Model Theory, list 4.

1. Prove that if T is q.e., then
(a) for all $M, N \models T$, $M \subseteq N$ (i.e. "M is a substructure of N") implies $M \prec N$,
(b) every monomorphism $f : M \to N$ is elementary.

2. Prove that if T is κ-stable, then $|S_n(A)| \leqslant \kappa$ for every $n < \omega$ and every A of power $\leqslant \kappa$.

3. Desrcibe complete 1-types over M (facultatively, for fans, complete n-types over M) and topology of the space $S_1(M)$ for $M \models T$ for the following theories T:
(a) $T = Th(\mathbb{Q}, \leqslant)$,
(b) $T = Th(\mathbb{N}, S)$, (c) $T = Th(\mathbb{Z}, S)$,
(d) the theory of independent predicates (Problem 3.12)
(e) the theory of independent equivalence relations with 2 equivalence classes (Problem 3.11)
(f) the theory of independent equivalence relations with infinitey many classes (first axiomatize it, prove it is q.e.),
(g) the theory of vector spaces over a fixed infinite field K,
(h) the theory of vector spaces of infinite dimension over a fixed finite field K.

4. Investigate stability of theories from Problem 3.

5. Prove that if κ is regular and T is κ-stable, then T has a saturated model of power κ (comment: the regularity assumption is not needed, but without it the problem is hard, in fact it is an early result of Shelah)

6. * Prove that if $\kappa < 2^{\aleph_0}$ and $p_\alpha, \alpha < \kappa$ is a family of complete non-isolated 1-types over $\emptyset$, then there is a (countable) model of T omitting all of them. (comment: this is a result of Shelah. Hint: Instead of trying to counstruct a model right away, construct a family of $2^{\aleph_0}$ countable models of T such that no complete non-isolated 1-type over $\emptyset$ is realized in two models of this family. In the construction use Skolem functions, after suitabe Skolemization.)

7. Which of the theories from Problem 3 have a prime model? Is it minimal?

8. Assume a prime model of T exists and is not minimal. Prove that there is an uncountable atomic model of T. Also, prove the reverse implication.

9. * Is the minimal model always prime ? (hint: no)