Model Theory, list 3.

1. Assume that $p(x) \in S(\emptyset), p' \subset p$ and $p'(x) \vdash p(x)$. Prove that $p' \equiv p$.

2. Assume that $p(x)$ is a type over A. Prove that p is isolated over $A \iff p(\mathcal{M})$ contains a non-empty subset definable over A.

3. Assume that $X \subset \mathcal{M}$ is a definable set (over some parameters) and for every $f \in Aut(\mathcal{M}/A)$, $f[X] = X$. Prove that X is definable by some formula with parameters from A.

4. Assume that $p(x), q(x)$ are types and $\varphi(x), \psi(x)$ formulas over A. Prove that
(a) $p(x) \vdash q(x) \iff \exists M\ |A|^+$-saturated, $p(M) \subset q(M)$.
(b) $p(x) \vdash q(x) \iff$ for every $\varphi(x) \in q(x)$ there is a finite $p_0(x) \subseteq p(x)$ such that $p_0(x) \vdash \varphi(x)$.
(c) $\varphi(x) \vdash \psi(x) \iff T(A) \vdash \forall x(\varphi(x) \to \psi(x))$.

5. Assume that $p(x), q(x)$ are types over A and $\mathcal{M} = p(\mathcal{M}) \cup q(\mathcal{M})$ and $p(\mathcal{M}) \cap q(\mathcal{M}) = \emptyset$. Prove that the sets $p(\mathcal{M})$ and $q(\mathcal{M})$ are definable over A.

6. Assume that $a, b \in \mathcal{M}$. Prove that
$$tp(a/\emptyset) \vdash tp(a/b) \iff tp(b/\emptyset) \vdash tp(b/a).$$

7. Prove that a theory T is $\forall$-axiomatizable (that is, axiomatizable by sentences of the form $\forall \overline{x} \varphi(\overline{x})$, where φ is open) $\iff$ every substructure of a model of T is a model of T.(hint: it is proved in the book by Sacks, Saturated model theory, but it is good if you think about it yourself).

8. Prove that if T is $\forall\exists$-axiomatizable, then the union of any chain of models of T is a model of T.

9. Give axioms of the theory of linear spaces over an infinite field K. Prove that it is complete and q.e. Describe definable sets in models of T. (hint: by a vector space over K we mean a structure $(V, +, 0, r)_r \in K$, where $r : V \to V$ are unary functions, scalar multiplications by $r \in K$).

10. The same as in the previous problem, for infinite linear spaces over a finite field K.

11. Give axioms of the theory $Th(2^\omega, E_n)_{n<\omega}$, where $f\ E_n\ g \iff f(n) = g(n)$. Prove that it is q.e.

12. The same for $Th(2^\omega, P_n(x))_{n<\omega}$, where $P_n(f) \iff f(n) = 0$.

13. We say that a theory T is model complete if $T \cup D_{at}(M)$ is complete for every $M \models T$. Prove that T is model complete $\iff$ for every formula φ there is an existential formula ψ (that is, of the form $\exists \overline{x} \psi'$, where ψ' is open, i.e. without quantifiers) such that $T \vdash \varphi \leftrightarrow \psi$.

14. Prove that T is model complete $\iff$ for all $M, N \models T$, $M \subseteq N$ implies $M \prec N$.