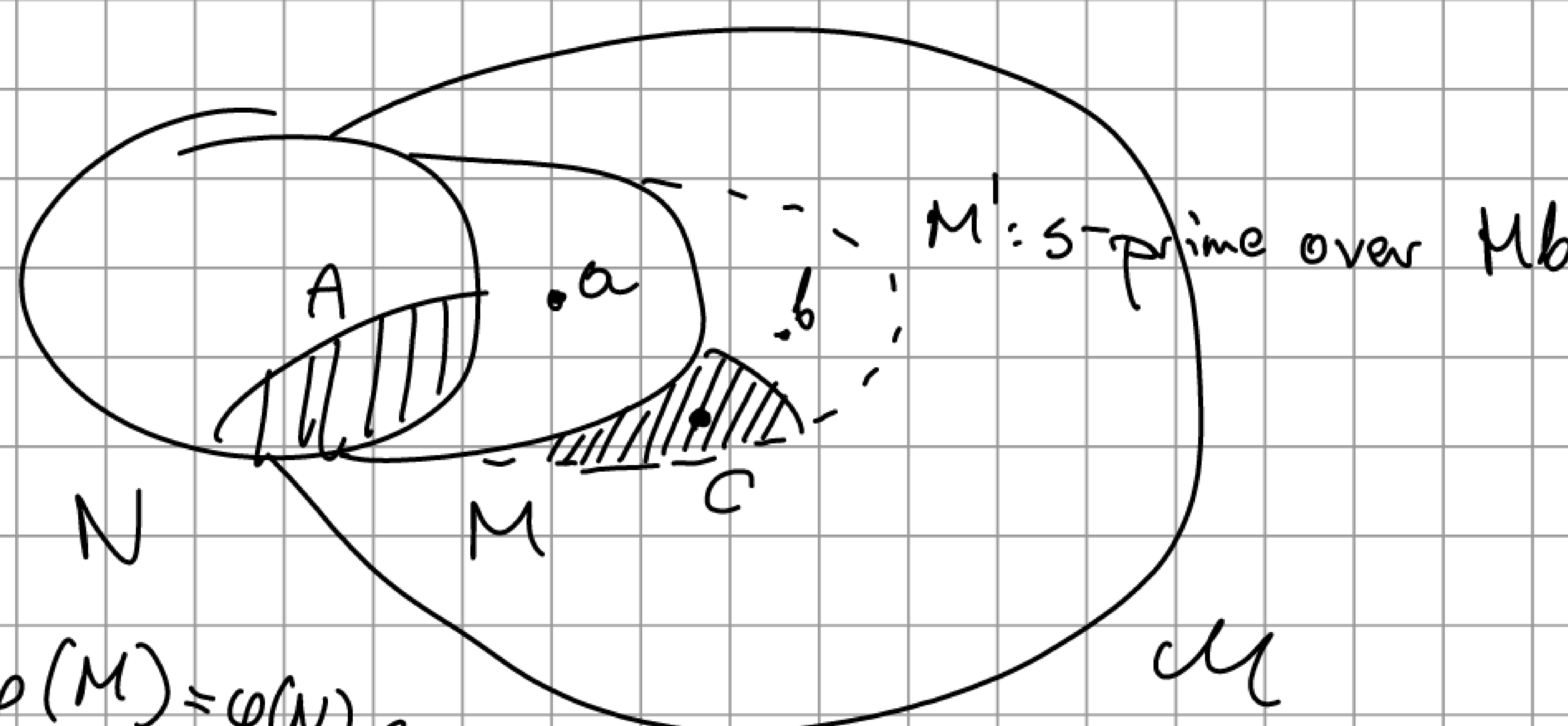
16.05.2022

Lemma 2 (Stretching Vaughtian pairs table)

T: $\aleph_0$-stable, (M, N, φ): Vaughtian triple
(M, N: $\aleph_0$-saturated)

$\Rightarrow \exists M' \succnsim M$ (M': $\aleph_0$-saturated) (M', N, φ): Vt.

Proof



$A = \varphi(M) = \varphi(N) \subseteq \varphi(M')$

$tp(b/M) \supseteq tp(a/M)$, $RM(b/A) = RM(b/M) = RM(a/N)$

Claim: $\varphi(M) = \varphi(M')$ ▨

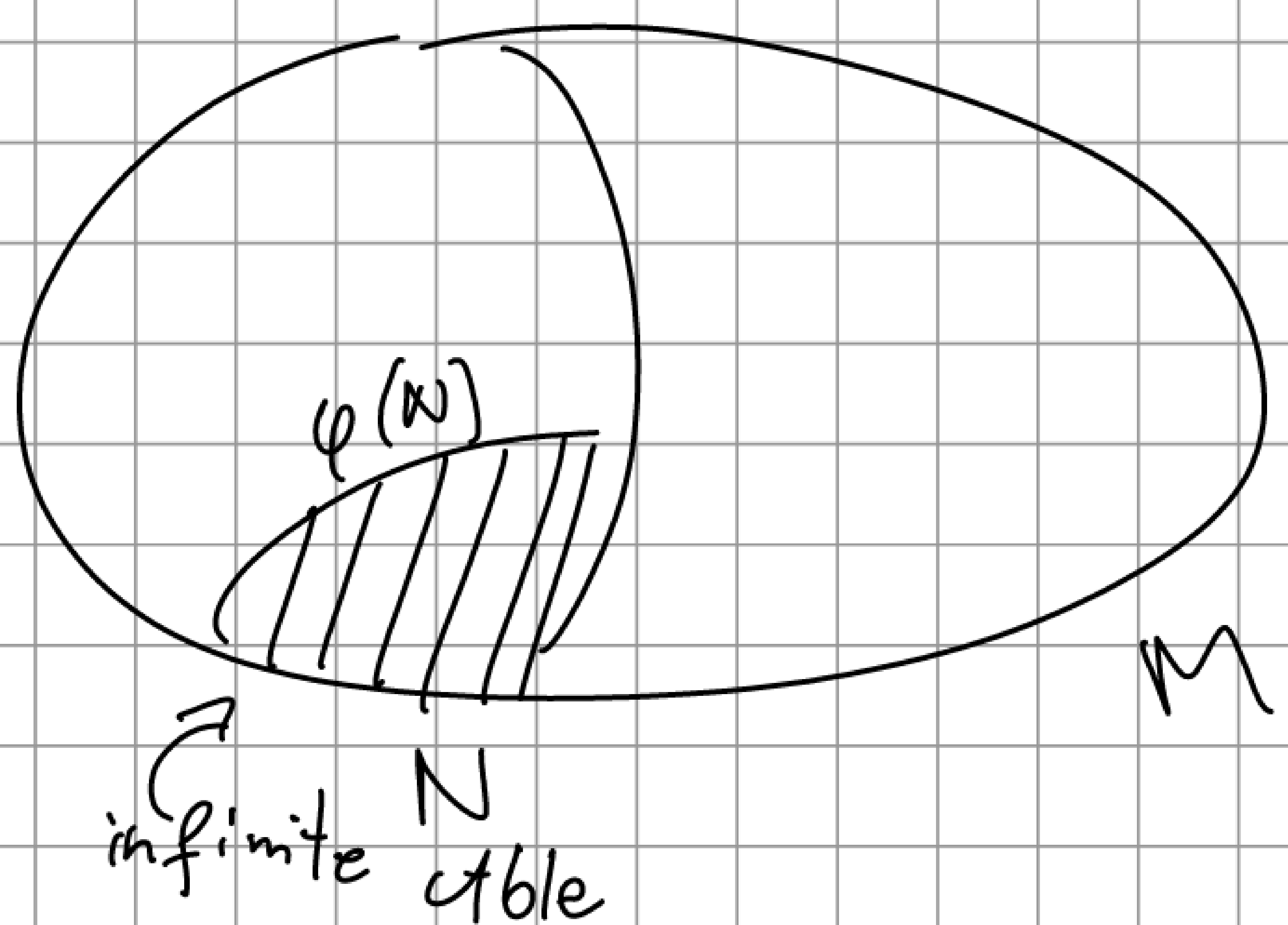
Corollary Suppose $\varkappa > \aleph_0$, T: $\varkappa$-categorical.
T has no Vaughtian pair and T: $\aleph_0$-stable.

Proof T: $\aleph_0$-stable: proved earlier.

No Vaughtian pair: (A.c.) suppose (M, N, φ): Vt.

By lemma 1 wlog M, N are $\aleph_0$-saturated, N: ctble

by lemma 2 wlog $\|M\| = \kappa$, so M is not saturated



$p(x) = \{\varphi(x)\} \cup \{x \neq a\}_{a \in \varphi(N)}$ $\Rightarrow$ not realised in M.

But T: $\aleph_0$-stable $\rightsquigarrow$ $\exists M'$ $\|M'\| = \kappa$ saturated

$$M \not\cong M' \quad \lightning$$

<u>Theorem</u> If T: $\aleph_0$-stable without a Vaughtian pair.

Then $\forall \kappa > \aleph_0$ T: κ-categorical.

<u>Corollary</u> (Morley thm, 1964) Let $\kappa > \aleph_0$, $\Rightarrow$

(1) T: κ-categorical

(2) T: $\aleph_1$-categorical

Lemma 3 (T: $\aleph_0$-stable with no Vaughtian pair).

$\exists$ a $\underbrace{\text{strongly minimal}}_{RM(\varphi)=Mlt(\varphi)=1}$ formula $\varphi(x,\bar{c})$, $\bar{c} \subseteq \underset{\text{prime}}{M} \models T$

Proof Problem from list 6 ■

Lemma 4 (T: $\aleph_0$-stable, no V. pair) Assume

$M \models T$, $\underbrace{\varphi(x)}_{s.m.} = \varphi(x, \underset{M}{\bar{a}})$, then:

(a) M is prime and minimal $/ \varphi(M) \cup \{\bar{a}\}$

(b) If $\|M\| > \aleph_0$, then $\dim(\varphi(M)/_{\bar{a}}) = \|M\|$

Explanation 1. $A \subseteq M \prec \mathcal{M}$, M: minimal $/A$

$\Leftrightarrow \neg\exists \underset{A}{M'} \precneqq M$

2. acl-dimension: Assume $\mathcal{U} = \varphi(M)$, where

$\varphi(x,\bar{a})$: s.m., $acl_{\bar{a}}: P(\mathcal{U}) \to P(\mathcal{U})$,

$acl_{\bar{a}}(\underset{\mathcal{U}}{A}) = acl(A\bar{a}) \cap \mathcal{U}$

3. $(\mathcal{U}, \underbrace{acl_{\bar{a}}}_{cl})$ is a pregeometry, i.e.

- $cl(cl(A)) = cl(A) \supseteq A$

- $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$
- $cl(A) = \bigcup_{\substack{A_0 \subseteq A \\ \text{finite}}} cl(A_0)$
- $a \in cl(Ab) \setminus cl(A) \Rightarrow b \in cl(Aa)$

$\rightsquigarrow$ Basis $\mathcal{B} \subseteq \mathcal{U}$ cl-independent set, $\dim(\mathcal{U}) = |\mathcal{B}|$.

In the case of $acl_{\bar{a}}$ on $\varphi(M)$ additionally:

Let $\underset{\text{s.m.}}{p(x)} \in S(\bar{a}) \cap [\varphi(x)]$ i.e. $RM(p) = MH(p) = \underline{1}$

Then an $acl_{\bar{a}}$-independent set $I \subseteq \varphi(M)$ is a Morley sequence in $p \Rightarrow$ indiscernible over $\bar{a}$.

Pf (lemma 4)

(a) Suppose M prime and is not minimal $/ \varphi(M) \cup \bar{a}$.

So $\varphi(M) \cup \bar{a} \subseteq N \not\cong M$

N: prime $/ \varphi(M) \cup \bar{a}$

But then (M, N, φ): a V. triple $\lightning$

(b) Let $I \subseteq \varphi(M)$ an $acl_{\bar{a}}$-basis of $\varphi(M)$. Then $\bar{a} \cup \varphi(M) \subseteq acl_{\bar{a}}(I) \cup \bar{a} \subseteq M$

$$\|M\| \overset{(a)}{=} |\bar{a} \cup \varphi(M)| = |I| = \dim(\varphi(M)/\bar{a})$$

$\uparrow$ $\varphi(M)$ uncountable

Proof of Morley thm

Let $\kappa > \aleph_0$, $\underset{\curlyvee}{M}, \underset{\curlyvee}{N} \models T$ of power κ.

M_0 : prime

$\overset{\cup}{\bar{a}}$ $\quad \varphi(x, \bar{a})$: s.m. (lemma 3)

By lemma 4 : M : prime & minimal $/ \varphi(M, \bar{a}) \cup \bar{a}$

N : —— $\| $ —— $/ \varphi(N, \bar{a}) \cup \bar{a}$

Let $I \subseteq \varphi(M)$ bases.

$J \subseteq \varphi(N)$

Let $p \in S(\bar{a}) \cap [\varphi(x)]$: a s.m. type

I, J : Morley sequences in p. By L46

$|I| = |J| = \kappa$. $f : I\bar{a} \to J\bar{a}$ s.t.

$$f = \begin{cases} f \restriction \bar{a} = id_{\bar{a}} \\ f : I \xrightarrow[\text{onto}]{1-1} J \end{cases}$$

f is elementary

$\Downarrow$

$$f : \mathrm{acl}(I\bar{a}) \xrightarrow{\cong} \mathrm{acl}(J\bar{a})$$

$$\underset{\varphi(M)_{\bar{a}}}{\cup\!\!\!\mid} \qquad\qquad \underset{\varphi(N)_{\bar{a}}}{\cup\!\!\!\mid}$$

$$f\restriction_{\varphi(M)\bar a} : \underset{\substack{\cap\!| \\ M}}{\varphi(M)\bar a} \xrightarrow{\cong} \underset{\substack{\cap\!| \\ N}}{\varphi(N)\bar a}$$

$\cap\!|$

$$f' : M \xrightarrow{\cong} N$$

(lemma 4a) ∎

<u>Thm</u> (Baldwin, Lachlan, 1971) Assume T: $\aleph_1$-categorical not $\aleph_0$-categorical. Then:

(1) T has $\aleph_0$ many ctble models

(2) Every model of T is homogenous.

$\aleph_1$-categorical theories:

- $\aleph_0$-stable
- have prime model
- no V. pairs
- s.m. sets
- M. sequences, "bases"

Stable formulas

T: fixed complete theory with infinite models

$\underset{\text{ctble}}{L} \ni \varphi = \varphi(\overset{x}{\bar{x}}, \overset{y}{\bar{y}})$

distinguished variables

parameters variables

Def Let $\delta(x,y) \in L$.

(1) $\delta(x,c)$, $c \subseteq \mathcal{M}$: an <u>instance</u> of δ

(2) $\varphi(x)$ is <u>positive δ-formula</u> if

$\models \varphi \leftrightarrow$ [positive] boolean combination of instances of δ

(3) $(A \subseteq \mathcal{M})$ <u>δ-type</u> = a type consisting of δ-formulas,

$L_\delta(A) = \{\delta\text{-formulas over } A\}$

(4) $S_\delta(A) = \{\text{complete } \delta\text{-types over } A\}$ (ultrafilters in $L_\delta(A)$)

<u>Remark</u> Let $\varphi \in L(A)$, $A \subseteq M$. Then

φ is a δ-formula $\Leftrightarrow \models \varphi \leftrightarrow$ boolean comb. of instances of δ over M

<u>Pf</u> ($\Rightarrow$)

$$\mathcal{M} \models \varphi(\bar{x}) \leftrightarrow \bigwedge_i \bigvee_j \delta(x, c_{ij})^{\varepsilon_{ij}} \quad \varepsilon_{ij} \in \{0,1\}, \quad c_{ij} \in \mathcal{M}$$

$$\mathcal{M} \models (\exists y_{ij})\left(\varphi(\bar{x}) \leftrightarrow \bigwedge_i \bigvee_j \delta(x, y_{ij})\right)$$

$$\curlyvee$$

$$M \models \quad \text{———} \quad '' \quad \text{———}$$

So c_{ij} may be taken from M.

<u>Example</u> T: the theory of a single equivalence relation E with two infinite classes. Let $\delta(x, y) = E(x, y)$.

$\models \underbrace{x = x}_{\sim} \leftrightarrow E(x,a) \vee E(x,b)$

is a δ-formula $/\emptyset$, where $a, b \in \mathcal{M} \models T$, $\models \neg E(a,b)$

<u>Def</u> (1) $\delta(x,y)$ has <u>order property</u>, if

$\exists \underset{i<\omega}{a_i, b_i} \subseteq \mathcal{M} \quad \forall i,j<\omega \models \delta(a_i, b_j) \Leftrightarrow i \leq j$

(2) $\delta(x,y)$ is <u>stable</u> if it does not have the order property

<u>Lemma 1</u> (1) $\varphi(x,y), \psi(x,z)$: stable

$\Longrightarrow \neg\varphi(x,y), (\varphi \vee \psi)(x, yz), (\varphi \wedge \psi)(x, yz)$

are stable

(2) Let $\psi(y,x) = \varphi(x,y)$, then

φ stable $\Leftrightarrow$ ψ stable

(3) φ : stable $\Leftrightarrow \exists n<\omega \ \neg \exists a_i b_i \ (i \leq n)$

$\bigwedge_{i,j} \models \varphi(a_i, b_j) \leftrightarrow i \leq j$

<u>Pf.</u> excercise

Def Let $p \in S(M)$ or $p \in S_\delta(M)$.

A <u>δ-definition</u> of p: a formula $\psi(y) \in L(M)$ s.t. $\forall c \in M$ $(\delta(x,c) \in p \iff \models \psi(c))$

(i.e. $\{c \in M : \delta(x,c) \in p\} = \psi(M)$)

Lemma 2 Assume $\delta(x,y)$ is stable, $p \in S(M)$ or $p \in S_\delta(M)$. Then:

(1) $p(x)$ has a δ-definition $\psi(y)$ that is a positive δ^*-formula, where $\delta^*(y,x) = \delta(x,y)$

(2) $A \subseteq M$ and M is $|A|^+$-saturated, then $\exists c_1, c_2, \ldots \in M$ $c_i \models p|_{Ac_{<i}}$ and the δ-definition of p is equivalent to a positive boolean combination of formulas $\delta(c_i, y)$, $i < \omega$.