9.05.2021 **Thm (Morley, Shelah)** T: $\aleph_0$-stable $\Rightarrow$

T has a saturated model of power κ

Pf. • $M = \bigcup_{\alpha<\kappa} M_\alpha \leftarrow$ elementary chain of models of T of power κ
$\|M_\alpha\| = \kappa$

• $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$ when $\delta \in Lim$

• $\alpha \to \alpha+1$: $M_{\alpha+1} \succ M_\alpha$ s.t.

(a) $\forall \underset{\text{finite}}{A \subseteq M_\alpha}$ $\forall p \in S(A)$ $p(M_{\alpha+1}) \neq \emptyset$

(b) $\forall \underset{\text{finite}}{A \subseteq M_\alpha}$ $\forall \underset{\text{stationary}}{p \in S(A)}$ $\exists I \subseteq M_{\alpha+1}$ $(|I| = \kappa \wedge$ I is a Morley sequence in $p)$

<u>Claim</u> M is saturated

• M is $\aleph_0$-saturated (easy)

• M is κ-saturated: (a.c.) let $A \subseteq M$, $|A| < \kappa$, $p \in S(A)$, $p(M) = \emptyset$. Choose A and p so that $(RM(p), Mlt(p))$ is lexicographically minimal. Then $Mlt(p) = 1$.

pf. let choose $\varphi \in p$ with $RM(p) = RM(\varphi)$, $Mlt(p) = Mlt(\varphi)$.

- If $Mlt(\varphi) > 1$, then

(*) $\exists \psi(x) \in L(\mathcal{M})$ $RM(\varphi \wedge \psi) = RM(\varphi \wedge \neg\psi) = RM(\varphi)$

So $\psi(x) = \psi(x, \underset{\mathcal{M}}{\underset{\cap}{\bar{c}}})$, $\varphi(x) = \varphi(x, \underset{M}{\underset{\cap}{\bar{a}}})$. Choose $\bar{c}' \subseteq M$ s.t. $tp(\bar{c}'/\bar{a}) = tp(\bar{c}/\bar{a}) \in S_k(\bar{a})$, $(k = |\bar{c}|)$

(we can choose it by (a))

(*) holds for ψ': $\psi(x, \bar{c}')$ in place of ψ

Let $A' = A \cup \bar{c}' \subseteq M$, $|A'| < \kappa$

$$\begin{array}{ccc} S(A) & & S(A') \\ \cup\!\!\!\!| & & \\ p & \begin{array}{l} \nearrow \\ \rightarrow \\ \searrow \end{array} & \begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_l \end{array} \end{array} \quad \text{s.t. } RM(p_i) = RM(p) \quad (l \le Mlt(p))$$

Also (*) $\Rightarrow$ $Mlt(\varphi) = Mlt(\varphi \wedge \psi') + Mlt(\varphi \wedge \neg\psi')$

Look at p_1: either $\varphi \wedge \psi \in p_1$ or $\varphi \wedge \neg\psi \in p_1$

$\Downarrow$

$(RM(p_1), Mlt(p_1)) <_{lex} (RM(p), Mlt(p))$

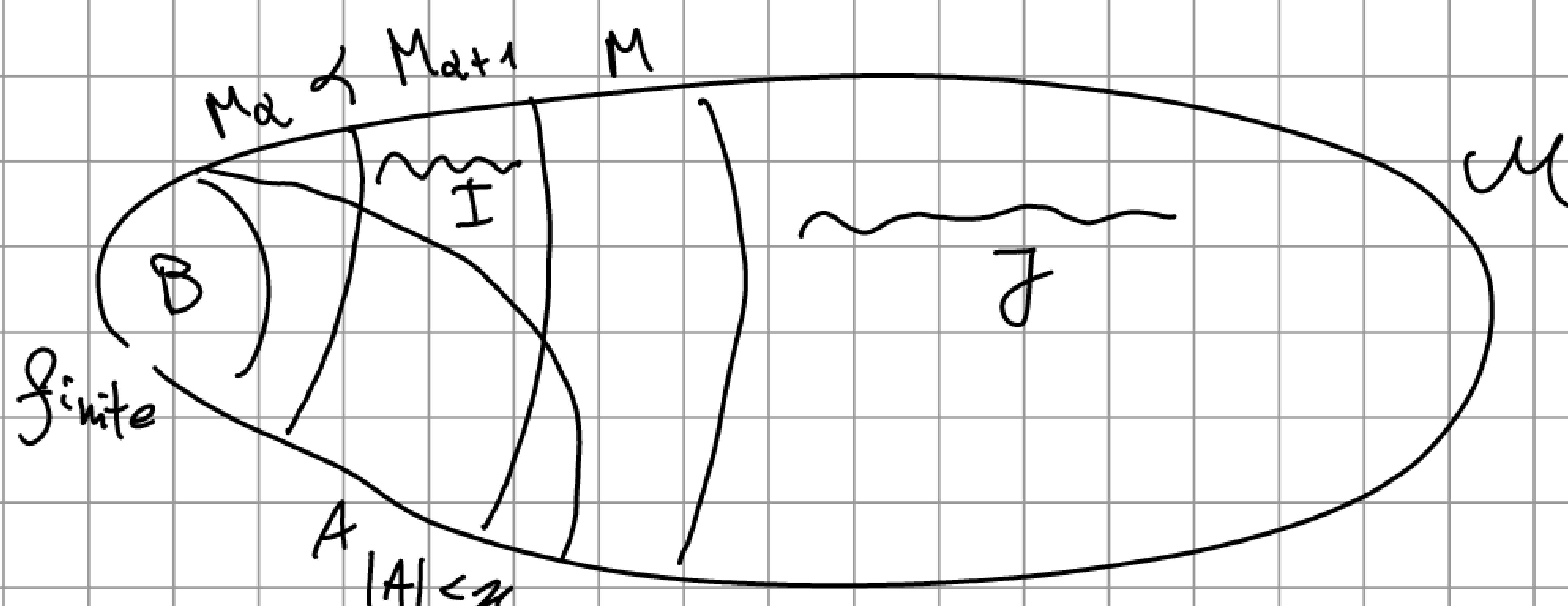
$p_1(M) \subseteq p(M) = \emptyset \Rightarrow p_1(M) = \emptyset$ ↯

Therefore $Mlt(\varphi) = 1$. Choose a finite $B \subseteq A$ s.t. $RM(p) = RM(p|B)$, $Mlt(p) = Mlt(\underbrace{p|B}_{p'})$

(enough that $\varphi \in L(B)$)

By (b) $\exists I \subseteq M$: a Morley sequence in p', $|I| = \varkappa$. Let $\underset{\mathcal{M}}{\overset{\cap}{J}} = \{a_\alpha : \alpha < \varkappa\}$: a M. sequence in $\underset{p}{\overset{\cup}{p'_{AI}}} \in S(AI)$

Then $I \cup J$ is a Morley sequence in p'_{AI}.



Let $\chi(x) \in p \subseteq p'_{AI} \Rightarrow \underset{\text{infinite}}{J} \subseteq \chi(\mathcal{M})$,

However $I \cup J$ indiscernible $\Rightarrow \{i \in I : \models \chi(i)\}$ is also cofinite in I (by the lemma from prev. lecture)

$I \cup J = (I \cup J)^+_\chi \cup \underset{\text{finite}}{(I \cup J)^-_\chi}$

$$\left|\bigcup_{\chi\in p} I_\chi^-\right| < \kappa \Rightarrow \left|\bigcap_{\chi\in p} I_\chi^+\right| = \kappa \Rightarrow \left|\bigcap_{\chi\in p} I_\chi^+\right| \neq \emptyset.$$

$(|p|<\kappa)$ Any $c \in \bigcap_{\chi\in p} I_\chi^+$ realises p in M $\blacksquare$

So: if $\kappa > \aleph_0$, T: κ-categorical

$$\Downarrow$$

$$T: \aleph_0\text{-stable}$$

$$\Downarrow$$

$$\exists \underset{\text{saturated}}{M} \models T \quad \|M\| = \kappa$$

S-isolation (s = "set")

Def. (1) $p \in S(A)$ is s-isolated if $\exists \underset{\text{finite}}{B \subseteq A}$ $p|_B \vdash p$.

(2) M is s-atomic over A if

$\forall \underset{\text{finite}}{\bar{a} \subseteq M}$ $tp(\bar{a}/A)$ is s-isolated

(3) M is an s-model if $\forall \underset{\text{finite}}{A \subseteq M}$ $\forall p \in S(A)$ $p(M) \neq \emptyset$

s-model $\equiv$ $\aleph_0$-saturated

(4) M is s-prime over A if M is s-model

and $\forall \underset{\hat{\mathcal{M}}}{N} \supseteq A$ ($\aleph_0$-saturated) $\exists f: M \xrightarrow[A]{\equiv} N$ $(f|_A = id_A)$

__Remark__ T: $\aleph_0$-stable, $p \in S(B)$, $\underset{\text{finite}}{B \subseteq A}$

$\Rightarrow \exists q \in S(A)$ $\quad q$: s-isolated
$\quad p \subseteq q$

__Proof__ (A.c.) Suppose there's no such q.

(1) $p \not\vdash$ a type in $S(A)$. So there's $B_0 \subseteq A$, $B \subseteq B_0$

and $p_0 \neq p_1 \in S(B_0)$ extending p.

(2) $p_i \not\vdash$ a type in $S(A)$. So: there is $B_1 \subseteq A$, $B_0 \subseteq B_1$
$i = 0, 1$

and $p_{00}, p_{01}, p_{10}, p_{11} \in S(B_1)$ (with $p_0 \subseteq p_{00}, p_{01}$ and $p_1 \subseteq p_{10}, p_{11}$)
pairwise distinct

(3) ....

$B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$

$$\begin{tikzcd}[column sep=small]
 & & & p \in S(B) \arrow[dll, dash] \arrow[drr, dash] & & & \\
 & p_0 \arrow[dl, dash] \arrow[d, dash] & & & & p_1 \arrow[d, dash] \arrow[dr, dash] & \in S(B_0) \\
p_{00} & p_{01} & & & & p_{10} \quad p_{11} & \in S(B_1)
\end{tikzcd}$$

Let $B_\omega = \bigcup_n^{\uparrow} B_n \subseteq A$: ctble set, but $|S(B_\omega)| \geq \aleph_1$

<u>Property of s-isolation:</u>

$tp(ab/A)$ s-isolated $\Leftrightarrow$ $tp(a/A)$ is s-isolated and $tp(b/Aa)$ is s-isolated.

(exercise)

<u>Corollary</u> T : $\aleph_0$-stable $\Rightarrow$ $\forall A \subseteq \mathcal{M}$ $\exists M \supseteq A$

M s-prime over A

<u>Proof</u> (sketch) $M = A \cup \underbrace{\{a_\alpha : \alpha < ?\}}_{\text{s-construction}}$ s.t.:

(1) $tp(a_\alpha / Aa_{<\alpha})$ is s-isolated

(2) M is $\aleph_0$-saturated

it terminates at some point

Then M is s-prime

The model M constructed this way is s-constructible $/A$ and s-primary $/A$, it's unique up to $\underset{A}{\cong}$.

Corollary T: $\aleph_0$-stable, M: s-prime $/A$

$\Rightarrow M$: s-atomic

Proof Let N: s-primary $/A$

$\Downarrow$ property of s-isolation

N: s-atomic

M: s-prime $/A \Longrightarrow \exists f : M \xrightarrow[A]{\equiv} N$

$\Downarrow$

M: s-atomic

Def (M, N) is a Vaughtian pair for T, if

$N \precneqq M \models T$ and for some $\varphi(x) \in L_1(N)$ non-algebraic consistent

$\varphi(N) = \varphi(M)$

(here (M, N, φ): Vaughtian triple)

Lemma 1 Assume T: $\aleph_0$-stable (enough T: small). If T has a Vaughtian pair, then there's $Vp\ (M, N)$ s.t. M, N: ctble and saturated.

Proof Let (M_0, N_0, φ) : Vaughtian triple.

$\varphi(\bar{x}, \bar{a})$

$\bar{a} \subseteq N_0$

Let $L' = L \cup \{\bar{a}\} \cup \{P(x)\}$

new constant symbols　　new predicate symbol

T': a complete theory in L' s.t.

(0) $T' \supseteq Th(N_0, \bar{a}) = Th(M_0, \bar{a})$ and for any $M' \models T'$:

(1) $N := P^{M'}(M')\restriction_L \prec M := M'\restriction_L$

(2) $\bar{a}^{M'} \subseteq P(M')$: $\bigwedge_i P(a_i) \in T'$

(3) (M, N, φ): a V. triple :

$\varphi = \varphi(x, \bar{a}^{M'})$

- $[\forall x\, (\varphi(x) \longrightarrow P(x))] \in T'$

- $[\exists x\, \neg P(x)] \in T$

Ad (1): Let $\psi(\bar{x}, y) \in L$.

$T' \ni [\forall \bar{x} [\bigwedge_i P(x_i) \wedge \exists y\, \psi(\bar{x}, y) \longrightarrow \exists y (P(y) \wedge \psi(\bar{x}, y))]]$

<u>Fact</u> $\exists M' \models T'$ ($M := M'\restriction_L$ and $N : P(M')\restriction_L$ are both ctble and saturated)

ctble

<u>Pf</u> (fact) $M' = \bigcup_{n<\omega} M'_n$: models of T' elem. chain

- M'_0: arbitrary
- $n \rightsquigarrow n+1$: $\underset{\text{ctble}}{M'_{n+1}} \succ M_n$ such that:

(i) $\forall A \underset{\text{finite}}{\subseteq} M'_n \ \forall p \in S^L(A)\ p(M'_{n+1}) \neq \emptyset$

(ii) $\forall A \underset{\text{finite}}{\subseteq} P(M'_n)\ \forall p \in S^L(A)$ $p(x) \cup \{P(x)\}$ is realised in M'_{n+1}.

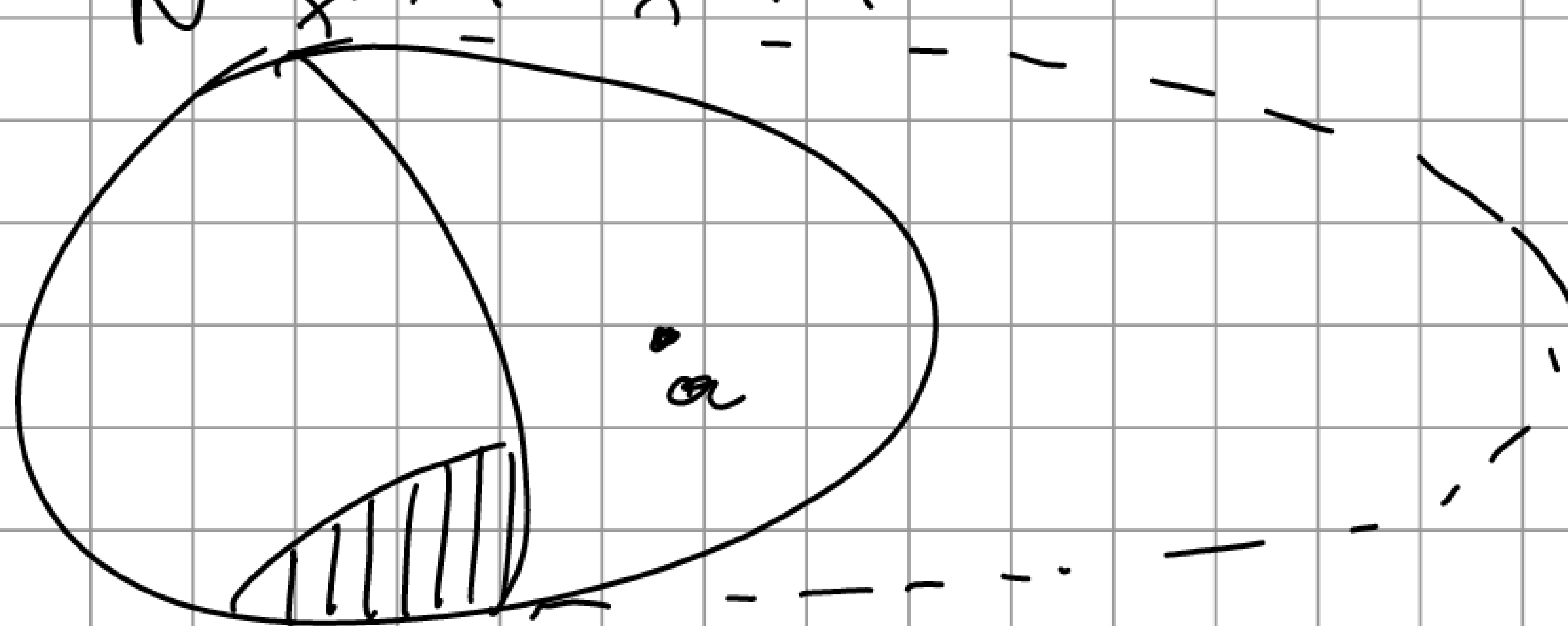
Fact, Lemma 1 ∎

Lemma 2 (stretching Vaughtian pair)

Let (M, N, φ): V. triple, M, N: $\aleph_0$-saturated, T: $\aleph_0$-stable. Then $\exists M' \succneq M$ (M', N, φ) is a V. triple

$\aleph_0$-saturated

$N \precneq M \precneq M'$



$\varphi(N) = \varphi(M) = \varphi(M')$

Proof Let $a \in M \setminus N$, $p = tp(a/N)$, $MR(p) = 1$.

So $p \subseteq q \in S(M)$, $RM(q) = RM(p)$

unique

⋮