Model theory, list 6.

1. Prove that every model $M \prec \mathcal{M}$ is algebraically closed (as a subset of $\mathcal{M}$).

2. – Recall the proof of existence of an infinite order-indiscernible set, based on Ramsey theorem.

3. (Lascar) Let $p \in S_1(M)$ is non-algebraic. Let U be an ultrafilter on M containing the family $\{\varphi(M) : \varphi \in p\}$. For $A \supseteq M$ we define

$$p_A = \{\psi(x) \in L_1(A) : \psi(\mathcal{M}) \cap M \in U\}$$

(when T is stable, p_A does not depend on the choice of U, proof later).
(a) Prove that $p_A \in S(A)$ is non-algebraic, contains p and for $A \subseteq B$, $p_A \subseteq p_B$.
(b) We define the sequence $\{a_n, n < \omega\} \subseteq \mathcal{M}$ so that $a_n \models p_{Ma_{<n}}$. Prove that $\{a_n, n < \omega\}$ is infinite order-indiscernible (this is an alternative way to prove existence of an order-indiscernible set).
(c) Prove that if a_0 i a_0' realize the same type over M, then there is a sequence $\{a_n : 0 < n < \omega\}$ such that both $\{a_n : n < \omega\}$ and $\{a_0'\} \cup \{a_n : 0 < n < \omega\}$ are order-indiscernible.

4. Give an example of a 5-element indiscernible set (in a stable theory) that can not be extended to an infinite indiscernible set.

5. Prove that if $\{a_n, n < \omega\}$ is order-indiscernible, then there are $\{a_q : q \in \mathbb{Q} \setminus \mathbb{N}\}$ such that $\{a_q, q \in \mathbb{Q}\}$ is order-indiscernible.

6. We say that a definable set $X \subseteq M$ is strongly minimal (briefly: s.m.) if $RM(X) = 1$ and $Mlt(X) = 1$.
(a) Prove that X is strongly minimal $\iff$ $\forall \varphi(x, \overline{y}) \in L \exists k < \omega \forall \overline{a} \subseteq M$, $\varphi(M, \overline{a}) \cap X$ or $\neg\varphi(M, \overline{a}) \cap X$ has $\leq k$ elements.
(b) From now on in this problem we assume X is 0-definable s.m. Prove that the operator acl has the exchange property on X, that is for every $A \cup \{a, b\} \subseteq X$, $a \in acl(A \cup \{b\}) \setminus acl(A) \Rightarrow b \in acl(A \cup \{a\})$.
(c) Assume that the set $A \subseteq X$ jest acl-independent (that is for $a \in A$, $a \notin acl(A \setminus \{a\})$). Prove that A is indiscernible.
(d) By a basis of a set $A \subseteq X$ we mean a maximal acl-independent subset of A. Prove that all bases of A equinumerous. (therefore we define the dimension $dim(A)$ as the power of any basis of A)

7. We say that T is strongly minimal if some (equivalently: every) model of T is strongly minimal.
(a) Prove that a strongly minimal theory has a unique model in every uncountable power (up to isomorphism).
(b) Prove that a strongly minimal theory has 1 or $\aleph_0$ countable models.
(c) Give examples of strongly minimal theories that have 1 and $\aleph_0$ countable models.