Preliminaries

T : a complete consistent theory, in language L (countable)
<u>with infinite models.</u>

that is, $T = Th(M)$, M : L-structure.
infinite

L denotes also the set of formulas of language L

$M = (|M|; \dots)$, but M also denotes $|M|$.
$\emptyset \neq$ universe of M — (for brevity)

usually we omit $|\cdot|$ in $|M|$.

$M \supseteq A$: a set of parameters.

$L_n(A) = \{\varphi(x_1, \dots, x_n, \bar{a}) : \varphi(\bar{x}, \bar{y}) \in L, \bar{a} \subseteq A\}$

$L(A) = \bigcup_n L_n(A)$, also $L(A)$: language L extended by names for elements of A.

$L_n(A)$: Lindenbaum algebra.

[formally: on $L_n(A)$: $\varphi \sim \psi \iff T(A) \vdash \varphi \leftrightarrow \psi$
$\iff M \models \varphi \leftrightarrow \psi$

here: $\underbrace{T(A)}$ $= Th(M, a)_{a \in A}$
a complete theory in language $L(A)$.

$L_n(A)/_\sim$: a Boolean algebra
(Lindenbaum algebra)

$[\varphi]_\sim \wedge [\psi]_\sim = [\varphi \wedge \psi]_\sim$ etc.

shortly : $L_n(A)$ denotes also $L_n(A)/_\sim$.

$S_n(A) = \{$complete n-types over A, in $M\}$.
in variables $x_1, \dots, x_n$

consistent n-type over A $\rightsquigarrow$ proper filter in $L_n(A)$

An n-type $p(\bar{x})$ over A is <u>complete</u> if
$\cap$
$L_n(A)$
- $p(\bar{x})$: consistent type
- $\forall \varphi(\bar{x}) \in L_n(A)$ $(\varphi(\bar{x}) \in p$ or $(\neg\varphi(\bar{x})) \in p)$

$S(A) := S_1(A)$
(default)

$S_n(A)$: topological space :
for $\varphi(\bar{x}) \in L_n(A)$
$$[\varphi] = \{p \in S_n(A) : \varphi \in p\}$$
basic open set [clopen]
$\shortparallel$
closed and open

$S_n(A)$: compact Hausdorff space, 0-dimensional
(i.e. basis of clopen sets)

complete n-types$/A$ $\leftrightsquigarrow$ ultrafilters in $L_n(A)$

So $S_n(A) = S(L_n(A))$, the Stone space of ultrafilters in $L_n(A)$

- the ~~topo~~ topology on $S_n(A)$ = the Stone space topology.

For $p(\bar{x}) \in S_n(A)$

$$p(M) = \{\bar{a} \in M^n : \underbrace{\bar{a} \text{ satisfies } p}_{\bar{a} \models p}\}$$

realizes p

$\bar{a} \models p$, i.e. $M \models \varphi(\bar{a})$ for every $\varphi(\bar{x}) \in p(\bar{x})$

- The same notation for arbitrary type (also incomplete)
- A formula $\varphi(\bar{x}) \in L(M)$: a special case of a type $\{\varphi(\bar{x})\}$.

$\varphi(M) = \ldots$

- When $p \in S_n(A)$, $\bar{a} \subseteq M$ and $\bar{a} \models p$, then

$$p = tp^M(\bar{a}/A) = tp(\bar{a}/A) = \{\varphi(\bar{x}) \in L_n(A) : M \models \varphi(\bar{a})\}.$$

<u>Example</u> Assume $p(\bar{x})$: a consistent type over M.

Then $\exists N \succcurlyeq M$, $\underbrace{p \text{ is realized in } N}$

i.e. $p(N) \neq \emptyset$.

From now on "a type" means "a consistent type".

<u>Def</u> A type $p(\bar{x})$ over A is <u>isolated</u>, if:

$\exists \varphi(\bar{x}) \in L_n(A)$ $\begin{cases} ① \ \varphi(\bar{x}) \text{ is consistent (with } T\text{), i.e.} \\ \quad \varphi(M) \neq \emptyset \ (\Leftrightarrow T(A) \vdash \exists \bar{x}\, \varphi(\bar{x})) \\ (2) \ \cancel{\varphi(M)} \\ \quad \forall \psi(\bar{x}) \in p(\bar{x}) \quad \varphi(M) \subseteq \psi(M) \\ \quad\quad \Updownarrow \\ \quad T(A) \vdash \varphi(\bar{x}) \to \psi(\bar{x}) \end{cases}$

symbolically: $\varphi(\bar{x}) \vdash p(\bar{x})$ $\longrightarrow$ (2)

- When $p(\bar{x})$: a complete type over A, then:

  $p(\bar{x})$ is isolated $\Longleftrightarrow$ p is isolated in $S_n(A)$ in the topological sense (i.e. $\{p\}$ is open)

<u>Tarski–Vaught test</u>

Assume $A \subseteq M$. Then $A = |N|$ for some $N \prec M$ <u>iff</u>

$\forall \varphi(x) \in L_1(A) \ [\varphi(M) \neq \emptyset \Rightarrow \varphi(M) \cap A \neq \emptyset]$

Construction of an elementary submodel of M containing A:

- $A_n \subseteq M$, $n < \omega$, increasing chain of sets

recursive construction:

$A_0 = A$

$A_n \subseteq A_{n+1} \subseteq M$ such that $\forall \psi(x) \in L_1(A_n)$
$[\psi(M) \neq \emptyset \Rightarrow \psi(M) \cap A_{n+1} \neq \emptyset]$

$A_\infty = \bigcup_{n<\omega} A_n$ satisfies TV-test.

Omitting types theorem

Assume $p_n(\bar{x}_n)$, $n<\omega$: a family of non-isolated types in theory T, over $\emptyset$. Then:

$(\exists M \models T)$ M omits every p_n [i.e. $p_n(M)=\emptyset$]

Assume $\underset{\overset{\cup|}{A}}{M}, N \models T$

Def. $f: A \to N$ is <u>elementary</u> ($f: A \xrightarrow{\equiv} N$) if:

$$\forall \bar{a} \subseteq A\ \forall \varphi(\bar{x}) \in L\ \left(M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(f(\bar{a}))\right)$$

$(\Leftrightarrow\ tp^M(\bar{a}) = tp^N(f(\bar{a}))\ )$

Elementary diagram of $A \subseteq M$:

$D_e(A) = T(A) = Th(M, a)_{a \in A}$

<u>Remark</u> $f: A \to N$ is elementary $\Leftrightarrow (N, f(a))_{a\in A} \models T(A)$

Atomic diagram of $A \subseteq M$:

$D_{at}(A) = \{\varphi \in D_{el}(A) : \varphi$ is a quantifier free sentence$\}$

$= \{\varphi(\bar{a}) \in L(A) : M \models \varphi(\bar{a})$ and $\varphi(\bar{a})$: q.f.-sentence$\}$

<u>Remark</u> $f: M \to N$ is a monomorphism (i.e: $f: M \xrightarrow{\cong} f(M) \subseteq N$)
$\uparrow$substructure

$\Leftrightarrow (N, f(a))_{a \in M} \models D_{at}(M)$.

Here always $f: M \to N$ denotes a <u>monomorphism</u>.

$M \subseteq N$: M is a submodel (substructure) of N

$M \prec N$: M is an elementary submodel of N, i.e:

$$M \subseteq N \text{ and } id_M : M \xrightarrow{\equiv} N$$

<u>Remark</u> Assume $M \prec N$, $A \subseteq M$.

(1) Assume $p(\bar{x}) \subseteq L_n(A)$. Then

$p(\bar{x})$ is a consistent type in $M \Leftrightarrow p(\bar{x})$ is a consistent type in N

(2) Assume $A \subseteq B \subseteq M$

- If $p(\bar{x})$: a type over B, then $p\restriction_A \overset{def}{=} p(\bar{x}) \cap L(A)$ a type over A
- Let $r: S_n(B) \to S_n(A)$, $r(p) \overset{def}{=} p\restriction_A$. Then r: continuous and "onto".

(3) If $p(\bar{x})$: a type over A, then $\exists q(\bar{x}) \in S_n(A)\ p(x) \subseteq q(\bar{x})$

Saturation, universality, (strong) homogeneity.

Let $\varkappa \in CN$, $\varkappa \geq \aleph_0$.

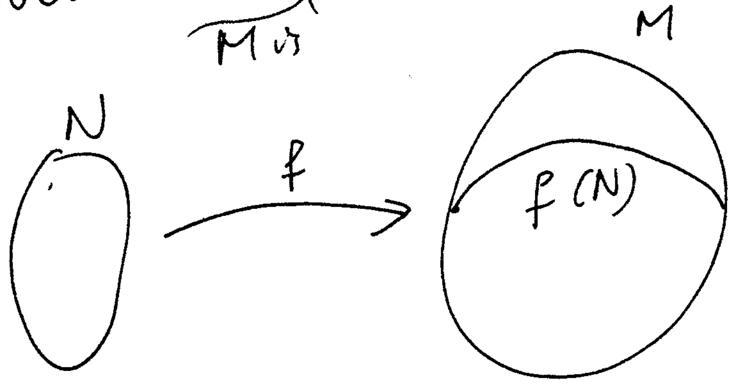
<u>Def.</u> (1) M $\varkappa$-<u>saturated</u> (nasycony) if $\forall A \subseteq M, |A| < \varkappa\ \forall p \in S_1(A)\ p(M) \neq \emptyset$

M is <u>saturated</u> if M is $\|M\|$-saturated

(2) M is $\varkappa$-universal if $\forall N \equiv M\ (\|N\| \leq \varkappa \Rightarrow \exists f: N \xrightarrow{\equiv} M)$

elementarily equivalent i.e. $Th(N) = Th(M)$

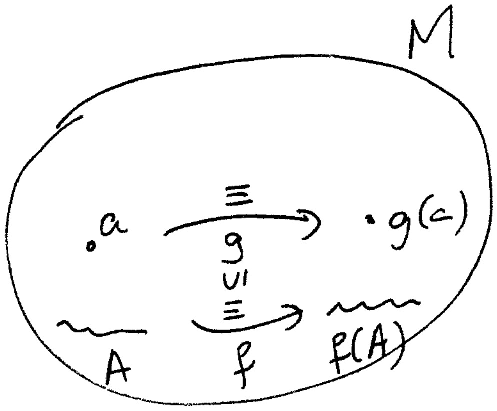
M: universal $\underset{M \text{ is}}{\iff}$ $\|M\|$-universal



$f : N \xrightarrow{\cong} f(N) \prec M$

(3) M : $\varkappa$-homogeneous if $\forall A \subseteq M$, $|A| < \varkappa$, $\forall a \in M$ $\forall f : A \xrightarrow{\equiv} M$ $\exists g \supseteq f : A \cup \{a\} \xrightarrow{\equiv} M$

homogeneous = $\|M\|$-homogeneous.



4. M strongly $\varkappa$-homogeneous if $\forall A \subseteq M$, $|A| < \varkappa$, $\forall f : A \xrightarrow{\equiv} M$ $\exists g \supseteq f : M \xrightarrow{\cong} M$

strongly homogeneous =: strongly $\|M\|$-homogeneous.

5. M is $\varkappa$-compact if ($\forall$ 1-type $p(x)$ over M)
$(|p| < \varkappa \Rightarrow p(M) \neq \emptyset)$

Elementary chains of structures

Def $\langle M_\alpha : \alpha < \mu \rangle$, $\mu \in \mathrm{Ord}$, : an ~~e~~ elementary chain of structures if $(\forall \alpha < \beta < \mu)\, M_\alpha \prec M_\beta$.

Union of chain (when $\mu \in \mathrm{Lim}$)

$$M_\mu = \bigcup_{\alpha < \mu} M_\alpha \quad :$$

- $|M_\mu| := \bigcup\limits_{\alpha<\mu}^{\nearrow} |M_\alpha|$
- $c \in L$ constant symbol $c^{M_\mu} = c^{M_\alpha}$ for $\alpha < \mu$
- P: relation symbol

$$\underset{\substack{\cap \\ |M_\mu|}}{P^{M_\mu}(a_1,\dots,a_n)} \iff M_\alpha \models P(a_1,\dots,a_n) \text{ for } \alpha<\mu$$
sufficiently large
[so that $\bar{a} \subseteq M_\alpha$]

- $f^{M_\mu}(\bar{a}) = b \iff M_\alpha \models f(\bar{a}) = b$ for $\alpha < \mu$ sufficiently large

Fact (Tarski) $M_\alpha \prec M_\mu$ for all $\alpha < \mu$.

<u>Proof</u> (1) $M_\alpha \subseteq M_\mu$ (substructure): exercise

(2) $\forall \varphi(\bar{x}) \in L \ \forall \alpha<\mu \ \forall \bar{a} \subseteq M_\alpha \ (M_\alpha \models \varphi(\bar{a}) \Leftrightarrow M_\mu \models \varphi(\bar{a}))$

(a) φ atomic: $M_\alpha \subseteq M_\mu$ ✓

(b) $\varphi = \psi_1 \wedge \psi_2$, $\varphi = \neg\psi$: easy

(c) $\varphi(\bar{x}) = \exists y \, \psi(\bar{x}, y)$

$$M_\alpha \models \varphi(\bar{a}) \Rightarrow M_\alpha \models \psi(\bar{a}, b) \text{ for some } b \in M_\alpha$$

$\Downarrow$ ind. assumption for ψ

$$M_\mu \models \psi(\bar{a}, b)$$

$$\Downarrow$$

$$M_\mu \models \varphi(\bar{a})$$

$$M_\mu \models \underbrace{\varphi(\bar a)}_{\exists y\, \psi(\bar a, y)} \Rightarrow M_\mu \models \psi(\bar a, b) \text{ for some } b \in M_\mu$$

$$\Downarrow \text{ ind. assumption} \qquad\qquad b \in M_\beta \text{ for some } \alpha \leq \beta < \mu$$

$$M_\beta \models \psi(\bar a, b)$$

$$\Downarrow$$

$$M_\beta \models \varphi(\bar a)$$

$$\Downarrow\ M_\alpha \prec M_\beta$$

$$M_\alpha \models \varphi(\bar a)$$

<u>Elementary directed systems of structures:</u>

Let $(I, \leq)$: a <u>directed set</u>, i.e.:

(1) $\leq$: partial order on I

(2) $(\forall a, b \in I)(\exists c \in I)(a \leq c \wedge b \leq c)$

<u>Example</u> J : a set $\rightsquigarrow$ $([J]^{<\omega}, \subseteq)$: directed set.

Directed system of structures:

- $\mathcal{M} = (M_i, f_{ij})_{i \leq j \in I}$

connecting functions $f_{ij} : M_i \longrightarrow M_j$, $f_{ii} = id_{M_i}$ such that

$(\forall i \leq j \leq t \in I)$ $f_{it} = f_{jt} \circ f_{ij}$

(compatibility)

$$\begin{tikzcd} & M_t \\ M_i \arrow[ur, "f_{it}"] \arrow[r, "f_{ij}"'] \arrow[r, phantom, shift left=4ex, "\#"] & M_j \arrow[u, "f_{jt}"'] \end{tikzcd}$$

- System $\mathcal{M}$ is <u>elementary</u> if all f_{ij} are elementary.

<u>Example</u> Elementary chain $(M_\alpha)_{\alpha<\mu}$

$\mathcal{M} = (M_\alpha, f_{\alpha\beta})_{\alpha\leq\beta<\mu}$ $\quad f_{\alpha\beta} = id_{M_\alpha} : M_\alpha \xrightarrow{\equiv} M_\beta$

elementary directed system of structures

Direct limit of a directed system $\mathcal{M}$: $M_\infty = \varinjlim \mathcal{M}$.

$(M_\infty, f_{i\infty})_{i\in I}$, where $f_{i\infty} : M_i \longrightarrow M_\infty$ such that

(1) $\forall i \leq j \in I \quad f_{i\infty} = f_{j\infty} \circ f_{ij}$ [compatible with connecting functions]

$$\begin{tikzcd} M_j \arrow[dr, "f_{j\infty}"] & \\ & M_\infty \\ M_i \arrow[uu, "f_{ij}"] \arrow[ur, "f_{i\infty}"'] \arrow[u, phantom, "\#"] & \end{tikzcd}$$

(2) $(\forall (M'_\infty, f'_{i\infty})_{i\in I}$ satisfying (1)) $\exists ! f : M_\infty \to M'_\infty$

(universality) $\qquad (\forall i \in I)\ f'_{i\infty} = f \circ f_{i\infty}$

$$\begin{tikzcd} & M_\infty \arrow[dd, "f"] \\ M_i \arrow[ur, "f_{i\infty}"] \arrow[dr, "f'_{i\infty}"'] \arrow[r, phantom, "\#"] & {} \\ & M'_\infty \end{tikzcd}$$

<u>Fact</u> M_∞ exists (and is unique up to $\cong$).

If $\mathcal{M}$ is elementary, then $f_{i,\infty} : M_i \xrightarrow{\equiv} M_\infty$.

<u>Proof</u> 1. Construction of M_∞:

$S := \dot{\bigcup}_{i\in I} |M_i|$: formally disjoint union.

$\sim$ on S : an equivalence relation

$$\begin{array}{ccc} M_i & & M_j \\ \Downarrow & & \Downarrow \\ x & \sim & y \end{array} \overset{\text{def}}{\iff} f_{it}(x) = f_{jt}(y) \text{ for some } (\equiv \text{every}) \ t \geq i, j$$

exercise: $\sim$ is transitive.

$$|M_\infty| := S/\sim$$

- $\sim \restriction_{|M_i|}$: the equality (because f_{ij}: 1-1 (monomorphism)
- $f_{i\infty}(\underset{\substack{\uparrow \\ M_i}}{x}) = x/\sim$, $\quad f_{i\infty} : |M_i| \xrightarrow{1-1} |M_\infty|$.

L-structure on $|M_\infty|$:

- $c^{M_\infty} = c^{M_i}/\sim$
- $p^{M_\infty}(a_{i_1}/\sim, \ldots, a_{i_n}/\sim) \iff M_t \models P(f_{i_1 t}(a_{i_1}), \ldots, f_{i_n t}(a_{i_n}))$
  $a_{i_j} \in M_{i_j}$ $\quad$ for $t \geq i_1, \ldots, i_n$
- f^{M_∞}: similarly

the rest is an exercise.