M}, a)$ for some $i \in \{0, \ldots, n\}$
- For every $a \subseteq A$, every $i \in \{0, \ldots, n\}$ $\exists \bar{c} \subseteq A$ $\varphi_i(\mathcal{M}, a) = \delta(\mathcal{M}, a, \bar{c})$

∎