Model Theory, List 2.

1. Give an example of two elementarily equivalent infnite linear orders and a partial isomorphism f between them that is not an elementary mapping.

2. Assume that T is a consistent theory (not necessarily complete). Prove that T is complete $\iff$ every two models of T have isomorphic elementary extensions. (hint: In the proof of $\Rightarrow$ construct isomorphic extensions as unions of suitabe elementary chains of models. Look at the proof of existence of a stronly κ-homogeneous model.)

3. Assume that $M \subseteq N$ are L-structures.
(a) Prove that if $\varphi(\overline{x})$ is an open formula (that is, quantifier free), then for all $\overline{a} \subseteq M$, $(M \models \varphi(\overline{a}) \iff N \models \varphi(\overline{a}))$.
(b) Prove that if $\varphi(\overline{x})$ is an $\exists$-formula (that is, of the form $\exists \overline{y} \psi(\overline{x}, \overline{y})$, where ψ is open), then for all $\overline{a} \subseteq M$, $(M \models \varphi(\overline{x}) \Rightarrow N \models \varphi(\overline{x}))$. Give an example, where the reverse implication fails.

4. Prove that if $S_n(\emptyset)$ is countable for all $n < \omega$, then T has a countable saturated model. Also prove the reverse implication.

5. Prove that a saturated model is strongly homogeneous.

6. Assume $N, M \prec \mathcal{M}$ and $f : M \to N$ is an isomorphism. Prove that f elementary (as a mapping between subsets of $\mathcal{M}$). (hint: There are hidden tacit assumptions here. Namely, M, N are *small*, see Lecture 3.)

7. Prove that every countable M has an elementary countable extension N that is homogeneous. (hint: construct N as the union of an elementary chain of models $M_n, n < \omega$. At step n ensure that a partial finite elemenary mapping $f : M_n \to M_n$ has a suitable extension in M_{n+1}.

8. * Let $f, g \in \mathbb{C}[x]$. Prove that if the structures $(\mathbb{C}, f)$ and $(\mathbb{C}, g)$ (for the language with one symbol of a unary function) are elementarily equivalent, then they are isomorphic.

9. Give axioms of the thery of successor function $Th(\mathbb{Z}, S)$. Prove it is decidable.(hint: Ehrenfeucht games)