

6.10.2021

Formalizacja metematyki: struktury, języki, spełnienie.

Uproszczony model rzeczywistości metematycznej:
struktura (I rodzaju)

Def. 1.1 $M = (A, f_1, f_2, \dots, f_k, P_1, \dots, P_n, c_1, \dots, c_l)$
model, struktura
 \uparrow
Zbiór $\neq \emptyset$
uniwersum struktury M
"arność" n_i

f_1, \dots, f_k - funkcje, $f_i: A \rightarrow A$

P_1, \dots, P_k - predykaty w A , $P_i \subseteq A^{m_i}$
(relacje)

c_1, \dots, c_l - stałe \cup , $c_i \in A$

Przykłady • gdy $n = 0$, M nazywamy strukturą algebraiczną (grupy, ciała)

• V , rodzinie zbiorów, (V, ϵ)
 \uparrow
relacja binarna

Zadania o \mathcal{M} :

w pewnym języku: f_i, P_j, c_t to symbole
oznaczające funkcje, relacje, stałe.

Odróżnienie między symbolem, a jego
znaczeniem.

f_i, P_j, c_t ← ta dolna kreska
ma zaznaczyć, że
mamy na myśli symbol,
 f_i, P_j, c_t ← ich znaczenia w
strukturze \mathcal{M}

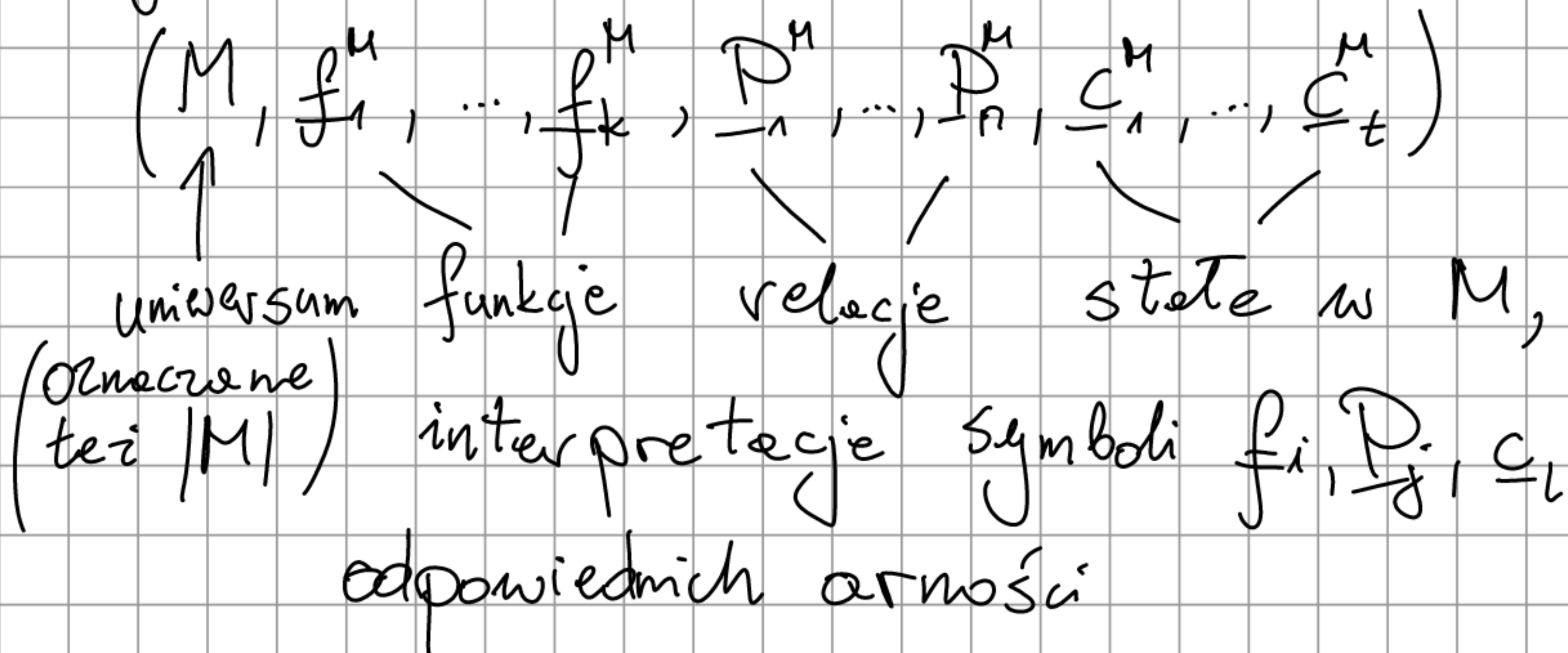
Def. 1.2 Język $L = \{f_1, \dots, f_k, P_1, \dots, P_n, c_1, \dots, c_t\}$
(Struktury \mathcal{M}) wraz z ornościami
tych funkcji i relacji

Inaczej: typ podobieństwa struktury \mathcal{M} ,
sygnatura \mathcal{M}

f_i, P_j, c_t to interpretacje f_i, P_j, c_t

Mówimy, że \mathcal{M} jest modelem dla L .

Def. 1.3 M : model dla L



Jak mówić w L ? Do tego służą:

- symbole języka,

- symbole logiczne: $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow$

$\Rightarrow, \Leftrightarrow$ to
META SYMBOLS

$\forall, \exists, x_i (i \in \mathbb{N})$
zmienne

- symbol równości: $=$

- symbole pomocnicze (nawiasy, przecinki...)

Def. 1.4 Wyrażenia języka L :

1. wyrażenia nazwowe (**termy**): zbiór

termów
cyjnych.

\mathcal{T}_L języka L , def. rekurren-

- zmienne, symbol stałej $\in \mathcal{T}_L$ (termy atomowe)

• $T_1, \dots, T_n \in \mathcal{T}_\alpha$, f : symbol funkcji z L , n -arny

Wtedy $f(T_1, \dots, T_n) \in \mathcal{T}_\alpha$

term zlozony

term **staly** =
term bez
zmiennych

2. formuly języka \mathcal{L} : \mathcal{F}_α

formuly
atomowe

• $(T_1 = T_2) \in \mathcal{F}_\alpha$
↑ term

• P_j - n -arny symbol relacyjny
 $P_j(T_1, \dots, T_n) \in \mathcal{F}_\alpha$

formuly
zlozone

• $\varphi, \psi \in \mathcal{F}_\alpha \Rightarrow (\neg\varphi), (\varphi \wedge \psi) \in \mathcal{F}_\alpha$

• v : zmienna, $\varphi \in \mathcal{F}_\alpha$, wtedy

$\exists v \varphi, \forall v \varphi \in \mathcal{F}_\alpha$

↑
"zasięg" $\exists v$ w $\exists v \varphi$

Przykład

$\forall x \exists y (x \in y \wedge \forall x (x \in y \Rightarrow x = y))$

w $\mathcal{L} = \mathcal{L} \cup \{ \in \}$ (binarny symbol rel.)

Def. 1.5 Niech $\varphi \in \mathcal{F}_L$, v : zmienna występująca w φ .

• Jeśli to wystąpienie v w φ jest w zasięgu pewnego kwantyfikatora Q w v , to spośród wszystkich takich wystąpień Qv w φ , w których v jest zasięgu, wybieramy to najbardziej na prawo i mówimy, że to Qv wiąże to wystąpienie v w φ .

• Jeśli takiego Qv nie ma, to mówimy, że v w φ jest wolne.

• v jest wolne w φ , gdy istnieje jej wolne wystąpienie.
Konwencja: zapis $\varphi(v_1, v_2, \dots, v_n)$ oznacza, że wszystkie zmienne wolne w φ są wśród v_1, \dots, v_n .

Def. 1.6 Zdaniem w L nazywamy formułę bez zmiennych wolnych.

Co to znaczy, że formuła jest prawdziwa w strukturze M dla L ?

(św. Tomasz: prawdziwy sąd to taki, który jest zgodny z rzeczywistością)

Def. 1.7 Model M dla L .

$$(M, \overset{M}{f}_1, \dots, \overset{M}{P}_1, \dots, \overset{M}{c}_1) \quad \overset{M}{\{f_1, \dots, f_k, P_1, \dots, P_m, c_1, \dots, c_k\}}$$

Niech $A = \{ \underline{a} : a \in M \}$, zbiór ^{NOWYCH} symboli statycznych.

Tworzymy nowy język $L(M) = L \cup A$. Interpretacje

termów statycznych σ w $L(M)$ w M :

• gdy $\mathcal{I} = \underline{c}_i \rightsquigarrow \mathcal{I}^M = \overset{M}{c}_i$

meta-symbol \nearrow

} termy atomowe

• gdy $\mathcal{I} = \underline{a} \rightsquigarrow \mathcal{I}^M = a$
 $a \in M$

• gdy $\mathcal{I} = f_i(\mathcal{I}_1, \dots, \mathcal{I}_n)$

$$\mathcal{I}^M = \overset{M}{f}_i(\overset{M}{\mathcal{I}}_1, \dots, \overset{M}{\mathcal{I}}_n)$$

Def. 1.8 Spletwienie zdań φ języku

$L(M)$ w M :

$M \models \varphi$: " φ jest prawdziwe w M "
spletwienie

1. Zdania atomowe

• $M \models \mathcal{I}_1 = \mathcal{I}_2 \iff \mathcal{I}_1^M = \mathcal{I}_2^M$

• $M \models \mathcal{D}(\mathcal{I}_1, \dots, \mathcal{I}_n) \iff (\mathcal{I}_1^M, \dots, \mathcal{I}_n^M) \in \mathcal{D}$

2. Zdania złożone

• $M \models \varphi \wedge \psi \iff M \models \varphi$ oraz $M \models \psi$

• $M \models \neg \varphi \iff$ nieprawda, że $M \models \varphi$.

• $M \models \exists v \varphi \iff$ istnieje $a \in M$ t. że

$M \models \varphi(\underline{a})$, gdzie

$\varphi(\underline{a})$ powstaje z φ przez

zastąpienie każdego wolnego

v przez symbol \underline{a} .

Def. 1.9 Jeżeli zdanie φ nie jest prawdziwe w M , to mówimy, że jest fałszywe w M .

Konwencja: spełnienie w M formuły $\varphi(x_1, \dots, x_n)$:
 $\langle a_1, \dots, a_n \rangle$ spełnia $\varphi(x_1, \dots, x_n)$, gdy
 $M \models \varphi(\underline{a}_1, \dots, \underline{a}_n)$

Def. 1.10 Uniwersalne domknięcie

formuły φ : $\bar{\varphi} = \forall x_1 \forall x_2 \dots \forall x_n \varphi$

Mówimy, że $M \models \varphi \iff M \models \bar{\varphi}$

[to znemy dobrze, np. kiedy mówimy o łączności działania w grupie, to

zwykle piszemy $(xy)z = x(yz)$, a

nie $\forall x \forall y \forall z ((xy)z = x(yz))$.]

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Def. 2.1 $\varphi \in \mathcal{F}_L$ jest tautologią (Klasycznego rachunku Logicznego), gdy $\forall M \text{ model dla } L \quad M \models \varphi$.

Piszemy $\models \varphi$.

Jak rozpoznać, że $\models \varphi$? NIEROZSTRZYGAJNE

Niekiedy jest Taut., np. $x = x$,

" \models jest relacją równoważności"

Def. 2.2 Formaty zdaniowe:

Z : zbiór zmiennych zdaniowych

$\hookrightarrow S$ - zbiór format zdaniowych nad Z .

Definicja rekurencyjna:

• $v \in Z \Rightarrow v \in S$

• $\alpha, \beta \in S \Rightarrow \neg \alpha, (\alpha \wedge \beta) \in S$

"Jakby to wszystko sprędy zowie, to może kumaniszi by stracili pracę"

NIE MÓWIĆ
"TAUTOLOGIA
TO ZDANIE
ZA WSZĘDZIE
PRAWDZIWIE"

Potocznie tautologia to zdanie, którego prawdziwość wynika z jego struktury.

Def. 2.3 Wartościowanie logiczne: dowolna funkcja

$$v: S \rightarrow \{0, 1\} \text{ t.j. } v(\neg \alpha) = 1 - v(\alpha),$$

$$v(\alpha \wedge \beta) = \min\{v(\alpha), v(\beta)\} \text{ dla dowolnych } \alpha, \beta \in S.$$

Def. 2.4 $\alpha \in S$ jest tautologią klasycznego rachunku zdań, gdy

$$\forall v: S \rightarrow \{0, 1\} \quad v(\alpha) = 1 \quad (\models \alpha)$$

wartościowanie

Przykład. $\models \neg(\alpha \wedge \neg \alpha)$ dla każdego $v \in S$.

Istnieje algorytm rozstrzygający, czy $\models \alpha$.

Def. 2.5 Zet. że $\alpha \in S$ jest zbudowane $\alpha(p_1, \dots, p_n)$

ze zmiennych zdaniowych p_1, \dots, p_n oraz

spójników logicznych. Niech $\varphi_1, \dots, \varphi_n \in \mathcal{F}_L$.

Zet., że $\varphi \in \mathcal{F}_L$ powstał przez podstawienie

wszędzie φ_i za p_i . Mówimy wtedy, że

φ jest przykładem formuły α .

Tw. 2.6 Jeśli $\models \alpha$ oraz φ : przykłada α , to $\models \varphi$.
Ćw.

Def. 2.7 Reguły wnioskowania:

$\frac{\varphi_1, \dots, \varphi_n}{\varphi} \leftarrow$ przesłanki (premises)
 $\varphi \leftarrow$ konkluzja, teza $\varphi_i, \varphi \in \mathcal{F}_L$

Podobnie $\frac{\alpha_1, \dots, \alpha_n}{\alpha}, \alpha_i, \alpha \in S$.

Def. 2.8 Reguła $\frac{\varphi_1 \dots \varphi_n}{\varphi}$ jest poprawna, (sound),

gdy $\forall M$ dla L ($M \models \varphi_1, \dots, M \models \varphi_n \Rightarrow M \models \varphi$).

Dla formuł zdaniowych reguła $\frac{\alpha_1 \dots \alpha_n}{\alpha}$

jest poprawna, gdy

$\forall v: S \rightarrow \{0, 1\} \quad (v(\alpha_1) = \dots = v(\alpha_n) = 1 \Rightarrow v(\alpha) = 1)$

Przykłady

- modus ponens
(reguła odrywania)
cut rule

$$\frac{\alpha, \alpha \rightarrow \beta}{\beta}, \quad \frac{\varphi, \varphi \rightarrow \psi}{\psi}$$

• reguła generalizacji
 \forall -rule ("A" rule)

$$\frac{\varphi}{\forall v \varphi}$$

v : zmienna

Aksjomatyczne ujęcie KRZ

A0. przykłady tautologii KRZ

A1. $\forall v (\varphi \rightarrow \varphi) \rightarrow (\varphi \rightarrow \forall v \varphi)$ (gdy v nie jest zm. wdn φ)

A2. $\forall v \varphi \rightarrow \varphi(v/t)$

pod warunkiem,
 że żadne

$[t_i: t_i \in \mathcal{I}_L, \text{którego}$

z takich występi v w φ nie jest w zasięgu

podstawiamy w φ za każde wolne wystąpienie zmiennej v

kwantyfikatora wiążącego jakąś zmienną w t

Istotne zastrzeżenie:

$$\varphi: \exists y \ x \neq y, \quad t = y, \quad \varphi(x/t): \exists y \ y \neq y$$

$$\exists y (x \neq y) \rightarrow \exists y (y \neq y) \quad \text{nie jest}$$

tautologią.

Aksjomaty równości $(v, v_1, v_2 - \text{zmienna})$
 $t - \text{term}$
 $\varphi - \text{formuła}$

• $v = v$

• $v_1 = v_2 \rightarrow t(\dots v_1 \dots) = t(\dots v_2 \dots)$

• $v_1 = v_2 \rightarrow (\varphi(\dots v_1 \dots) \rightarrow \varphi(\dots v_2 \dots))$

Pod warunkiem, że t występuje wolno w v_1
 pod t ówe podstawiamy v_2 nie jest
 w zasięgu kwantyfikatora wiążącego v_2 .

Def. 2.9 Zał. że $X \subseteq \mathcal{F}_L, \varphi \in \mathcal{F}_L$. Mówimy, że

$X \vdash \varphi \stackrel{\text{def.}}{=} \text{istnieje ciąg formuł}$

"dowodzi"

$\alpha_1, \dots, \alpha_n = \varphi$ t. że $\forall i \leq n:$

1° $\alpha_i \in X$ lub α_i jest aksjometem KR1

2° α_i wynika z $\{\alpha_1, \dots, \alpha_{i-1}\}$ na mocy
 modus ponens lub \forall -reguły, tzn.

$\exists j, t < i \quad \alpha_t = (\alpha_j \rightarrow \alpha_i)$

lub $\exists j < i \quad \alpha_i = \forall x \alpha_j$ (wtedy $\frac{\alpha_j, \alpha_i}{\alpha_i}$ przykład MP)

Def. 2.10 $\vdash \varphi$, gdy $\emptyset \vdash \varphi$.
 (φ jest tezą KRL)

Uwaga: $X \vdash \varphi \Leftrightarrow \exists X_0 \subseteq X \quad X_0 \vdash \varphi$
 ↑
 skończone

Podział $\vdash \forall x \varphi \rightarrow \exists x \varphi$

$\beta: \forall x \varphi \rightarrow \neg \forall x \neg \varphi$

Gdyby struktura była pusta, to niedzieta. Czym?

$\alpha_1: \forall x \varphi \rightarrow \varphi(x/y)$
 $\alpha_2: \forall x \neg \varphi \rightarrow \neg \varphi(x/y)$ } tu y : zmienna
 spoza φ

$\alpha_3: \alpha_2 \rightarrow (\varphi(x/y) \rightarrow \neg \forall x \neg \varphi(x))$ } AO, prawo transpozycji

$\alpha_4: \varphi(x/y) \rightarrow \neg \forall x \neg \varphi(x)$ } MP, α_2, α_3

$\alpha_5: \alpha_1 \rightarrow (\alpha_4 \rightarrow \beta)$ [AO] $(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma))$

$\alpha_6: \beta$ (2x MP, $\alpha_1, \alpha_4, \alpha_5$)

Uwaga $\vdash \varphi \Rightarrow \models \varphi$, tzn. KRL jest poprawny (sound)

Tw. 2.11 (Gödel o pełności KRL)
 completeness theorem

$\models \varphi \Rightarrow \vdash \varphi$

Def. 2.12 $X \subseteq \mathcal{F}_L$. $Cn(X) = \{ \varphi \in \mathcal{F}_L : X \vdash \varphi \}$.
konsekwencje

X jest teorią, gdy $X = Cn(X)$.

Fakt 2.13 $\varphi \vdash \bar{\varphi}$ i $\bar{\varphi} \vdash \varphi$
d-d \forall -regula

$\vdash \bar{\varphi} \rightarrow \varphi$

Def. 2.14 $X, Y \subseteq \mathcal{F}_L$ są równoważne, gdy
 $Cn(X) = Cn(Y)$

Wniosek 2.15 X i $\{ \bar{\varphi} : \varphi \in X \}$ są równoważne.

Def. 2.16 X jest spreczny, gdy istnieje zdanie φ t. że $X \vdash \varphi \wedge \neg \varphi$. W przeciwnym razie mówimy, że X jest niespreczny

Def. 2.17 (1) $M \models X \Leftrightarrow \forall \varphi \in X M \models \varphi$
 M jest modelem X

(2) X jest zupełny $\Leftrightarrow \forall$ zdanie φ ($X \vdash \varphi$ lub $X \vdash \neg \varphi$)

(3) X jest rozstrzygalny \Leftrightarrow istnieje algorytm rozstrzygający, czy $X \vdash \varphi$ dla $\varphi \in \mathcal{F}_L$

Przykład (1) $\text{Th}(M) = \{\varphi \in \mathcal{F}_L : M \models \varphi\}$: niespójna,
(teoria str. M) zupełna

(2) \emptyset jest niespójny

Tw. 2.18 (Gödel, o istnieniu modelu) Jeżeli S

jest niespójnym zbiorem zdań, to S

ma model (mocy $\leq \aleph_0 + |S|$)

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D-d. TW. Gödla o modelu (Leon Henkin)
gdzie L i S przeliczalne.

Niech $L' = L \cup \{c_n : n < \omega\}$ (zbiór nowych symboli)

Niech $\{ \varphi_n(x) : n < \omega \}$: numeracja \downarrow statych
formuł $\approx \{ \mathcal{F}_i \}$ ze zmienną wolną x .

Pomocniczo funkcja $f: \omega \rightarrow \omega$ rosnąca

i t.ż. $c_{f(n)}$ nie występuje w formułach

$\varphi_0(x), \dots, \varphi_n(x)$ dla $n=0, 1, \dots$

Niech $S_n = S \cup \{ \exists x \varphi_i(x) \rightarrow \varphi_i(c_{f(i)}) : i < n \}$
 $S = S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots$
Aksjomat Henkina

Niech $S_\omega = \bigcup_n S_n$.

Fakt S_ω jest niespreczny.

D-d. nie wprost

Skoro S_ω spreczne, to pewnie S_n spreczne.

Niech n to najmniejsze takie że S_{n+1} sprz.

($n \geq 0$ bo $S_0 = S$ niespreczne)

(tw. o dedukcji: φ zdanie, φ formuła, $X \subseteq \mathcal{F}_L$)
 $(X \cup \{\varphi\} \vdash \psi) \Leftrightarrow X \vdash \varphi \rightarrow \psi$

$$\begin{aligned} S_n \vdash (\exists x \varphi_n(x) \rightarrow \varphi_n(c_{f(n)})) &\longrightarrow \alpha \wedge \neg \alpha \\ \Downarrow \text{AO, MP} & \\ (p \rightarrow (q \wedge \neg q)) \rightarrow \neg p & \quad (\alpha = \text{zd.}) \end{aligned}$$

$$S_n \vdash \neg (\exists x \varphi_n(x) \rightarrow \varphi_n(c_{f(n)}))$$

$$\Updownarrow \text{AO, MP}$$

$$S_n \vdash \exists x \varphi_n(x) \wedge \neg \varphi_n(c_{f(n)})$$

$$\Downarrow$$

$$S_n \vdash \exists x \varphi_n(x) \quad \text{oraz} \quad S_n \vdash \neg \varphi_n(c_{f(n)})$$

Z wyboru f symbol $c_{f(n)}$ nie występuje

w zdaniach z S_n . W dowodzie $\neg \varphi_n(c_{f(n)})$

z S_n możemy zastąpić każde wystąpienie

$c_{f(n)}$ przez nową zmienną y nie występującą

w formułach dowodu. Dostajemy dowód

$$\neg \varphi_n(y) \text{ z } S_n. \text{ Stąd } S_n \vdash \neg \varphi_n(y).$$

$$\text{Na mocy } \forall\text{-reguły } S_n \vdash \forall y \neg \varphi_n(y)$$

Na mocy A2, MP $S_n \vdash \forall x \neg \varphi_n(x)$

Ale $S_n \vdash \exists x \varphi_n(x)$, stąd S_n sprzeczne.

Ale n miało być minimalne.

FAKT \square

Z tw. Lindenbauma niech $S' \supseteq S_\omega$

będzie niesprzecznym, zupełnym zb. zdań.

$C = \{c_n : n < \omega\}$ ← ta def. relacji równoważności:

$$c_n \sim c_m \iff S' \vdash c_n = c_m.$$

(Ćw. sprawdzić że to rel. równoważności)

Budujemy model M z klas abstrakcji:

$|M| = \bigcup C/n$. Musimy zinterpretować symbole

języka L' w M :

$$(1) \underline{R}_i^H \left([c_{i_1}]_n, [c_{i_2}]_n, \dots, [c_{i_k}]_n \right)$$

$$\stackrel{\text{def}}{\iff} S' \vdash R_i(c_{i_1}, \dots, c_{i_k})$$

$$(2) f_j^H \left([c_{i_1}]_n, \dots, [c_{i_k}]_n \right) = [c_{i_{k+1}}]_n$$

$$\stackrel{\text{def}}{\iff} S' \vdash f_j(c_{i_1}, \dots, c_{i_k}) = c_{i_{k+1}}$$

$$(3) \quad \underline{C}_n^M = [c_n]_n, \quad \underline{C}^M = [c_n]_n \quad \text{gdzie } S' \vdash \underline{c} = c_n$$

Poprawność (2):

$$S' \vdash \exists x \underbrace{f_j(c_{i_1}, \dots, c_{i_k})}_{\bar{c}} = x \quad \text{bo } \geq A2$$

$$\forall x (x \neq f_j(\bar{c})) \rightarrow f_j(\bar{c}) \neq f_j(\bar{c})$$

\Downarrow AO

$$f_j(\bar{c}) = f_j(\bar{c}) \rightarrow \exists x (x = f_j(\bar{c}))$$

$$\vdash f_j(\bar{c}) = f_j(\bar{c})$$

\Downarrow MP

$$\vdash \exists x \underbrace{f_j(\bar{c}) = x}_{\varphi_n(x)}$$

" $\varphi_n(x)$ dla pewnego n "

Ponadto $S' \vdash \exists x \varphi_n(x) \rightarrow \varphi_n(c_{f(n)})$

Zatem z MP $S' \vdash \varphi_n(c_{f(n)})$.

(3) Analogicznie

Lemat. \forall zdania $\alpha \in \mathcal{F}_L$ ($M \models \alpha \Leftrightarrow S' \vdash \alpha$)

D-d. Ind. względem dt. α .

1° α : atomowe

(a) $\alpha: t_1 = t_2$

$M \models t_1 = t_2$

$\Leftrightarrow t_1^M = t_2^M$

$\Leftrightarrow [c_n]_2 = [c_m]_2$

$\Leftrightarrow S' \vdash c_n = c_m \stackrel{\text{cw.}}{\Leftrightarrow} S' \vdash t_1 = t_2$

Ćw. istnieją c_n, c_m t. że

$S' \vdash t_1 = c_n, S' \vdash t_2 = c_m$

\Downarrow ind. wzg. dt. S'
 $t_1^M = [c_n]_2$

(b) $\alpha: R(t_1, \dots, t_k)$

$M \models R_i(t_1, \dots, t_k) \Leftrightarrow R_i^M([c_{i_1}]_2, \dots, [c_{i_k}]_2)$

\uparrow
 $S' \vdash t_l = c_{i_l}$

\Downarrow
 $t_l^M = [c_{i_l}]_2$

def. M

$\Leftrightarrow S' \vdash R_i(c_{i_1}, \dots, c_{i_k})$

aks. =

$\Leftrightarrow S' \vdash R_i(t_1, \dots, t_k)$

2° spójniki

(a) negacja: $M \models \neg \alpha \Leftrightarrow \neg (M \models \alpha)$

$\Leftrightarrow \neg (S' \vdash \alpha) \stackrel{2 \text{ zat. ind.}}{\Leftrightarrow} S' \vdash \neg \alpha$ S' : niesprzeczny
zupelny

$$S' \vdash \neg \alpha$$

(b) koniunkcja: podobnie

3° kwantyfikatory

$$\exists: \alpha: \exists x \varphi_n(x). \quad M \models \exists x \varphi_n(x) \Rightarrow M \models \varphi_n([c_k]_n)$$

$$\stackrel{\text{z zol. ind.}}{\Longleftrightarrow} S' \vdash \varphi_n(c_k) \stackrel{A2}{\Rightarrow} S' \vdash \exists x \varphi_n(x)$$

$$S' \vdash \exists x \varphi_n(x) \Rightarrow S' \vdash \varphi_n(c_{f(n)}) \stackrel{\text{zol. ind.}}{\Rightarrow} M \models \varphi_n(c_{f(n)})$$
$$\Downarrow$$
$$M \models \exists x \varphi_n(x)$$

lewat ~~■~~

Z tego skoro $M \models S'$, to $M \models S$

rw. ~~■~~

Wn. 2.13 tw. Gödla o pełności KRL

$$\models \varphi \Rightarrow \vdash \varphi$$

D-d. BSO φ : zdanie (patrz φ)

Zat. nie wprost że $\not\vdash \varphi$. Wtedy z tw.

o dedukcji $\not\vdash \varphi$ niespreczny. Z tw.

o istnieniu modelu $\exists M \models \neg \varphi$ więc $\not\vdash \varphi$.

~~■~~

Tw. 2.20 (o zwartości)

Jeżeli każdy skończony podzbiór $X \in \mathcal{F}_L$ ma model, to X ma model.

Wn. 2.21 $X \models \varphi \Leftrightarrow (\forall M \models X) M \models \varphi$ ▀

Ustalamy (przeliczalny) L , rozważamy tylko struktury dla L .

Def. 3.1 $M \equiv N$ (elementarnie równoważnie)

\Updownarrow def.

$$\text{Th}(M) = \text{Th}(N) \Leftrightarrow \forall \varphi \in \mathcal{F}_L (M \models \varphi \Leftrightarrow N \models \varphi)$$

Def. 3.2 $M \subseteq N \Leftrightarrow |M| \subseteq |N|$ oraz

\uparrow
podstruktura

$$\forall i, j \in I \quad f_i^M = f_i^N \upharpoonright |M|$$

$$\mathcal{P}^M = \mathcal{P}^N \upharpoonright |M|$$

$$c_t^M = c_t^N$$

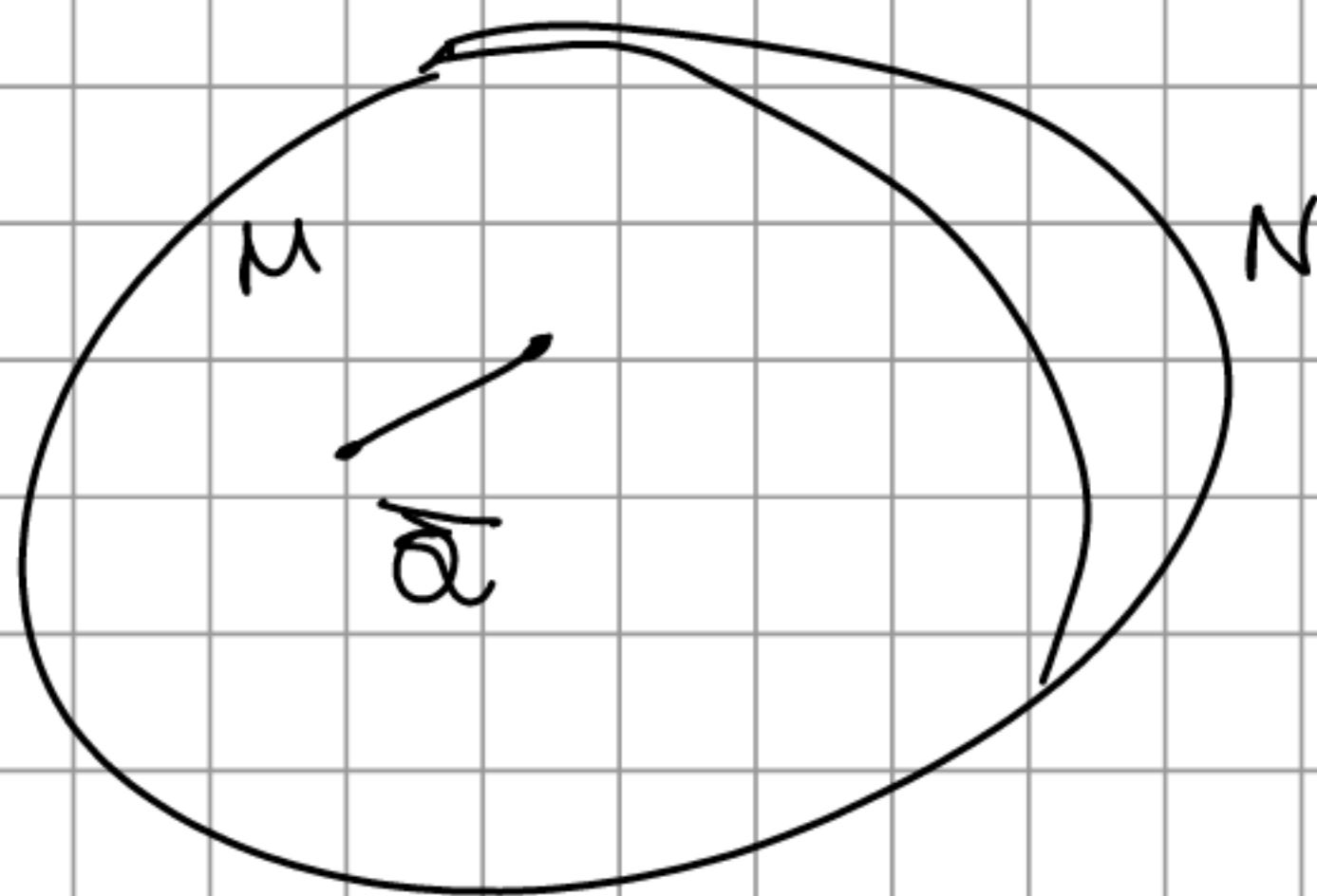
Def. 3.3 $g: M \xrightarrow{\cong} N \Leftrightarrow g: |M| \xrightarrow[nv]{1-1} |N|$
 izomorfizm oraz $\forall \bar{a} \in |M|$
 $\forall b \in |M|$ $\bar{a} = (a_1, \dots, a_n)$
 $g(\bar{a}) = (g(a_1), \dots, g(a_n))$

$\forall i, j, t$ $\left\{ \begin{array}{l} M \models \mathcal{R}_i(\bar{a}) \Leftrightarrow N \models \mathcal{R}_i(g(\bar{a})) \\ M \models f_j(\bar{a}) = b \Leftrightarrow N \models f_j(g(\bar{a})) = g(b) \\ M \models c_t = b \Leftrightarrow N \models c_t = g(b) \end{array} \right.$

Def. 3.4 $M \cong N \Leftrightarrow \exists g: M \xrightarrow{\cong} N$
 izomorficzne

Def. 3.5 $M \prec N$ gdy $M \subseteq N$ oraz
 elementarne podstruktura

$\forall \bar{a} \in M \quad \forall \varphi(\bar{x}) \in \mathcal{F}_L \quad (M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(\bar{a}))$



Def. 3.6 $f: M \xrightarrow{\equiv} N \iff \forall \bar{a} \in M \forall \varphi(\bar{x}) \in \mathcal{F}_L$
 elementarne $M \models \varphi(\bar{a}) \iff N \models \varphi(f(\bar{a}))$

Test Tarskiego - Vaughta Złożymy, że $A \subseteq |M|$.

Wtedy A jest uniwersum elementarnej podstruktury M wtedy, gdy spełnia warunki:

(*) Dla każdej formuły $\varphi(x, \bar{y}) \in \mathcal{F}_L, \bar{a} \in A$
 jeśli $M \models \exists x \varphi(x, \bar{a})$, to dla pewnego $b \in A, M \models \varphi(b, \bar{a})$

D-d. " \Rightarrow " $A = |N|, N \leq M, \bar{a} \in A$

$$M \models \exists x \varphi(x, \bar{a}) \Rightarrow N \models \exists x \varphi(x, \bar{a})$$

$$\iff \text{dla pewnego } b \in A \quad N \models \varphi(b, \bar{a})$$

" \Leftarrow "

$$M \models \varphi(b, \bar{a})$$

(a) $A = |N|$ dla pewnego $N \subseteq M$.

tzn. że $f_i^M[A] \subseteq A$. Tak jest, bo:

$$\text{dla } \bar{a} \in A \quad M \models \exists x f_i(\bar{a}) = x$$

∇ (*) $M \models f_i(\bar{a}) = b$ dla $b \in A$, tzn. $f_i^M(\bar{a}) = b$

(b) Dla każdej formuły $\varphi(\bar{y}) \in \mathcal{F}_i$ i $\bar{a} \subseteq A$
 $M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(\bar{a})$ (to chcemy pokazać)

indukcja względem dt. φ :

• φ : atomowy: OK, bo $N \subseteq M$

• przejście przez spójniki: Trivialne

• przejście przez kwantyfikatory

\exists : $\varphi(\bar{y}) = \exists x \psi(x, \bar{y})$

$M \models \exists x \psi(x, \bar{a}) \stackrel{(*)}{\Leftrightarrow} M \models \psi(b, \bar{a}) \stackrel{\text{zól. ind.}}{\Leftrightarrow} N \models \psi(b, \bar{a})$
dla pewn. $b \in A$ dla pewn. $b \in A$

$\Leftrightarrow N \models \exists x \psi(x, \bar{a})$

□

Wn. 3.7 (tw. Löwenheima - Skolem, Mostowski)

(a) Jeśli $A \subseteq |M|$, to $(\exists N \prec M) (A \subseteq |N|)$,
(Ddne)

$|N| \leq |A| + |L|$
" def.
 $|L|$

(b) Jeśli M : nieskończony, to $\exists N \not\subseteq M$
(Górne) \uparrow
dowolnie dużej
mocy

D-d. (a) $\kappa = |A| + |L|$. Konstruujemy ciąg
 $a_\alpha \in M, \alpha < \kappa$ ($a_{<\alpha} = \{a_\beta : \beta < \alpha\}$)

Rekurencyjnie

• Na kroku α rozważamy formułę

$\varphi_\alpha(x) \in \mathcal{F}_L(Aa_{<\alpha})$ t. że $M \models \exists x \varphi_\alpha(x)$

Jako a_α bierzemy dowodny świadek tego \exists

20.10.2021

Proof (upward \aleph -S, Malcev thm)

Consider theory $T = \text{Th}(M, \underline{a}^M)$ in
language $L(M)$.

↑
elemental diagram of M

let κ be any cardinal number.

let $\{c_\alpha : \alpha < \kappa\}$ be a set of new

constant symbols. let $T' = T \cup \{c_\alpha \neq c_\beta : \alpha < \beta < \kappa\}$

in $L' = L(M) \cup \{c_\alpha : \alpha < \kappa\}$. We'll prove
that T' is consistent.

let $X \subseteq T'$ be finite. We expand $(M, a)_{a \in M}$

interpreting finitely many c_α 's appearing
in X as distinct elements of M

↯

We get a model of X , so

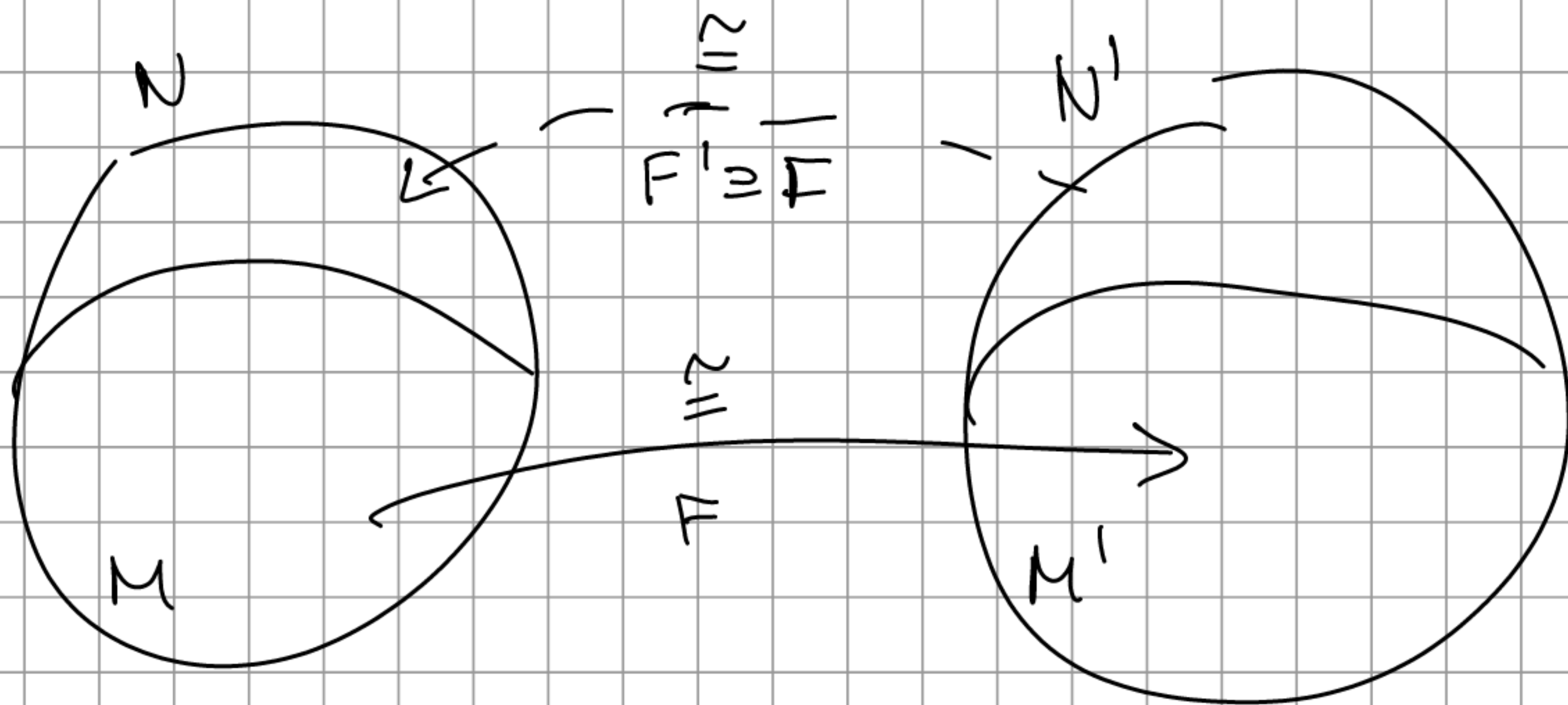
X is consistent, so T' is

(compactness thm)

Let $N' = T' \supseteq T$. Also $N' = C_\alpha \neq C_\beta$ for
 $(N', \underline{a}^N)_{a \in |M|}$ all $\alpha < \beta < \kappa$. Hence $\|N'\| \geq \kappa$

Exercise: • $\{ \underline{a}^N : a \in |M| \} = |M'|$ for some $M' \prec N'$

• $M \cong M'$
 $a \mapsto \underline{a}^N$



• we can extend F so that
 N is isomorphic to N' .

Löwenheim-Skolem paradox

"We cannot express uncountability"

My interpretation of this paradox:

Let $(V, \in) \models \text{ZFC}$ be some "big" model that contains \mathbb{N} and is transitive

Take subset of ZFC (call it ZFC^* s.t. it's finite and $\forall A, B \in V$
 $A \subseteq B \Leftrightarrow A \in B$)

$\text{ZFC}^* \models$ "For every X we have $|P(X)| \neq |X|$ "

(This is Cantor's theorem, it is true in ZFC and we can take some

finite number of ZFC axioms that prove this).

Take some $\mathbb{N} \subseteq M \subseteq V \models \text{ZFC}^*$ (L-S thm).
↑
countable

We also need M to be transitive
(just add some axioms to ZFC^*).

Now, $M \models "P(N) \neq N"$ which can
be written as

$$M \models \forall X \subseteq N (X \in A \rightarrow |A| \neq |N|)$$

We now show that $A \in M (\Leftrightarrow A \subseteq M)$,

M is countable, but proves that

A is uncountable! (which is not
true in V).

Problem How to determine if $M \equiv N$

or $M \cong N$?



Ehrenfeucht - Fraïssé games

Assume, that L is a finite relational language. We consider L -structures M, N .

Def. 4.1 $\Gamma_n(M, N)$ is a game consisting of n moves (each move is one move for a player).

We call the players
 $\hookrightarrow I$ (spoiler) $\hookrightarrow II$ (prover)

At each move players pick one element of M and N (one from each).

On move i ($1 \leq i \leq n$) the players have already picked elements $a_1, \dots, a_{i-1} \in M$, $b_1, \dots, b_{i-1} \in N$. Now I picks one of M, N and then one element from it.

Player II responds with picking an element from the other model: $a_i \in M, b_i \in N$.

After n moves: $\{a_1, \dots, a_n\} \subseteq M$, $\{b_1, \dots, b_n\} \subseteq N$.

$$f: a_i \mapsto b_i$$

Player II wins if f is an isomorphism of the induced substructures.

Example When $M \cong N$, then II obviously win.

Just take $F: M \xrightarrow{\cong} N$ and pick according to it.

Example $(\mathbb{Z}, \leq) \not\cong (\mathbb{Z}_1, \leq) \sqcup (\mathbb{Z}_2, \leq)$

$$\left(\begin{array}{c} \dots \\ \mathbb{Z}_1 \end{array} \right) < \left(\begin{array}{c} \dots \\ \mathbb{Z}_2 \end{array} \right)$$

For instance: $n=5$

$$\begin{array}{ccc} \text{I: } 1 & \longrightarrow & \text{II: } 1_1 \\ \text{II: } 32 & \longleftarrow & \text{I: } 1_2 \\ & & \vdots \end{array}$$

Arbitrary number greater than 5 wins!

Proof exercise. ▀

Thm 4.2 (Ehrenfeucht - Fraïssé)

$M \equiv N \iff \forall n \ll \mathbb{N}$ has a winning strategy
in $\Gamma_n(M, N)$

Def. 4.3 X is a set of axioms of theory T
when $Cn(X) = T$.

Examples of theories

Let $L = \{ \leq \}$.

1. LO: axioms " \leq is a linear order".
Linear order

2. DLO: axioms of LO and:
dense LO $(\forall x, y) (x < y \rightarrow \exists z \ x < z < y)$

3. DLO₀: axioms of DLO and:
DLO without endpoints $\forall x \exists y, z \ y < x < z$

DLO₀ is consistent, because it has a
model: \mathbb{Q} . Moreover, it is complete
and decidable.

Def. 4.3 Theory T is κ -categorical if for every $M, N \models T$ s.t. $\|M\| = \|N\| = \kappa$ we have $M \cong N$.

Remark 4.4 DLO_0 is \aleph_0 -categorical

Proof (Back-and-forth argument)

Assume $M = \{a_n, n < \omega\} \models DLO_0$

$N = \{b_n, n < \omega\} \models DLO_0$

We construct functions f_n and sets $A_n \subseteq M$, $B_n \subseteq M$ such that

(a) $f_n \subseteq f_{n+1}$ (which implies $A_n \subseteq A_{n+1}$, $B_n \subseteq B_{n+1}$)

(b) $f_n: A_n \xrightarrow{\cong} B_n$ of induced structures

(c) $a_n \in A_{2n+1}$, $b_n \in B_{2n+2}$

Construction We will construct those objects

recursively:

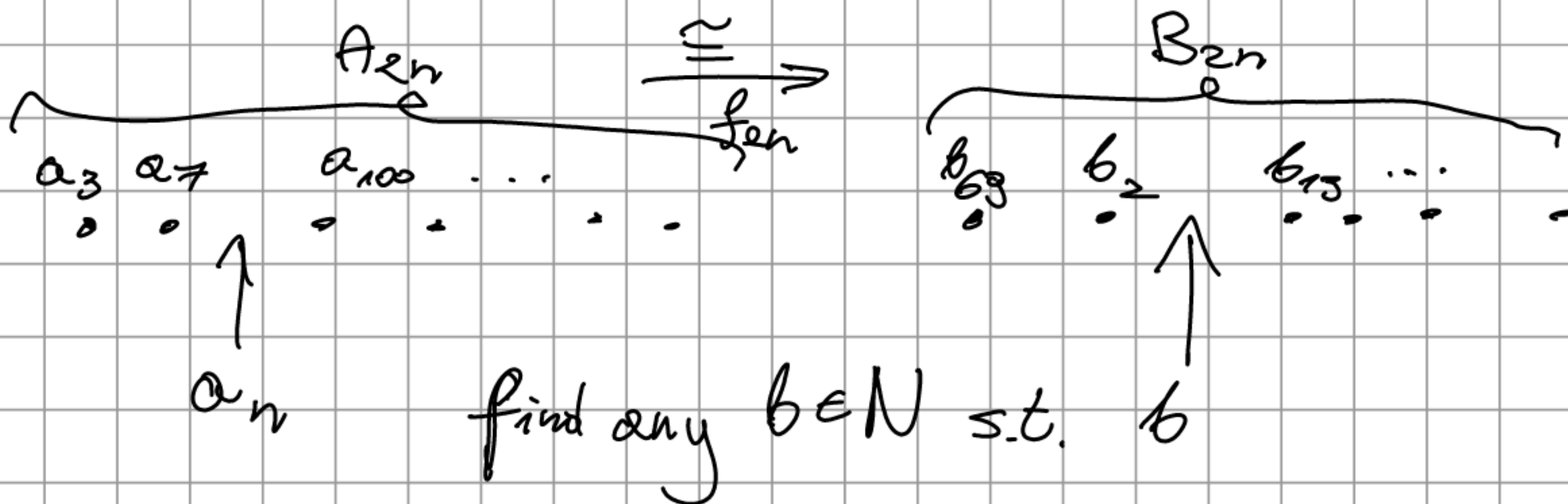
• $f_0 = A_0 = B_0 = \emptyset$

• recursive step: $2n \rightarrow 2n+1 \rightarrow 2n+2$

We have $f_n: A_{2n} \rightarrow B_{2n}$

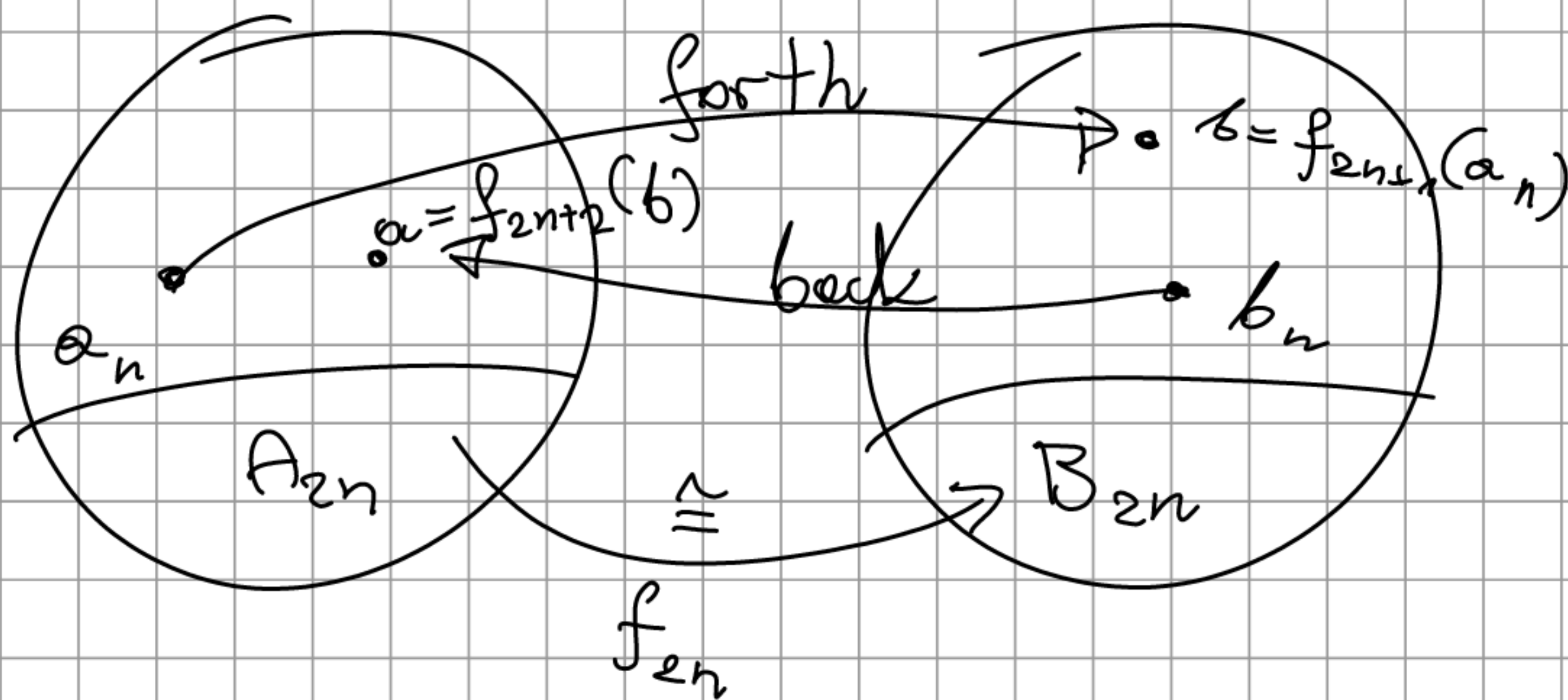
look at a_n . If $a_n \in A_{2n}$, then $f_{2n+1} = f_{2n}$.

Else then $A_{2n+1} = A_{2n} \cup \{a_n\}$. Now:



Now $B_{2n+1} = B_{2n} \cup \{b\}$, $f_{2n+1} = f_{2n} \cup \langle a_n, b \rangle$

Similarly we chose a for b_n and construct f_{2n+2} , A_{2n+2} , B_{2n+2} .



Now $f = \bigcup_n f_n$, by (c) $\text{Dom} f = M$, $\text{Rng} f = N$ and by (b) f is an isomorphism.

Corollary 4.5 DLO_0 is complete

Proof Let $\varphi \in \mathcal{F}_L$ be a sentence. Suppose
(ad absurdum) that $DLO_0 \not\models \varphi$ and $DLO_0 \not\models \neg \varphi$.

Hence, by " $\models \Leftrightarrow \models$ " there is $M, N \models DLO_0$

s.t. $M \models \varphi, N \models \neg \varphi$

\downarrow \downarrow , but $M_0 \models \varphi$ \downarrow
 $N_0 \models \neg \varphi$ \downarrow
(downward) \cong countable \downarrow
L-S

27.10.2021

Henkin's proof of Gödel's model existence thm.

Why Henkin's axiom? Recall:

S - a consistent set of formulas

\cap

S' in $L' = L \cup \{c_n : n \in \mathbb{N}\}$

consistent
complete

Contains $\Pi_n : \exists x \varphi_n(x) \rightarrow \varphi_n(c_{f(n)})$

Enumeration of all formulas in \mathcal{F}_L

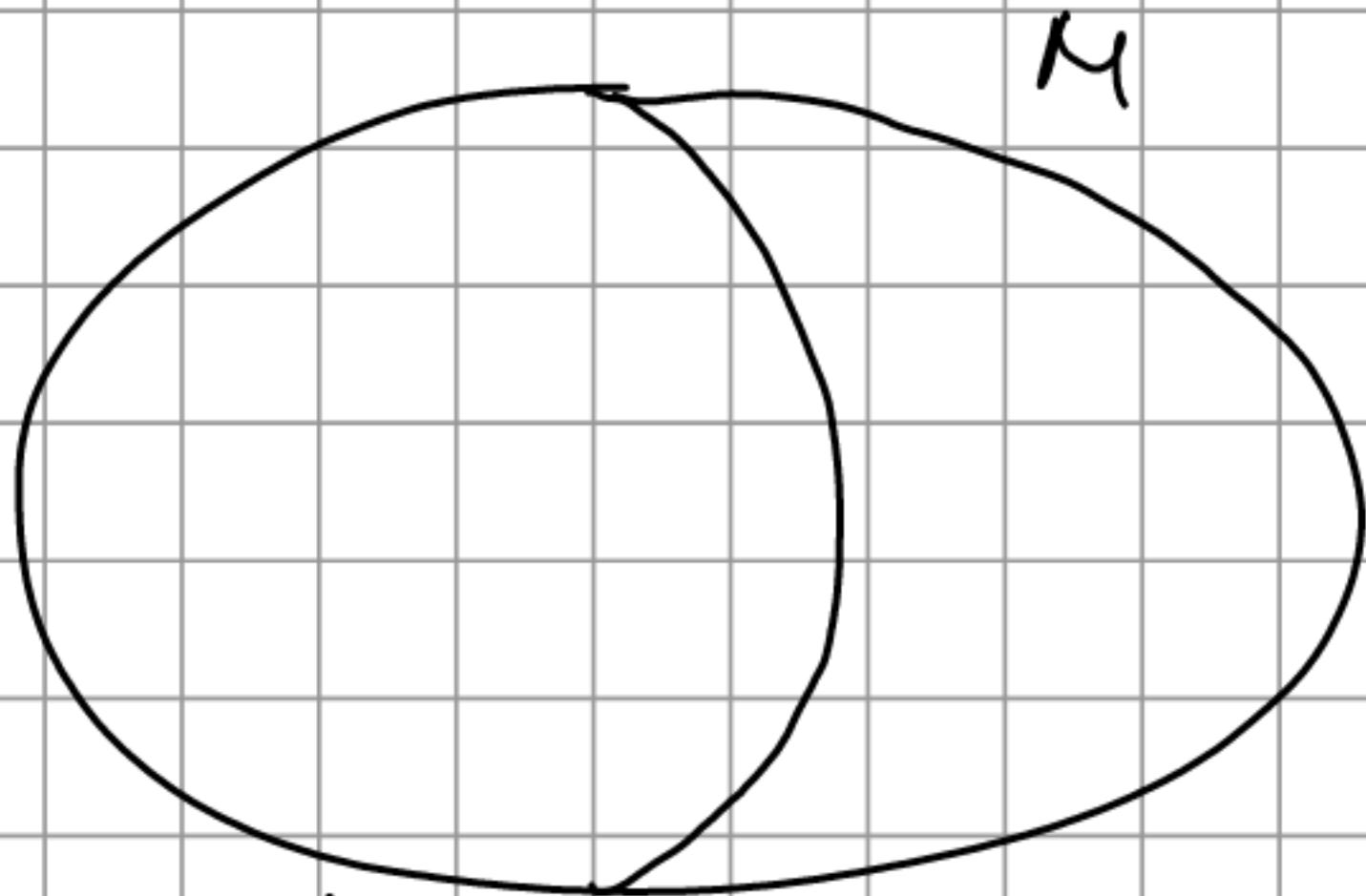
Suppose $M \models S'$

Henkin's axioms + completeness of S' ensures

that N satisfies T-V test

$M \models \exists x \varphi(x, c)$

$\varphi_n(x)$ for some n



$N = \{c_n^M : n < \omega\}$

S' is complete, so

$S' \vdash \exists x \varphi_n(x)$

Also $S' \vdash \forall_n \stackrel{MP}{=} \Rightarrow S' \vdash \varphi_n(c_{f(n)}) \Rightarrow M \models \varphi_n(c_{f(n)})$

So $N \preceq M, N \models S'$

the witness

S' contains the "atomic diagram" of N from N

- $P_i(\bar{c})$ is decided in S'
 - $f_j(\bar{c}) = c'$ — " —
 - $c_n = c_m$ — " —
- this is enough to determine structure of N

DLO_0 : Π_0 -categorical \Rightarrow complete

Corollary 4.6 . DLO_0 is decidable

Proof(a) let $\varphi \in \mathcal{F}_1$: a sentence

Algorithm: generate proofs in $DLO_0: \bar{\alpha}_0, \bar{\alpha}_1, \dots$
(in steps $0, 1, \dots$)

It exists because set of axioms is finite

In step $i, \bar{\alpha}_i$ is generated.

We verify if the conclusion of

$\bar{\alpha}_i$ happens to be φ or $\neg\varphi$.

Countably many

If YES, then we get either $DLO_0 + \varphi$
or $DLO_0 + \neg\varphi$, STOP.

If NOT, proceed to step $i+1$.

There is some proof of φ or $\neg\varphi$, because
 DLO_0 is complete. \blacksquare

Remark 4.7 If T is a complete theory
with recursively enumerable set of
axioms, then T is decidable.

There is an effective way of generating these axioms in a list

Proof (b) (more practical)

Fact 4.8 DLO_0 is quantifier ^(q.e.) eliminable

that is: $\forall \varphi \in \mathcal{F}_L \exists \psi \in \mathcal{F}_L \text{ } DLO_0 + \varphi \leftrightarrow \psi$

with no quantifiers
(quantifier free)

Proof (Fact 4.8) Let $\varphi(\bar{x}) \in \mathcal{F}_L$.

There are finitely many configurations $c_1(\bar{x}), \dots, c_n(\bar{x})$

$C(\bar{x})$: a conjunction of atomic or negated atomic formulas about \bar{x} completely describing ordering $<$ on \bar{x}

$$\text{DLO}_0 \vdash \bigvee_{i=1}^n c_i(\bar{x})$$

exclusive
alternative

$$(\mathbb{Q}, \leq) \models \bigvee_{i=1}^n c_i(\bar{x})$$

Exercise T is complete consistent
theory, $M \models T$, $\varphi \in \mathcal{F}_L$, then
 $T \vdash \varphi \Leftrightarrow M \models \varphi$

$$\text{Also } \forall i \left[\text{DLO}_0 \vdash c_i(\bar{x}) \rightarrow \varphi(\bar{x}) \text{ or } \text{DLO}_0 \vdash c_i(\bar{x}) \rightarrow \neg \varphi(\bar{x}) \right]$$

because: if $\bar{a}, \bar{b} \in \mathbb{Q}$, both satisfying $c(\bar{x})$, then

$$\exists f \in \text{Aut}(\mathbb{Q}, \leq) f(\bar{a}) = \bar{b} \quad \text{fact } \blacksquare$$

$$\text{Let } I = \{ i \in \{1, \dots, n\} : \text{DLO}_0 \vdash c_i(\bar{x}) \rightarrow \varphi(\bar{x}) \}$$

exercise: $\text{DLO}_0 \vdash \varphi(\bar{x}) \Leftrightarrow \bigvee_{i \in I} c_i(\bar{x})$

$$\Downarrow$$

φ : quantifiers free

DLO_0 is decidable

(We transform φ to φ and look
if it holds in (\mathbb{Q}, \leq)) \blacksquare

Examples of theories with q.e.

(1) ACF_p = the theory of algebraically closed fields
of char. $p \geq 0$

(2) RCF = the theory of real closed fields
 $Th(\mathbb{R}, +, \cdot, 0, 1, \leq)$

(3) Many easy theories:

e.g. the theory T of independent unary predicates $P_n(x)$
 $n < \omega$

axioms: $A_{I,J} : \exists x \left(\bigwedge_{n \in I} P_n(x) \wedge \bigwedge_{n \in J} \neg P_n(x) \right)$

$I, J \subseteq \omega$
finite,
disjoint

A model M of T :

$$M = (2^\omega, P_n)_{n < \omega}, \quad P_n(f) \iff f(n) = 0$$

(4) Abelian groups, modules over a fixed ring
They have only reduction of quantifiers
to the level of pp-formulas

positive primitive

$$\exists \bar{y} A \bar{y} = \bar{x}$$

↑
matrix

Def 5.1 An algebra of sets: $\mathcal{C} \subseteq \mathcal{P}(X)$ st.

(1) $\emptyset, X \in \mathcal{C}$

(2) \mathcal{C} closed under Boolean operations
 $\complement, \cup, \cap, \Delta, \dots$

$(\mathcal{C}, \cup, \cap, \complement, \mathbb{0}, \mathbb{1})$: a Boolean algebra
 $\mathbb{0} \stackrel{C}{=} \emptyset, \mathbb{1} \stackrel{C}{=} X$

Generally $L_{BA} = \{ \cup, \cap, ', \mathbb{0}, \mathbb{1} \}$ BA: the theory of Boolean algebras

Axioms of BA

(1) \cup, \cap : associative, commutative, distributive,

(2) $\mathbb{0}' = \mathbb{1}, \mathbb{1}' = \mathbb{0}, x'' = x, (x \cup y)' = x' \cap y'$

$(x \cap y)' = x' \cup y'$

(3) $x \cup (x \cap y) = x, x \cap (x \cup y) = x$

Thm (Stone) Every B. Algebra is \cong an algebra of sets

Let $A = (A, \wedge, \vee, ', \mathbb{0}, \mathbb{1})$ be a BA.

• For $x, y \in A$ $x \leq y \iff x \vee y = y$
 $\uparrow \iff x \wedge y = x$
a partial ordering

- $a \neq \mathbb{0} \in A$ is an atom if $\forall x \in A$ $x \leq a \implies x = \mathbb{0} \vee x = a$
- A is atomless if it has no atoms.

Example the algebra of sets $\mathcal{C} \subseteq \mathcal{P}(\mathbb{I}\mathbb{Q})$ generated by open intervals with rational endpoints is atomless BA.

Def. 5.2 $\emptyset \neq \mathcal{U} \subseteq A$ is a filter if

$$(1) x \leq y \text{ and } x \in \mathcal{U} \implies y \in \mathcal{U}$$

$$(2) x, y \in \mathcal{U} \implies x \wedge y \in \mathcal{U}$$

Def. 5.3 \mathcal{U} is a proper filter if $\mathbb{0} \notin \mathcal{U}$

Def. 5.4 \mathcal{U} is an ultrafilter in A if

it is a maximal proper filter in A

Remark 5.5 Every proper filter extends to an ultrafilter.

Def. 5.6 $S(A) = \{ \text{ultrafilters in } A \}$: a topological space
↑ Stone space of A

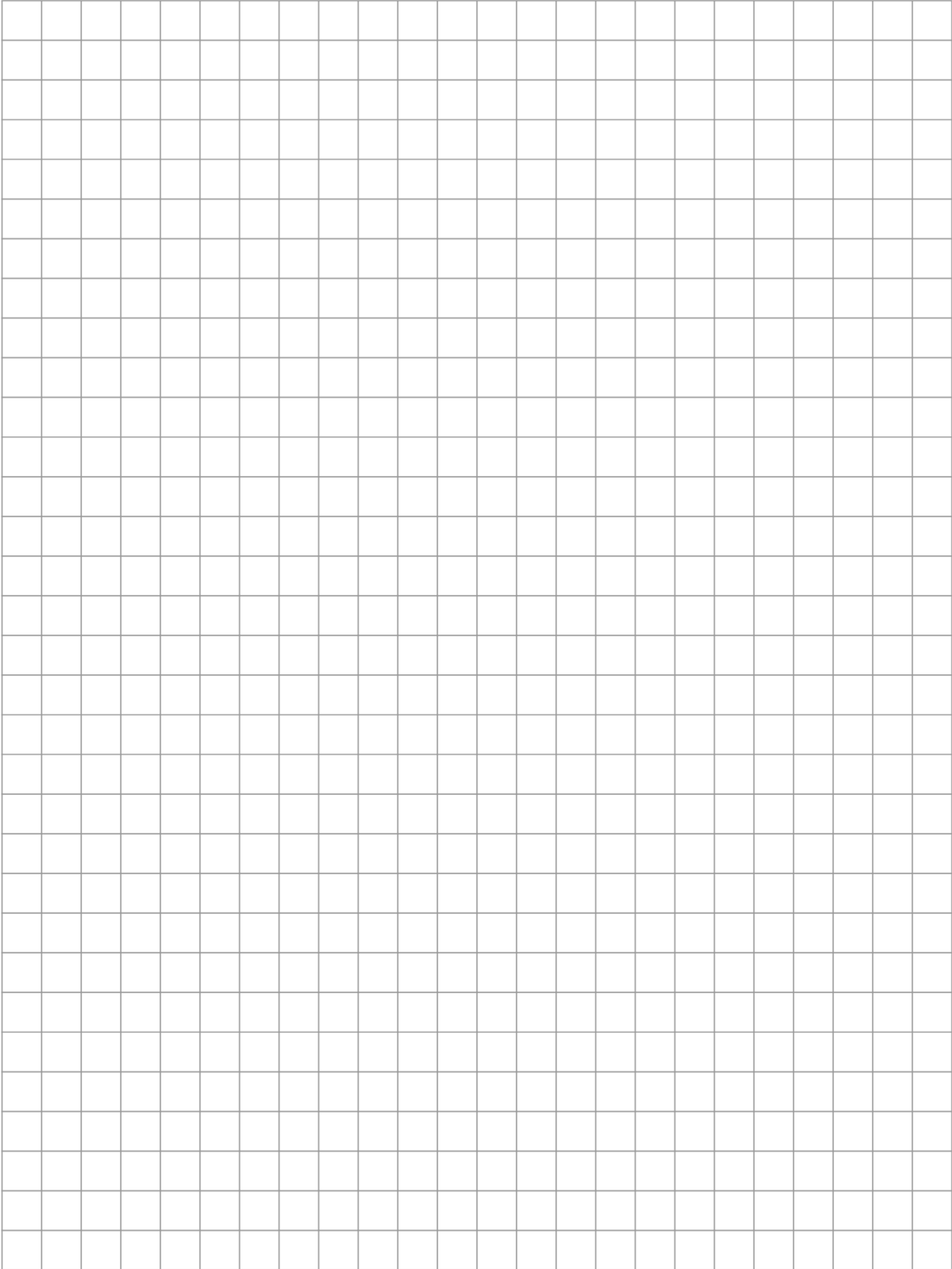
the basis of topology: $[a] = \{ \mathcal{U} \in S(A) : a \in \mathcal{U} \}$
(for $a \in A$) (it is clopen)

Proof of Stone representation thm

$$\mathcal{C} = \{ [a] : a \in A \} \subseteq \mathcal{P}(S(A))$$

$$A \ni a \xrightarrow{\cong} [a]$$





3.11.2021

Corollary BA = the theory of algebras of sets
= $\{ \varphi \in \mathcal{F}_{L_{BA}} : \varphi \text{ holds in every algebra of sets} \}$

Proof: ex.

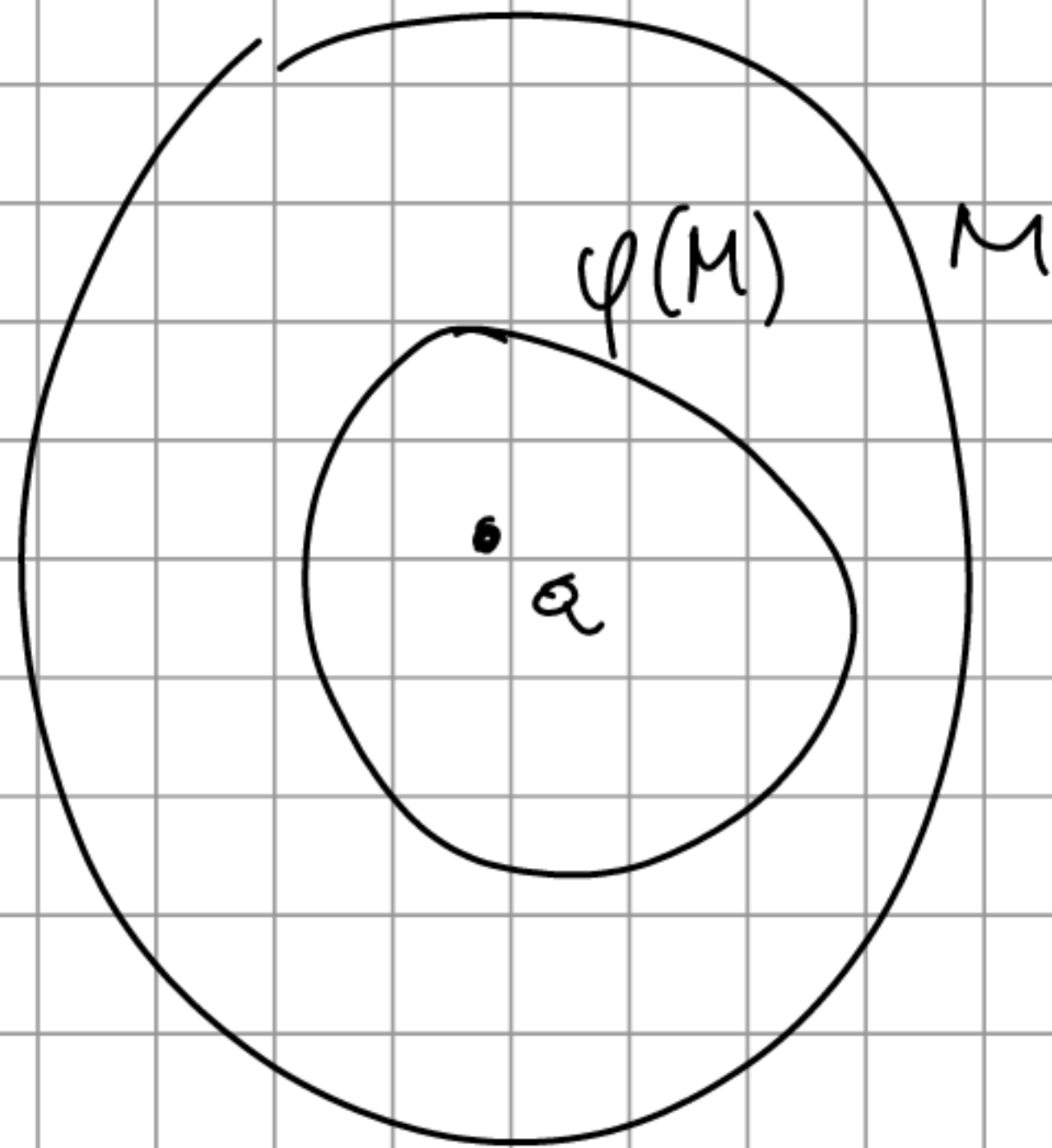
Types, definable sets. L : fixed (contbl), T : complete consistent theory in L .

$M \models T \quad a \in M$

$\varphi(x) \in \mathcal{F}_L$

$\varphi(M) = \{ b \in M : M \models \varphi(b) \}$

↑
definable (sub)set of M



Def. $tp^M(a) = \{ \varphi(x) \in \mathcal{F}_L : M \models \varphi(a) \}$

↑
type of a in M

Def. $\mathcal{F}_L(x) = \{ \text{formulas of } \mathcal{F}_L \text{ of the form } \varphi(x) \}$

Likewise: $\mathcal{F}_L(\bar{x})$

x_1, \dots, x_n

Def. $\varphi(x) \in \mathcal{F}_L(x)$ is T -consistent

$$\begin{array}{c} \Leftrightarrow \\ T \vdash \exists x \varphi(x) \end{array}$$

$$\begin{array}{c} \Leftrightarrow \\ M \models \exists x \varphi(x) \end{array}$$

$$\Leftrightarrow \varphi(M) \neq \emptyset$$

Def. $p(x) \subseteq \mathcal{F}_L(x)$ is T -consistent

$p(x)$ is a type in variable x

$\Leftrightarrow \forall p_0(x) \subseteq p(x)$ (finite $\Leftrightarrow T \vdash \exists x \bigwedge_i p_0(x)_i$ is T -consistent)

$\bigwedge_i p_0(x)_i$ is T -consistent

$\Leftrightarrow T \vdash \exists x \bigwedge_i p_0(x)_i$

a 1-type

Likewise: $p(\bar{x}) \subseteq \mathcal{F}_L(\bar{x})$: a k -type in variables \bar{x}

Def. A type $p(x)$ is complete $\Leftrightarrow p$ is consistent and $\forall \varphi(x) \in \mathcal{F}_L(x)$ ($\varphi \in p(x) \vee \neg \varphi \in p(x)$)

Example $tp^M(a)$ is a complete type.

Remark $a \in M < N \Rightarrow tp^M(a) = tp^N(a)$

A generalization: $A \subseteq |M|$, $L(A) = L \cup \{ \underline{a} : a \in A \}$
a set of parameters ↑
new constant symbols

$T(A) := \text{Th}(M, \underset{a}{\overset{a^M}{\parallel}})_{a \in A}$ - a structure in $L(A)$

A type over A (in T) = a type in $T(A)$

Def. A type $p(x)$ is realized in M by a , if
 $\forall \varphi(x) \in p(x) \quad M \models \varphi(a) \Leftrightarrow p(x) \in \text{tp}^M(a)$

Remark Assume $p(x)$ is a consistent type over A .

Then there is an elementary extension $N \supseteq M$
s.t. $p(x)$ is realized in N .

Proof Let $T' = \text{Th}(M, m)_{m \in M} \cup \{ \varphi(c) : \varphi \in p(x) \}$
↑
a new constant symbol

T' is consistent (compactness theorem)

•
•
•

Corollary Every consistent type extends to a complete type.

Proof $p(x)$ — $p(x)$ is realised in $N \succ M$
over A in M by c^N c^N

$$\Rightarrow p(x) \subseteq \underbrace{tp^N(c^N/A)}_{\text{a complete type over } A} := tp^N(c^N) \text{ in } (N, a)_{a \in N}$$

in $T = \{ \varphi(x) \in \mathcal{F}_{L(A)}(x) : M \models \varphi(a) \}$

Def. $\text{Def}(M) = \{ \text{definible subsets of } M \}$

Remark

(1) $\text{Def}(M)$ is BA over subsets of M

(2) For every $a \in M$ $\{ \varphi(M) : \varphi(x) \in tp^M(a) \}$ is an ultrafilter in $\text{Def}(M)$

\sim on $\mathcal{F}_L(x) : \varphi \sim \psi \Leftrightarrow T \vdash \varphi(x) \leftrightarrow \psi(x)$

$$\Leftrightarrow \varphi(M) = \psi(M)$$

Proof " \Rightarrow " is clear. " \Leftarrow " (ad absurdum)

suppose $T \not\vdash \varphi(x) \leftrightarrow \psi(x) \Leftrightarrow T \not\vdash \forall x (\varphi(x) \leftrightarrow \psi(x))$

$\Leftrightarrow T \not\vdash \exists x (\varphi(x) \Delta \psi(x)) \Leftrightarrow M \models \exists x (\varphi(x) \Delta \psi(x)) \quad \downarrow$
 T is complete

$$\underline{L_n(\emptyset)} = (\mathbb{F}_L(x)/\sim, \wedge, \vee, ', \mathbb{0}, \mathbb{1})$$

Lindenbaum algebra

$$\bullet [\varphi]_n \wedge [\psi]_n = [\varphi \wedge \psi]_n$$

$$\bullet [\varphi]_n \vee [\psi]_n = [\varphi \vee \psi]_n$$

$$\bullet [\varphi]_n' = [\neg \varphi]_n$$

$$\bullet \mathbb{0} = [x \neq x]_n$$

$$\bullet \mathbb{1} = [x = x]_n$$

Remark

$$(1) [\varphi(x)]_n \xrightarrow{F} \varphi(M)$$

$$\text{gives } F: L_n(\emptyset) \xrightarrow{\cong} \text{Def}(M)$$

(2) $\text{tp}^M(a)/\sim$: an ultrafilter in $L_n(\emptyset)$

(3) $p(x)/\sim$ is an ultrafilter in $L_n(\emptyset)$ for every complete type $p(x)$ in T .

(4) $\{ \text{complete types in } T, \text{ in } x \} \xrightleftharpoons[\text{onto}]{1-1} S(L_n(\emptyset))$

$\{ \text{consistent types in } T, \text{ in } x \} \leftrightarrow \text{filters in } L_n(\emptyset)$

Ad 4 Assume that $p(x)$ is a complete type in T . Then $p(x)$ is closed under \sim

$$\Rightarrow p(x)/\sim \in S(L_1(\emptyset))$$

Vice versa, if $U \in S(L_1(\emptyset))$, then

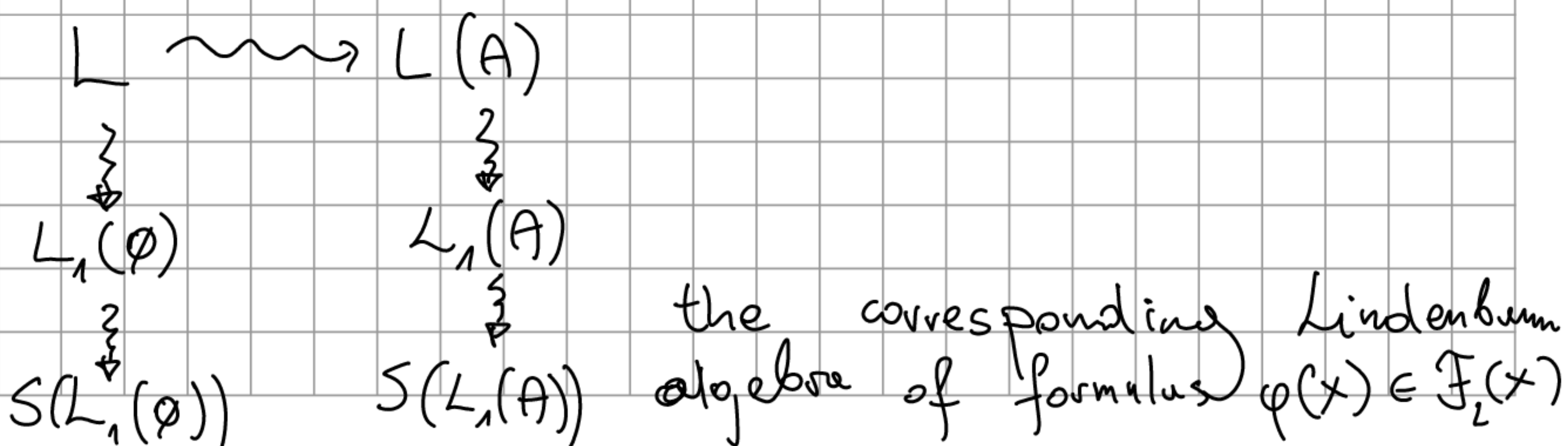
$p_U = \{ \varphi(x) \in \mathcal{F}_L(x) : [\varphi]_{\sim} \in U \}$ is a complete consistent theory in T , in X .

For $A \subseteq M \rightsquigarrow \text{Def}_A(M) = \{ \varphi(M) : \varphi(x) \in \mathcal{F}_{L(A)} \}$

$\varphi(\bar{x}, \bar{a}) \in \mathcal{F}_{L(A)}(\bar{x}) \rightsquigarrow \varphi(M) = \{ \bar{b} \in M^k : M \models \varphi(\bar{b}, \bar{a}) \}$

where $\varphi(\bar{x}, \bar{y}) \in \mathcal{F}_L$ } set definible
in M over parameters A
[A -definible in M]

$\text{Def}_A(M)$ - the algebra of A -definible subsets of M .



$S_1(A)$ = the space of all complete 1-types
over A in T .

$$S_1(A) \longleftrightarrow S(L_1(A))$$

Stone space topology

Recall assume $A : BA$

$$\downarrow$$
$$S(A)$$

$$A \ni a \rightsquigarrow [a] = \{ \mu \in S(A) : a \in \mu \}$$

$\{ [a] : a \in A \}$ is a basis of Stone

topology in $S(A)$. It is compact,

Hausdorff, 0-dimensional, $[a]$ -clopen

Topology on $S_1(A)$:

Basic open sets: $[\underset{\uparrow}{\varphi}] = \{ p(x) \in S_1(A) : \varphi(x, \bar{a}) \in p(x) \}$

$\mathcal{F}_{L(A)}(x)$

Likewise: $L_k(A)$, $S_k(A)$ k -types over A in T
in variables \bar{x} .

Def Let κ be any cardinal number. We say
that M is κ -saturated if $(\forall A \subseteq M)$
 $|A| < \kappa$
 $(\forall p \in S_n(A))$ p is realised in M .

M is saturated if it is $|M|$ -saturated

Corollary (1) $\forall M \forall \kappa \exists N \succ M$ N is κ -saturated

(2) If κ is regular and $2^{<\kappa} = \kappa$ $\left| \begin{array}{l} 2^{<\kappa} = \sup_{\mu < \kappa} 2^\mu \\ \kappa = \kappa \end{array} \right.$

(I, \leq) a linear ordering.

$\text{cf}(I) = \min \{ |J| : J \subseteq I \}$
cofinality cofinal
unbounded

$\kappa \in \text{Ord}$ $\kappa = \{ d \in \text{Ord} : d < \kappa \}$

\uparrow
 $\mathbb{C}\mathbb{N}$

$(\kappa, <)$ well-ordered

Lecture 10. 11. 2021

Recall: $\kappa \in \mathbb{C}N, \kappa \geq \aleph_0, L: \text{dble}, T: \text{complete consistent theory in } L$

Def.

(1) $M \models \kappa$ -saturated, if $\forall A \subseteq M \forall p \in S_1(A) \quad p$ is realized in M
 $|A| < \kappa$

(2) M is saturated if M is $|M|$ -saturated.

Corollary (1) $\forall \kappa \forall M \in \mathcal{N} \forall N \in \mathcal{N} \kappa < N \leq M \implies N$ is κ -saturated

(2) If $\kappa > \aleph_0$ is regular and $2^{<\kappa} = \kappa$, then $\forall M \in \mathcal{N} \forall N \in \mathcal{N} \kappa < N \leq M \implies N$ is κ -saturated

$\downarrow \leftarrow \text{exercise}$
 $(\kappa < \kappa = \kappa)$

$(|N| = \kappa \text{ and } N \text{ is saturated})$

Proof

$S_1(A)$

Remark. For $A \subseteq M, |S(A)| \leq 2^{|A| + \aleph_0}$

Proof. $|S(A)| \leq |P(L_1(A))| \leq 2^{|L_1(A)|}, |L_1(A)| = |A| + \aleph_0$

Idea of the proof of (1).

Let $\mu = 2^\kappa$. Then $\text{cf}(\mu) > \kappa$

We construct a chain of models $N_\alpha, \alpha < \mu$, of power $\leq \mu$ such that: elementary each

(1) $M = N_0 \prec N_1 \prec \dots \prec N_\alpha \prec N_\beta \prec \dots$ for $\alpha < \beta < \mu$

Recursively:

(2) At step $\alpha = \beta + 1$: we have a model N_β .

We choose $N_\alpha \succ N_\beta$ so that:

$(\forall B \subseteq N_\beta) \forall f \in S_1(B) \quad p$ is realized in N_α .
 $|B| < \kappa$

and $\|N_\alpha\| \leq \mu$.

(2)

- there are $\leq \mu$ types p to consider
- there is a model $N' \supset N_\beta$ realizing all of them
- by downward Löwenheim-Skolem, can assume $\|N'\| \leq \mu$.

(3) Limit step: assume $\alpha \in \text{Lim}$ (a limit ordinal) and N_β already are chosen for all $\beta < \alpha$.

Then $N_\alpha = \bigcup_{\beta < \alpha} N_\beta$.

Then let $N = \bigcup_{\alpha < \mu} N_\alpha$

• $M = N_0 \prec N$

• N is κ -saturated.

Proof. Assume $A \subseteq N$, $|A| < \kappa$, $p \in S_1^N(A)$

$\text{cf}(\mu) > \kappa \Rightarrow (\exists \alpha < \mu) A \subseteq N_\alpha$

because: Say $A = \{a_\gamma : \gamma < \nu\}$, where $|A| = \nu < \kappa$.

For ~~$\forall \gamma < \nu$~~ $\gamma < \nu$ let $\alpha_\gamma < \mu$ s.t. $a_\gamma \in N_{\alpha_\gamma}$.

~~$\{\alpha_\gamma : \gamma < \nu\} \subseteq \mu$~~

$\left. \begin{array}{l} \{\alpha_\gamma : \gamma < \nu\} \subseteq \mu \\ \text{of power } \leq \nu < \kappa < \text{cf}(\mu) \end{array} \right\} \Rightarrow \exists \alpha < \mu$

$\{\alpha_\gamma : \gamma < \nu\} \subseteq \alpha$.

that is: $(\forall \gamma < \nu) a_\gamma \in N_{\alpha_\gamma} \prec N_\alpha$
 $a_\gamma \in N_\alpha$

So: $A \subseteq N_\alpha$.

So: $A \subseteq N_\alpha \prec N \Rightarrow S_1^N(A) = S_1^{N_\alpha}(A)$

So $p \in S_1^{N_\alpha}(A)$,

By the successor step: p is realized in $N_{\alpha+1}$
by some c .

$N_{\alpha+1} \prec N$, hence c realizes p also in N . \square

(2) A similar proof.

Auxiliary lemma on \prec :

(1) \prec is transitive [exercise]

(2) If γ is a limit ordinal and $(N_\alpha : \alpha < \gamma)$ is
an elementary chain of structures, then there

[i.e. for $\alpha < \beta < \gamma$ $N_\alpha \prec N_\beta$]

is a structure N_γ [called ~~the~~ union of the chain]

s.t.

$$(a) |N_\gamma| = \bigcup_{\alpha < \gamma} |N_\alpha|$$

(b) $N_\alpha \prec N_\gamma$ for all $\alpha < \gamma$. [Tarski]

Proof

Definition of N_γ : $|N_\gamma| = \bigcup_{\alpha < \gamma} |N_\alpha|$

• Interpretations of L -symbols in N_γ :

(i) P : relational symbol:

let $a_1, \dots, a_n \in |N_\gamma|$.

$P^{N_\gamma}(a_1, \dots, a_n) \Leftrightarrow P^{N_\alpha}(a_1, \dots, a_n)$ holds for

some $[\equiv \text{every}] \alpha < \gamma$ with $a_1, \dots, a_n \in |N_\alpha|$ (4)

[does not depend on the choice of α ,
because $(N_\alpha)_{\alpha < \gamma}$: elementary].

• f : a function symbol of L :

$$a_1, \dots, a_n, a_{n+1} \in |N_\gamma|$$

$$f^{N_\gamma}(a_1, \dots, a_n) = a_{n+1} \Leftrightarrow \text{for some } \alpha < \gamma \text{ with}$$

every $a_1, \dots, a_{n+1} \in |N_\alpha|$

$$f^{N_\alpha}(a_1, \dots, a_n) = a_{n+1}$$

• c : a constant symbol of L .

$$c^{N_\gamma} = c^{N_\alpha} \text{ for every } \alpha < \gamma.$$

(b) : $N_\alpha < N_\gamma$ for every $\alpha < \gamma$

Inductive statement:

$$(\underbrace{\forall \varphi \in \mathcal{F}_L}_{\varphi(\vec{x})} \forall \alpha < \gamma \forall \vec{a} \subseteq N_\alpha [N_\alpha \models \varphi(\vec{a}) \Leftrightarrow N_\gamma \models \varphi(\vec{a})])$$

(*)

Proof by induction on $|\varphi|$:

1. φ quantifier-free.

Then (*) true, because for every $\alpha < \gamma$, $N_\alpha \subseteq N_\gamma$
substructure.

2. Induction step:

Assume $\varphi(\bar{x}) = \exists y \psi(\bar{x}, y)$.

shorter, so (*) holds for ψ .

$$N_\alpha \models \exists y \psi(\bar{a}, y) \Leftrightarrow N_\alpha \models \psi(\bar{a}, b) \text{ for some } b \in N_\alpha$$

↓ induction hypothesis

$$N_\gamma \models \psi(\bar{a}, b)$$

↓

$$N_\gamma \models \exists y \psi(\bar{a}, y)$$

$$N_\gamma \models \exists y \psi(\bar{a}, y) \Leftrightarrow N_\gamma \models \psi(\bar{a}, b) \text{ for some } b \in N_\gamma$$

Choose $\beta < \gamma$ s.t. $b \in N_\beta$. ↓ induction assumption

$$N_\beta \models \psi(\bar{a}, b)$$

↓

$$N_\beta \models \exists y \psi(\bar{a}, y)$$

↓ $N_\alpha < N_\beta$

$$N_\alpha \models \exists y \psi(\bar{a}, y)$$

Example.

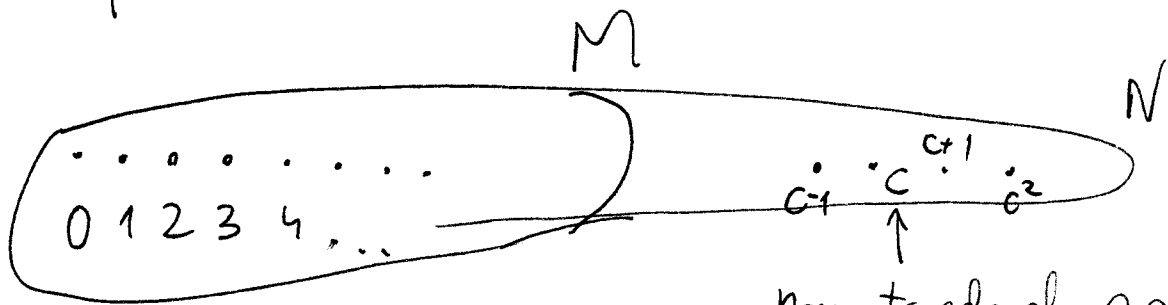
$$M = (\mathbb{N}, \overset{L}{+}, \cdot, 0, 1, <) \quad TA = Th(M) \text{ true arithmetic.}$$

For $n \in \mathbb{N}$ let $\underline{n} = \underbrace{1 + \dots + 1}_n$ (an L-term) not a number
(numeral n)

Let $p(x) = \{x > \underline{n} : n \in \mathbb{N}\}$. : a consistent type in TA.

let $N \supset M$ countable s.t. p is realized in N , by some $c \in N$.

$$N \succ M$$



non-standard natural number, infinitely large

$$M \models (\forall x \neq 0) \exists y \ y+1 = x$$

\Downarrow

~~$N \models$ "there is a predecessor of c "~~

$N \models$ "there is a predecessor of c "

also infinitely large

Similarly: $c < c+1 < c+2 < \dots < c^2 \dots$

Example 2. Non-standard analysis.

$$\mathcal{R} = (\mathbb{R}, \overset{L}{+}, \cdot, 0, 1, f, r, R)$$

f : all functions: $\mathbb{R}^k \rightarrow \mathbb{R}$

R : all relations $\subseteq \mathbb{R}^m$

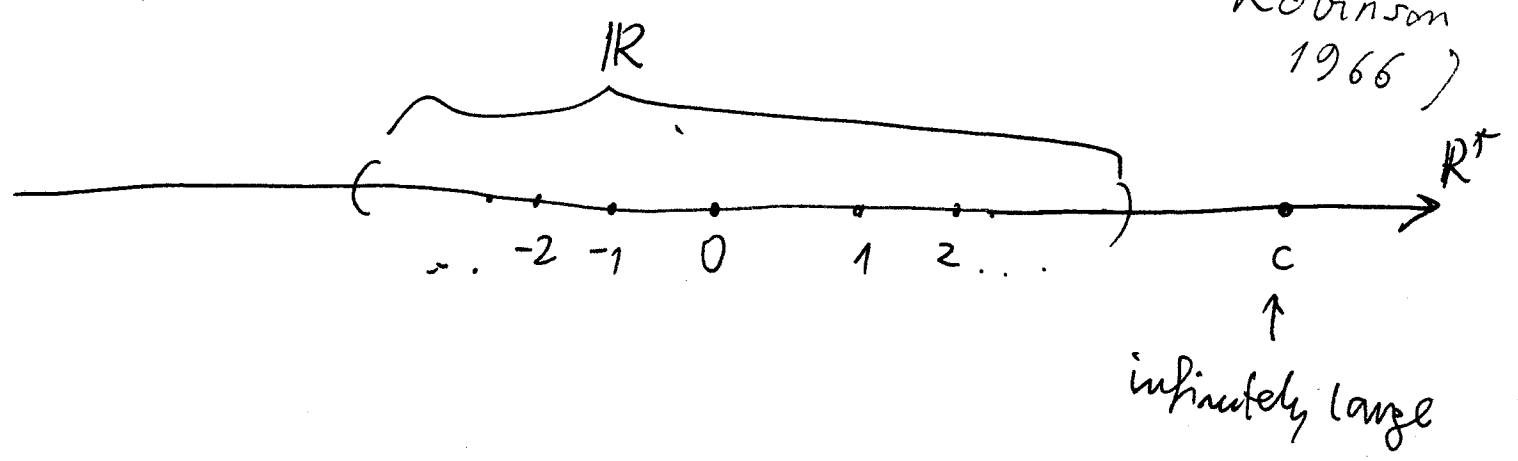
r : all reals

analysis

$$A = Th(\mathcal{R})$$

Let $M \models A$ $\overset{\mathbb{R}^+}{\mathbb{R}}$ -saturated, wlog $\mathbb{R} < M$

analysis in M : "non-standard analysis" (Abraham Robinson 1966)



- $\frac{1}{c} \in \mathbb{R}^*$: in \mathbb{R}^* :
 - positive, $\neq 0$
 - $< \frac{1}{n}$ for every $n \in \mathbb{N}$] infinitesimal

Omitting types

Assume T : a complete theory, $M \models T$, $p(x)$ a ^{consistent} type in T (over \emptyset)

Def M omits $p \Leftrightarrow \neg \exists a \in M$ a realizes p

Def $p(x)$ is isolated, if $\exists \varphi(x) \in \widehat{T}_L$

$$T \vdash \exists x \varphi(x)$$

and

Notation: $\varphi(x) \vdash p(x) \overset{\text{not}}{\rightsquigarrow} [T \vdash \varphi(x) \rightarrow \psi(x) \text{ for every } \psi(x) \in p(x)]$

[T in the background]

Remark (1). Assume $p(x)$ is a complete type.

Then p is isolated $\Leftrightarrow \{p\}$ is open in $S(\emptyset)$
[i.e. p is isolated in the topological sense].

(2) p is ~~is~~ isolated \Leftrightarrow the set $\{\varphi(x) \in S(\emptyset) : p(x) \subseteq \varphi(x)\}$ has nonempty interior in $S(\emptyset)$
[Stone topology]

Proof (1). $\varphi \vdash p \Leftrightarrow [\varphi] = \{p\}$ in $S(\emptyset)$ ~~for~~
(If $T \vdash \exists x \varphi(x)$)
(2): exercise.

Remark If $p(x)$ is isolated, then $\forall M \models T$ p is realized in M [i.e. $p(x)$ can not be omitted]. (8)

PF. $\varphi(x) \vdash p(x)$ $M \models T$ $M \models \exists x \varphi(x)$
 \Leftarrow
 C realizes $p(x)$. C ~~is the~~ witness.

Thm (A. Ehrenfeucht)

If T is countable, complete, consistent, ~~is~~
 $p(x)$ is non-isolated, then $\exists M \models T$ M omits p .

Proof (similar to Henkin)

Let $\{c_n : n \in \omega\}$: a set of new constant symbols,
 $L' = L \cup \{c_n\}$.

$\{\varphi_n(x) : n < \omega\}$: enumeration of $\mathcal{F}_{L'}(x)$.

$h: \omega \rightarrow \omega$ increasing, s.t. $c_{h(n)}$ does not appear in $\varphi_0, \dots, \varphi_n$.

Let $H_i = \{\exists x \varphi_i \rightarrow \varphi_i(c_{h(i)})\}$: Henkin's axiom

We define an increasing sequence of consistent

sets $T = T_0 \subseteq T_1 \subseteq \dots \subseteq \mathcal{F}_{L'}$ s.t.

(a) $T_{2i+1} = T_{2i} \cup \{H_i\}$

(b) $T_{2i+2} = T_{2i+1} \cup \{\neg \psi(c_i)\}$ for some $\psi(x) \in p(x)$.

Recursive construction:

• Suppose we have T_{2i} consistent.

then $T_{2i+1} = T_{2i} \cup \{H_i\}$ consistent (as in Henkin's proof)

• Assume $T_{2in} = T \cup \{\psi_j(\bar{c}, c_i) : j < k\}$

new constant symbols from among $c_n, n < \omega$

Suppose we can not pick $\psi(x) \in p(x)$ so that

$T_{2i+1} \cup \{\neg \psi(c_i)\}$ is consistent.

Hence $T_{2i+1} \vdash \psi(c_i)$ for every $\psi(x) \in p(x)$

Let $\Psi = \bigwedge_{j < k} \psi_j(\bar{c}, c_i)$

by deduction thm: $T \vdash \Psi(\bar{c}, c_i) \rightarrow \psi(c_i)$ for every $\psi(x) \in p(x)$

\bar{c}, c_i : new constant symbols

~~$T \vdash \Psi(\bar{y}, x) \rightarrow \psi(x)$~~

$T \vdash \Psi(\bar{y}, x) \rightarrow \psi(x)$ [if needed:

But: $\vDash [\Psi(\bar{y}, x) \rightarrow \psi(x)] \rightarrow (\exists \bar{y} \Psi(\bar{y}, x) \rightarrow \psi(x))$ \bar{y}, x : new variables]

So: $T \vdash (\exists \bar{y} \Psi(\bar{y}, x)) \rightarrow \psi(x)$ for every $\psi(x) \in p(x)$

But: $T \vdash \exists x (\exists \bar{y} \Psi(\bar{y}, x))$ [as: $T \cup \{\Psi(\bar{c}, c_i)\}$ consistent]

hence: the formula

$\exists \bar{y} \Psi(\bar{y}, x)$ isolates $p(x)$

Therefore can find T_{2i+2} !

10

Now as in Henkin's proof

let $T_\infty = \bigcup_i T_i$ and $S \supseteq T_\infty$

↑
consistent complete theory.

We construct a model M of S from constant symbols $c_n, n < \omega$.

By case ~~A~~ (b) ($T_{2i+1} \rightsquigarrow T_{2i+2}$):

$M \models \neg \psi(c_i)$ for some $\psi \in p$, so
no c_i realizes $p(x)$ and M omits p .

Remark.

It is much harder to omit types in uncountable theories.

Def. ($\kappa \in \mathbb{N}, \kappa \geq \aleph_0$)

T is κ -categorical, if $(\forall M, N \models T) (\|M\| = \|N\| = \kappa \Rightarrow$
[categorical in κ] $M \cong N)$

Corollary (Ryll-Nardzewski, Svenonius, Engeler) \square .

(1) T is \aleph_0 -categorical

(2) $\forall n, S_n(\emptyset)$ is finite

Proof (1) \Rightarrow (2) Suppose for some $n, S_n(\emptyset)$ is infinite.

Hence there is a non-isolated $p(\bar{x}) \in S_n(\emptyset)$.

omitting types thm $\Rightarrow \exists M_1 \models T$ omitting $p(\bar{x})$

realizing types thm $\Rightarrow \exists M_2 \models T$ realizing $p(\bar{x})$

$M_1 \not\cong M_2 \quad \downarrow$

(2) ⇒ (1)

Suppose $M = \{a_n : n < \omega\}$, $N = \{b_n : n < \omega\}$ countable models of T .
We will show: $M \cong N$.

We construct a sequence of finite functions

$$\emptyset = f_{-1} \subseteq f_0 \subseteq f_1 \subseteq \dots \subseteq f_n \subseteq \dots \quad n < \omega \text{ s.t.}$$

(a) $\text{Dom } f_i \subseteq M, \text{ Rng } f_i \subseteq N$

(b) $a_i \in \text{Dom } f_i, b_i \in \text{Rng } f_i$

(c) f_i is elementary, i.e. $\text{tp}^M(d_1, \dots, d_k) = \text{tp}^N(f(d_1), \dots, f(d_k))$,
where $\text{Dom } f_i = \{d_1, \dots, d_k\}$.

Recursively:

¶ Suppose we have f_i . We will find f_{i+1} ($i \geq -1$)

Step 1 (forth) ¶ $a_{i+1} \hookrightarrow \text{Dom } f_{i+1}$:

If $a_{i+1} \in \text{Dom } f_i$, then do nothing

If $a_{i+1} \notin \text{Dom } f_i$, then let

$$p = \text{tp}^M(\langle d_1, \dots, d_k, a_i \rangle)$$

$$p \in S_{k+1}(\emptyset) \leftarrow \text{finite} \Rightarrow p \text{ isolated} :$$

$$\varphi(x_1, \dots, x_k, y) \vdash p(x_1, \dots, x_k, y)$$

$$\text{so } \exists y \varphi(x_1, \dots, x_k, y) \in \text{tp}^M(d_1, \dots, d_k).$$

$$N \models \exists y \varphi(\cancel{d_1}, \dots, \cancel{d_k}, y) \ll \text{tp}^N(f_i(d_1), \dots, f_i(d_k))$$

Let b be a witness: $N \models \varphi(d_1, \dots, d_k, b)$.

$$N \models \varphi(f_i(d_1), \dots, f_i(d_k), b)$$

φ isolates type in $S_{k+1}(\emptyset)$

$$\text{so } \underbrace{tp^M(d_1, \dots, d_k, a_i)}_{\downarrow \varphi} = \underbrace{tp^N(\underbrace{f_i(d_1), \dots, f_i(d_k), b}_{\downarrow \varphi})}_{\downarrow \varphi}$$

Let $f' = f_i \cup \{ \langle a_i, b \rangle \}$. f' elementary.

Step 2. Replace the roles of M, N .

(back) Find $a \in M$ s.t. $f_{i+1} = f' \cup \{ \langle a, b_i \rangle \}$ elementary.

Let $f = \bigcup_i f_i$ $f: M \xrightarrow{\equiv} N$, $\text{Dom } f = M$
 (by (c)) $\text{Rng } f = N \Rightarrow f: M \xrightarrow{\cong} N$

17.11.2021

Assume M : a dbl structure $\Rightarrow \text{Aut}(M)$: the group of automorphisms of M .

$$\text{Aut}(M) \curvearrowright M : \underset{\uparrow}{\sigma} \cdot \underset{\uparrow}{a} = \sigma(a)$$

$\text{Aut}(M) \quad M$

also on M^n :

$$\sigma \cdot \langle a_1, \dots, a_n \rangle = \langle \sigma(a_1), \dots, \sigma(a_n) \rangle$$

Corollary (Ryll-Nardzewski) $\text{Th}(M)$ is Π_0^1 -categorical iff $\forall n$ on M^n there are only finitely many orbits of $\text{Aut}(M)$.

Proof " \Rightarrow " Fix n . $S_n(\emptyset) = \underbrace{\{p_1, \dots, p_k\}}_{\text{all isolated}}$.

Let $\varphi_i(\bar{x}) \vdash p_i(\bar{x})$, $i=1, \dots, k$

Clearly: $\bigcup \underbrace{[\varphi_i]}_{\text{clopen}} = S_n(\emptyset) \Rightarrow$ in $S_n(\emptyset)$

$$[\varphi_1 \vee \varphi_2 \vee \dots \vee \varphi_k] \sim \perp$$

Hence $T \vdash \varphi_1(\bar{x}) \vee \dots \vee \varphi_k(\bar{x})$

More: $[\varphi_i] \cap [\varphi_j] = \emptyset$ for $i \neq j$, so

$T \vdash \varphi_1(\bar{x}) \dot{\vee} \varphi_2(\bar{x}) \dot{\vee} \dots \dot{\vee} \varphi_k(\bar{x})$ $\{ \bar{a} \in M^n : M \models \varphi_k(\bar{a}) \}$

Hence $M^n = \varphi_1(M) \dot{\cup} \dots \dot{\cup} \varphi_k(M)$

i.e. $T \vdash \forall \bar{x} \left(\bigvee_i \varphi_i(\bar{x}) \wedge \bigwedge_{1 \leq i < j \leq k} \neg (\varphi_i(\bar{x}) \wedge \varphi_j(\bar{x})) \right)$

• $\varphi_1(M), \dots, \varphi_k(M)$: orbits of the $\text{Aut}(M)$ on M^n :

- let $\bar{a} \in \varphi_i(M), \bar{b} \in \varphi_j(M)$. If $i \neq j$ then

$\neg \exists \sigma \in \text{Aut}(M) \sigma(\bar{a}) = \bar{b}$.

If $i = j$, then $\exists \sigma \dots$ because

then $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$

\downarrow

construct σ

(back-and-forth arg)

" \Leftarrow " Fix n . Notice that $\mathcal{L}_n(\emptyset)$ finite.

If not, then there are pair-wise T -contradictory consistent
theory formulas $\varphi_i(\bar{x}), i < \omega$

$\mathcal{F}_L(\bar{x})$

This gives infinitely many orbits on M^n .

$\varphi_i(M)$: a union of orbits. \Downarrow

$(\bigwedge_n L_n(\emptyset) \text{ finite} \Rightarrow S_n(\emptyset) \text{ is finite}) \Rightarrow T \text{ is } \aleph_0\text{-categorical}$

Remark Assume $M = \{a_n : n < \omega\}$, $N = \{b_n : n < \omega\}$

: L -structures. s.t. $\forall n \text{tp}^M(a_0, \dots, a_{n-1}) = \text{tp}^N(b_0, \dots, b_{n-1})$

Then $M \cong N$
 $a_i \mapsto b_i$

"Another" proof of Gödel's model existence thm:

T : consistent $\Rightarrow T$ has a model
complete

We construct sequences:

- 1) of types $p_n(x)$
- 2) of new constant symbols c_n
- 3) of theories T_n s.t.:

$$a) T = T_0 \subseteq T_1 \subseteq \dots$$

$$b) p_n(x) \in S_1^{T_n}(\emptyset)$$

$$c) T_{n+1} = \text{Cn} \left(T_n \cup \{ \varphi(c_n) : \varphi(x) \in p_n(x) \} \right)$$

in language $L_{n+1} = L \cup \{ c_i : i \leq n \}$

T_n is complete and consistent: (ind. on n)

- T_0 is from assumptions

- consistency: $p_n(x)$ compl. type in T_n , so
for $p' \subseteq p_n$
finite

$$T_n \vdash \exists x \bigwedge p'(x) \Rightarrow T_n \cup \{ \bigwedge p'(c_{n+1}) \}$$

consistent

- completeness

Let $\sigma(c_n)$: a sentence in L_{n+1}

$$1^\circ \sigma(x) \in p_n(x) \text{ or } (\neg \sigma(x)) \in p_n(x)$$

in L_n \Downarrow \Downarrow

$$T_{n+1} \vdash \sigma(c_n)$$

$$T_{n+1} \vdash \neg \sigma(c_n)$$

Let $T' = \bigcup_n T_n$: consistent, complete in
 $\mathcal{L}' = \mathcal{L} \cup \{c_n : n < \omega\}$. We can ensure in
the construction that $\forall \varphi(x) \in \mathcal{F}_{\mathcal{L}'}$ (if $T' \vdash \exists x \varphi(x)$,
then $T' \vdash \varphi(c_n)$ for some n)
 $\varphi(x) \iff p_n(x)$

(how? choose $\varphi(x) \in \mathcal{F}_{\mathcal{L}'}$ s.t. $T' \vdash \exists x \varphi(x)$.

Then $\varphi \in \mathcal{F}_{\mathcal{L}_k}$ for some k , $T_k \vdash \exists x \varphi(x)$

for some $n \geq k$ we ensure $p_n(x) \equiv \varphi(x)$

$\implies T_{n+1} \vdash \varphi(c_n) \implies T' \vdash \varphi(c_n)$

Therefore $\{c_n : n < \omega\}$ form a model of T

Problem of model theory:

- how to: construct models?
- how to: describe models of T ?

Stability hierarchy of theories: Assume T complete, consistent with infinite models.

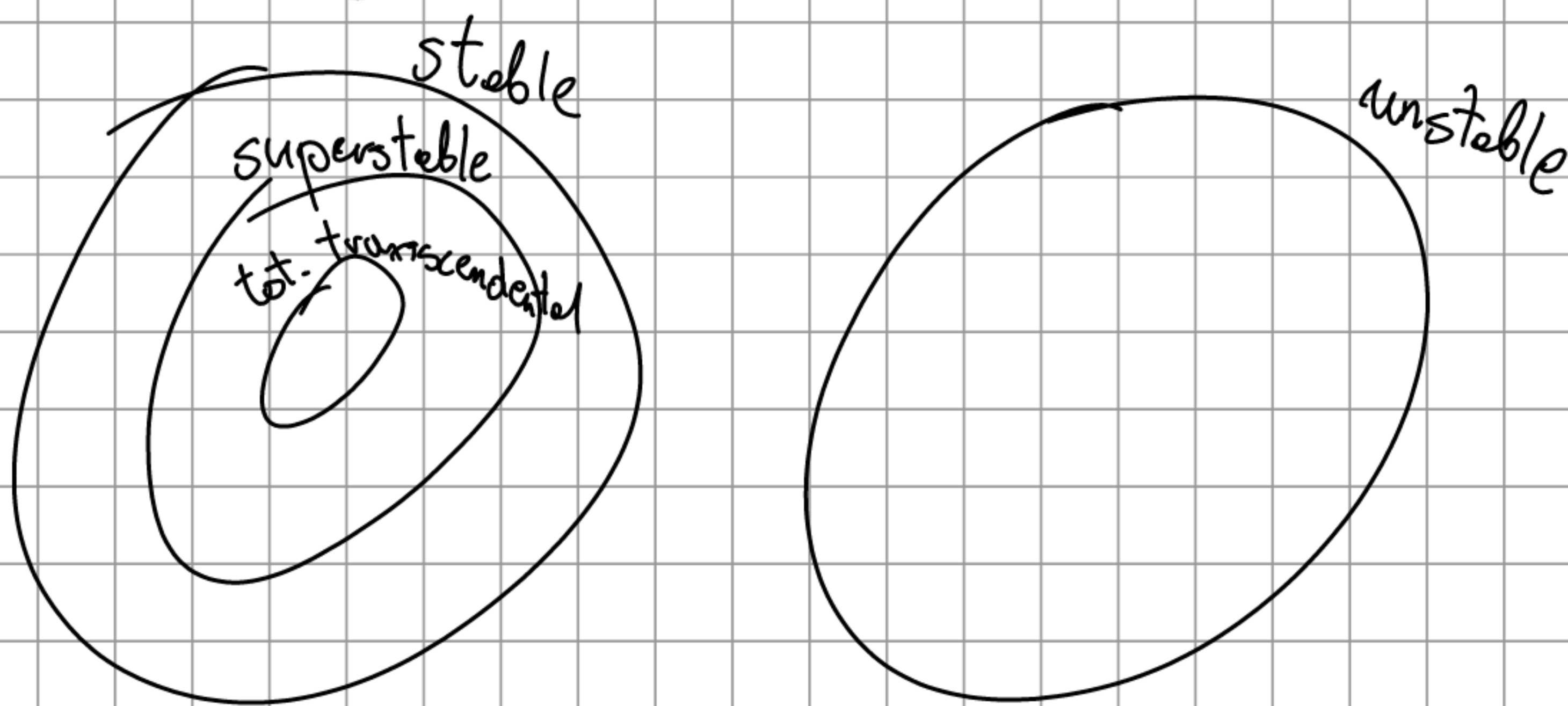
Let $\kappa \geq \aleph_0$.

(1) T is κ -stable $\Leftrightarrow \forall M \models T \forall A \subseteq M \left(|S_n(A)| \leq \kappa \right)$
 $|A| \leq \kappa$

(2) T is stable $\Leftrightarrow \exists \kappa (T \text{ } \kappa\text{-stable})$

(3) T is superstable $\Leftrightarrow \exists \mu \forall \kappa \geq \mu T$ is κ -stable

(4) T Totally transcendental $\Leftrightarrow \forall \kappa T$ κ -stable



Remark T is tot. trans. $\Leftrightarrow T$ is \aleph_0 -stable

Proof " \Rightarrow " obv.

" \Leftarrow " Suppose T is \aleph_0 -stable but not κ -stable for some κ

$M \models T$ s.t. $|S(A)| > |A| \geq \aleph_0$.

We shall find $A_0 \subseteq A$ with $|S(A_0)| > |A_0| + \aleph_0$.

Let $\varphi(x) \in \mathcal{F}_{L(A)}(x)$.

Def. φ is big iff $|S(A) \cap [\varphi]| > |A|$.

$p \in S(A)$ is big iff $\forall \varphi(x) \in p(x)$

φ is big.

Lemma If $\varphi(x)$ is big, then there is $\psi(x)$ s.t. both $(\varphi \wedge \psi)(x)$ and $(\varphi \wedge \neg\psi)(x)$ are big.

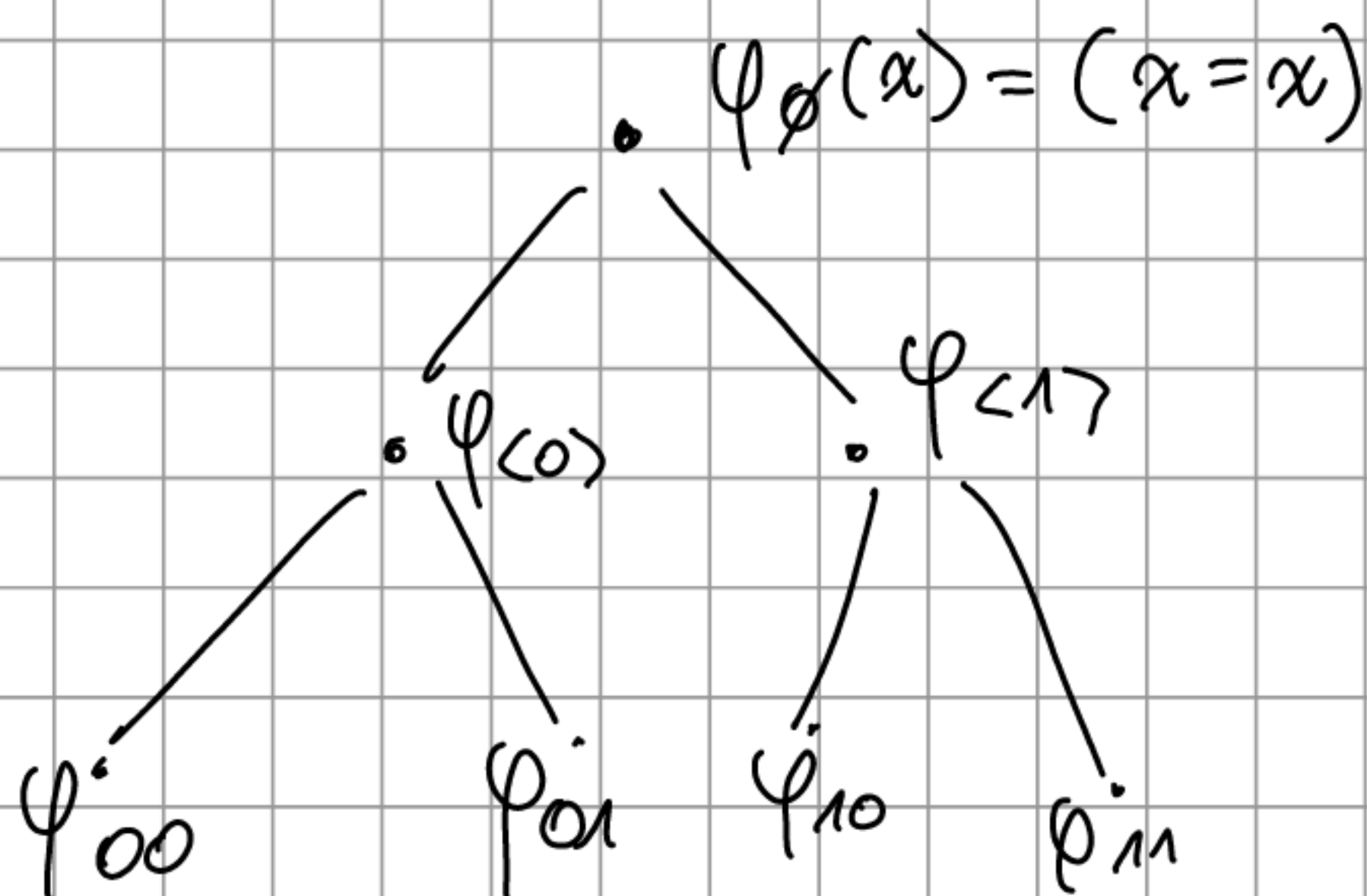
Proof (o.a.) If not, then there is exactly one big type $p(x) \in S(A) \cap [\varphi]$

$\{ \psi(x) \in \mathcal{F}_{L(A)}(x) : (\varphi \wedge \psi)(x) \text{ is big} \}$

big, $\in S(A) \cap [\varphi]$

⋮

From the lemma we construct a tree of big formulas $\varphi_\eta(x) \in \mathcal{F}_{L(A)}(x)$, $\eta \in 2^{<\omega}$



$$T \vdash \varphi_\eta(x) \Leftrightarrow \varphi_{\eta_0}(x) \dot{\vee} \varphi_{\eta_1}(x)$$

Let $A_0 \subseteq A$ including parameters of all $\varphi_\eta(x)$
ctd

For $\eta \in 2^\omega$ $\{ \varphi_{\eta|n}(x) : n < \omega \} \subseteq \{ \varphi_\eta(x) \in S(A_0) \}$
 a consistent type over A_0 in T .

$\eta \neq \eta' \Rightarrow p_\eta \neq p_{\eta'}$ so: $|S(A_0)| \geq 2^{\aleph_0} > |A_0| + \aleph_0$
 and T is not \aleph_0 -stable

Examples

(1) $\text{Th}(\mathbb{C}, +, \cdot) = \text{ACF}_0$: acl -stable

(2) $\text{Th}(\mathbb{Z}, +)$: superstable, but not tot. trans.

(3) $\text{SCF}_{p,l}$: the theory of separably closed fields of char. p and Erisov invariant l is stable, but not superstable

(4) $\text{Th}(\mathbb{Q}, \leq)$, $\text{Th}(\mathbb{Z}, +, \cdot)$, BA_0 , $\text{Th}(\mathbb{R}, ;, +, \leq)$
are unstable

(5) R : a ring with 1, M : R -module $\Rightarrow \text{Th}(M)$ is stable

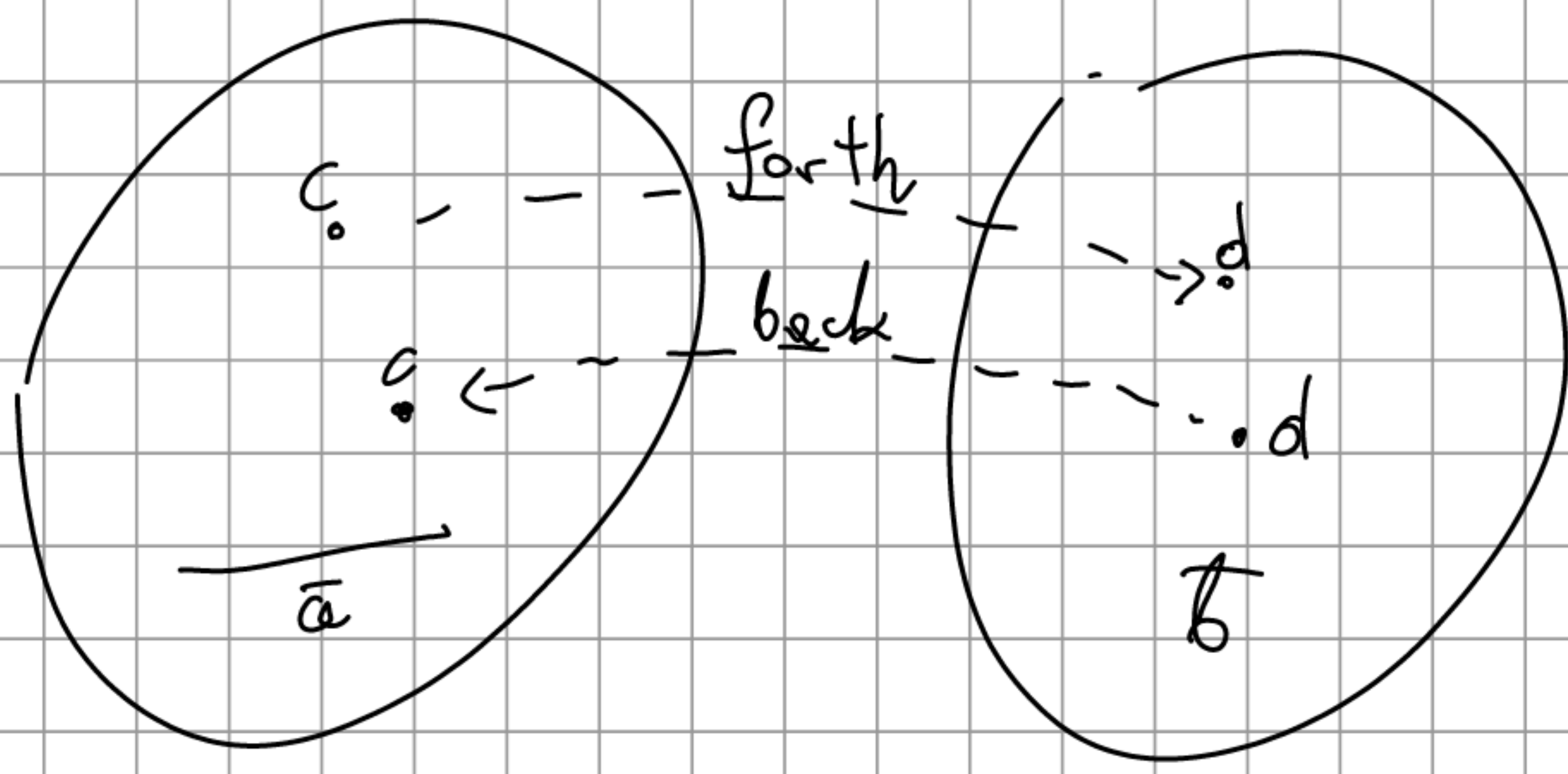
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First we define equivalence rel. $\cdot \equiv_\alpha \cdot$ ($\alpha \in \text{Ord}$)
on sets M^n , $n < \omega$ (M : fixed)

1) $\bar{a} \equiv_0 \bar{b} \iff \text{tp}^M(\bar{a}) = \text{tp}^M(\bar{b})$

2) $\delta \in \text{Lim} \quad \bar{a} \equiv_\delta \bar{b} \iff \forall \alpha < \delta \quad \bar{a} \equiv_\alpha \bar{b}$

3) $\bar{a} \equiv_{\alpha+1} \bar{b} \iff \left[\forall c \in M \exists d \in M \quad \bar{a}c \equiv_\alpha \bar{b}d \right] \wedge$
 $\left[\forall d \in M \exists c \in M \quad \bar{a}c \equiv_\alpha \bar{b}d \right]$



Fact \equiv_α is an equivalence relation on M^n

Proof Induction on α

1. $\alpha = 0$ obvious

2. Limit step. $\delta \in \text{Lim}$. We assume that $\forall \alpha < \delta$
 \equiv_α is an equiv. rel.

$$\Rightarrow \cdot \equiv_{\delta} \cdot = \bigcap_{\alpha < \delta} \cdot \equiv_{\alpha} \cdot$$

3. Successor step. Assume that \equiv_{α} is an equivalence relation. We prove that $\equiv_{\alpha+1}$ is an eq. rd.. Reflexive, symmetric is trivial. Transitive:

$\bar{a} \equiv_{\alpha+1} \bar{a}' \equiv_{\alpha+1} \bar{a}''$. We want to show that $\bar{a} \equiv_{\alpha+1} \bar{a}''$.

"forth": choose any $c \in M$. Since

$\bar{a} \equiv_{\alpha+1} \bar{a}'$ we have $c' \in M$ s.t.

$\bar{a}c \equiv_{\alpha} \bar{a}'c'$ and also $c'' \in M$ s.t.

$\bar{a}'c' \equiv_{\alpha} \bar{a}''c''$ ^{Ind. ass.} $\Rightarrow \bar{a}c \equiv_{\alpha} \bar{a}''c''$

"back": similar. ▀

Fact $\bar{a} \equiv_{\alpha} \bar{b}, \alpha > \beta \Rightarrow \bar{a} \equiv_{\beta} \bar{b} \left[\equiv_{\alpha} \subseteq \equiv_{\beta} \right]$

Proof exercise

Lemma Assume M is ctble.

$$(1) \forall \bar{a} \subseteq M \exists \alpha < \omega_1 \forall \beta \in (\alpha, \omega_1) [\bar{a}]_{\equiv_\alpha} = [\bar{a}]_{\equiv_\beta}$$

$$(2) \exists \alpha < \omega_1 \forall \bar{a}, \bar{b} \subseteq M (\bar{a} \equiv_\alpha \bar{b} \Rightarrow \bar{a} \equiv_{\alpha+1} \bar{b})$$

Proof

(1) $\langle [\bar{a}]_{\equiv_\alpha} : \alpha < \omega_1 \rangle$ a weakly decreasing sequence of ctble sets.

$[\bar{a}]_{\equiv_\alpha} \neq [\bar{a}]_{\equiv_\beta}$ It has to stabilize before ω_1 .

(2) From (1) for $\bar{a} \subseteq M$ we pick $\alpha_{\bar{a}} < \omega$ as in (1)

Let $\alpha = \sup_{\bar{a} \subseteq M} \alpha_{\bar{a}} < \omega_1$ α is good

in (2)

Def $SH(M) = \min \{ \alpha : \forall \bar{a}, \bar{b} \subseteq M (\bar{a} \equiv_\alpha \bar{b} \Rightarrow \bar{a} \equiv_{\alpha+1} \bar{b}) \}$

Scott height

$\forall \beta > \alpha \forall n \uparrow (\equiv_\beta = \equiv_\alpha \text{ on } M^n)$

$$SH(M) < \|M\|^+$$

??

Proof (3) $L_{\omega_1, \omega}$: the set of generalized formulas

(a) atomic formulas, terms: as before

(b) negation, quantifiers: as before

(c) we admit conjunctions, disjunctions of length $< \omega_1$. If $\{ \varphi_i : i < \omega \}$: $L_{\omega_1, \omega}$ formulas

$\bigvee_n \varphi_n, \bigwedge_n \varphi_n$: $L_{\omega_1, \omega}$ formulas

Example: Scott sentence (\mathbb{N}, \leq)

$\varphi_0(x)$: "x is minimal"

$\varphi_{n+1}(x)$: $\exists y (\varphi_n(y) \wedge x < y \wedge \neg \exists z (y < z < x))$

Analogously
 $L_{\kappa, \lambda}$
bound on len of conj.
bound on len of quant.

Scott sentence for (\mathbb{N}, \leq) φ : " \leq is an infinite LO"
 $\wedge \forall x \bigvee_{n < \omega} \varphi_n(x)$

Lemma 1 $\forall \alpha < \omega_1 \forall \bar{a} \subseteq M \exists \varphi_\alpha(x)$: a $L_{\omega_1, \omega}$ -formula

$\forall \bar{b} \subseteq M (\bar{a} \equiv_a \bar{b} \iff M \models \varphi_\alpha(\bar{b}))$

Proof Induction on α .

$$(1) \alpha = 0: \varphi_0(\bar{x}) = \bigwedge \text{tp}^M(\bar{a})(\bar{x})$$

$$(2) \delta \in \text{Lim}: \varphi_\delta(\bar{x}) = \bigwedge_{\alpha < \delta} \varphi_\alpha(\bar{x})$$

(3) $\alpha + 1$: For any $b \in M$ let $\psi_b(\bar{x}, y)$ be a formula good for $[\bar{a}b]_{\equiv_\alpha}$ class

$$\text{namely: } \forall \bar{c}d \in M (\bar{a}b \equiv_\alpha \bar{c}d \Leftrightarrow M \models \psi_b(\bar{c}, d))$$

(by ind. ass.)

$$\varphi_{\alpha+1}(\bar{x}) : \forall y \bigvee_{b \in M} \psi_b(\bar{x}, y) \left(\text{"} \forall y \exists b \bar{a}b \equiv_\alpha \bar{x}y \text{"} \right)$$

$$\wedge \bigwedge_{b \in M} \exists y \psi_b(\bar{x}, y) \left(\text{"} \forall b \exists y \bar{a}b \equiv_\alpha \bar{x}y \text{"} \right)$$

□

Lemma 2. $\forall \alpha < \omega_1 \forall M \left(\exists \varphi : \alpha \text{ } L_{\omega_1 \omega} \text{-sentence} \right) \forall N$
 $(M \equiv_\alpha N \Leftrightarrow N \models \varphi)$

Proof similar

Using lemmas 1 & 2 (3) follows.

Skolem functions, skolemization. T : a theory in L

Def. T has Skolem functions, if

$\forall \varphi(\bar{x}, y) \in \mathcal{F}_L$ there is a term

$t_\varphi(\bar{x}) \in \mathcal{T}_L$ st. $T \vdash \exists y \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, t_\varphi(\bar{x}))$

Def. Assume T has Skolem functions.

Assume $M \models T$ and $A \subseteq |M|$. Then

$\mathcal{H}(A)$ = the smallest subset X of M st.
the Skolem hull of A in M $A \subseteq X$ and $\forall t_\varphi(\bar{x}) \forall \bar{a} \subseteq X \quad t_\varphi^M(\bar{a}) \in X$

Remark 1. If T has Skolem functions, then T
has q.e.

Proof We prove: for every $\varphi \in \mathcal{F}_L$ there is a q.f.
 $\varphi' \in \mathcal{F}_L$ s.t. $T \vdash \varphi \leftrightarrow \varphi'$. Induction on $|\varphi|$.

- φ is atomic formula: OK

- the connectives: OK

- $\varphi(\bar{x}) = \exists y \varphi(\bar{x}, y)$

By the ind. ass. $T \vdash \psi \leftrightarrow \psi'$ q.f.
 wlog. ψ q.f.

$$T \vdash \exists y \psi(\bar{x}, y) \leftrightarrow \psi(\bar{x}, t_\psi(\bar{x}))$$

↑
for free



Remark 2 If T has Skolem functions and
 $A \in M \models T$, then $\mathcal{H}(A) \prec M$

Proof Apply T - V test. Let $\varphi(\bar{x}, y) \in \mathcal{F}_L$,
 $\bar{a} \in \mathcal{H}(A)$ st. $M \models \exists y \varphi(\bar{a}, y)$.

But $T \vdash \exists y \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, t_\varphi(\bar{x}))$

So $M \models \varphi(\bar{a}, t_\varphi(\bar{a}))$
 \uparrow
 $\mathcal{H}(A)$

Remark 3 Assume " \leq " $\in L$ and $M: L$ -str. s.t.

(M, \leq^M) is well ordered. Then in M we

have definable Skolem functions, i.e. if

for every $\varphi(\bar{x}, y) \in \mathcal{F}_L$ there is a definable

function f in M s.t.

$$M = \exists y \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, f(\bar{x}))$$

Proof. (idea) $f(\bar{x}) =$ the \leq -minimal y s.t. $\varphi(\bar{x}, y)$

Skolemization of:

- language L
- theory T (in L)
- structure M

① Single step: $L \rightsquigarrow L'$, $T \rightsquigarrow T'$, $M \rightsquigarrow M'$:

[here: $M \models T$, T : a theory in L].

For each formula $\varphi(\bar{x}, y) \in \mathcal{F}_L$ let $t_\varphi(\bar{x})$: a new function symbol.

$$L' = L \cup \{t_\varphi : \varphi \in \mathcal{F}_L\}$$

$$T' = \text{Con} (T \cup \{ \exists y \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, t_\varphi(\bar{x})) : \varphi \in \mathcal{F}_L \})$$

$M' = M$ expanded to an L' -structure:

t_φ^M such that $M' \models \exists y \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, t_\varphi(\bar{x}))$.

So: $M' \models T'$, hence T' : consistent.

② Iteration:

$$L^{(0)} = L, L^{(n+1)} = (L^{(n)})'$$

$$T^{(0)} = T, T^{(n+1)} = (T^{(n)})', \quad n = 0, 1, 2, \dots$$

$$M^{(0)} = M, M^{(n+1)} = (M^{(n)})'$$

③ Result: skolemization of:

$$L : L^S = \bigcup_n L^{(n)}, \quad T : T^S = \bigcup_n T^{(n)}, \quad M : M^S =$$

= M expanded to
an L^S -structure ... $M^{(n)}$, $n < \omega$.

Remark $M^s \models T^s$, T^s has Skolem functions.

LR2/2

Proof Let $\varphi(\bar{x}, y) \in \mathcal{F}_{L^s}$. Then $\varphi(\bar{x}, y) \in \mathcal{F}_{L(n)}$ for some n ,

hence $t_\varphi(\bar{x}) \in L^{(n+1)} \subseteq L^s$

and $T^{(n+1)} \vdash \exists y \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, t_\varphi(\bar{x}))$

\uparrow
 $T^s \vdash \exists y \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, t_\varphi(\bar{x}))$.

Fact. T^s is a conservative extension of T , that is:

If $\varphi \in \mathcal{F}_L$, then $T \vdash \varphi \Leftrightarrow T^s \vdash \varphi$.

Proof wlog $\varphi = \bar{\varphi}$: a sentence. (\Rightarrow) clear.

\Leftarrow : Suppose $T \not\vdash \varphi$. So there is $M \models T \cup \{\neg\varphi\}$.

\Downarrow
 $T^s \not\vdash \varphi \quad \Leftarrow \quad M^s \models T^s \cup \{\neg\varphi\}$

Ramsey theorem Assume $f: \underbrace{[N]^k}_{k \geq 1} \rightarrow \{0, 1\}$.

$\{X \in \mathcal{P}(N) : |X| = k\}$

Then $\exists X \subseteq N$ infinite $\exists t \in \{0, 1\} f \upharpoonright [X]^k \equiv t$

(X is called "homogeneous for f ").

Proof. Induction on k . $k=1$: OK.

and N may be replaced by any infinite set.

Induction step $k \mapsto k+1$:

Assume $f: [N]^{k+1} \rightarrow \{0, 1\}$.

We define $a_n \in N$ and $X_n \subseteq N$ (for $n \in N$) s.t.

(1) $a_i \in X_{i-1} \setminus X_i$ [and $X_{-1} = N$]

(2) $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$

(3) $\forall i \exists t_i \in \{0, 1\} \forall Y \in [X_i]^k f(\{a_i\} \cup Y) = t_i$

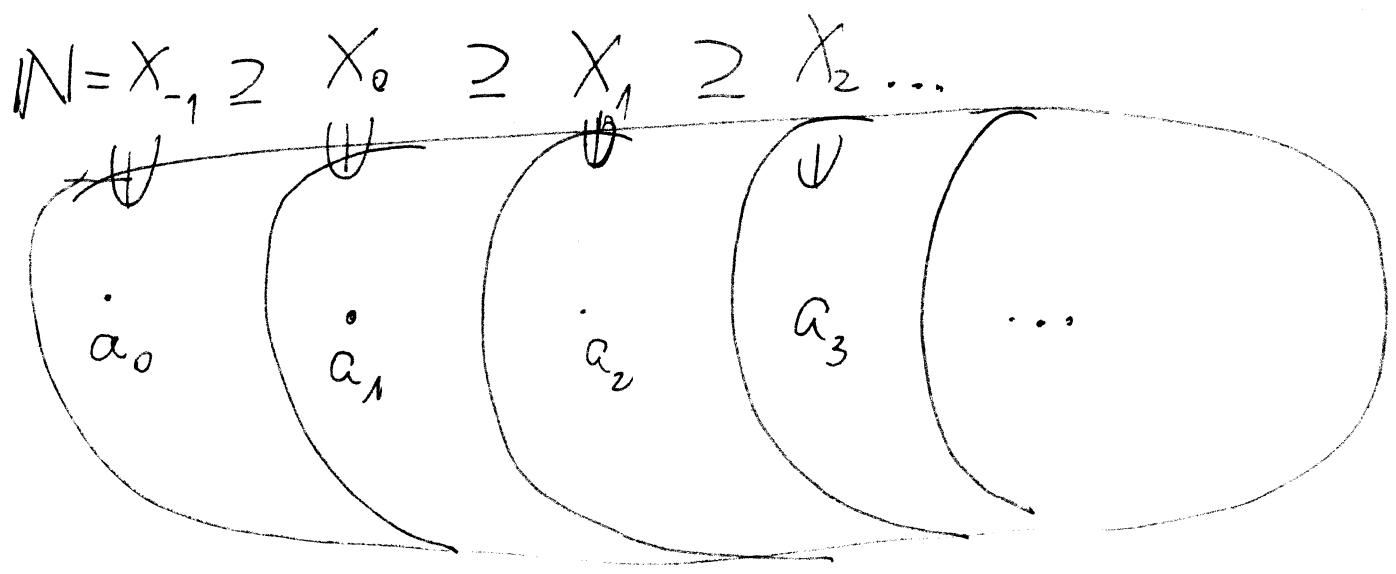
Construction:

Let $a_i \in X_{i-1}$. Define $f': [X_{i-1} \setminus \{a_i\}]^k \rightarrow \{0, 1\}$
by $f'(Y) = f(Y \cup \{a_i\})$

by inductive assumption

there is $X_i \subseteq X_{i-1} \setminus \{a_i\}$, homogeneous for f' .
 $\dots f' \upharpoonright [X_i]^k \equiv t_i$

Picture Drawing:



Let $t \in \{0, 1\}$ s.t. $\{i : t = t_i\}$ is infinite.

Then $X := \{a_i : t = t_i\}$ good for f

Why? Let $Y \in [X]^{k+1}$ then $f(Y) = t$, because:

$$Y = \{a_{i_1}, a_{i_2}, \dots, a_{i_{k+1}}\} = \{a_{i_1}\} \cup Y', \quad Y' \in [X_{i_1}]^k$$

$$\text{so: } f(\{a_{i_1}\} \cup Y') = \underset{\substack{\uparrow \\ \text{step } i_1}}{f}(\{a_{i_1}\}) = t_{i_1} = t.$$

Corollary (Ramsey thm)

$$f: [\mathbb{N}]^k \rightarrow \{0, 1, \dots, l\} \Rightarrow \exists X \subseteq \mathbb{N} \text{ infinite } \exists t \in \{0, 1, \dots, l\} \text{ s.t. } f \upharpoonright [X]^k = t.$$

many variants, like:

Corollary (finite version of Ramsey theorem)

~~$$\forall k, l \in \mathbb{N}^+ \exists m \in \mathbb{N}^+ \forall A \subseteq \mathbb{N} \text{ w/ } |A| \geq m \exists X \subseteq A \text{ w/ } |X| \geq m \text{ s.t. } f \upharpoonright [X]^k = \text{constant}$$~~

$$\forall k, l, m \in \mathbb{N}^+ \exists n \in \mathbb{N}^+ \forall A \subseteq \mathbb{N} \text{ w/ } |A| \geq n \exists X \subseteq A \text{ w/ } |X| \geq m \text{ s.t. } f \upharpoonright [X]^k = \text{constant}$$

order indiscernible sets

Def Assume (I, \leq) : a linear ordering (e.g. $I = \mathbb{N}$)

and $A = \{a_i : i \in I\} \subseteq M$ (an I -indexed set)

We say that:

A is order indiscernible if $\forall \varphi(x_1, \dots, x_n) \in \mathcal{F}_L$

$$\forall i_1 < \dots < i_n, j_1 < \dots < j_n \in I$$

~~$$M \models \varphi(a_{i_1}, \dots, a_{i_n}) \iff M \models \varphi(a_{j_1}, \dots, a_{j_n})$$~~

$$M \models \varphi(a_{i_1}, \dots, a_{i_n}) \iff \varphi(a_{j_1}, \dots, a_{j_n}). \quad (*)$$

Thm If T is a complete theory with infinite models and (I, \leq) is a linear order, then

$\exists M \models T \exists A = \{a_i : i \in I\} \subseteq M$ A infinite, order indiscernible.

Proof Let $L' = L \cup \{c_i : i \in I\}$
new constant symbols.

$$T' = T \cup \bigcup_{\varphi} S_{\varphi}, \quad \varphi = \varphi(x_1, \dots, x_n) \in \mathcal{F}_L$$

where $S_{\varphi} = \{ \varphi(c_{i_1}, \dots, c_{i_n}) \iff \varphi(c_{j_1}, \dots, c_{j_n}) : \left. \begin{array}{l} i_1 < \dots < i_n \in I \\ j_1 < \dots < j_n \in I \end{array} \right\}$

It is enough to show that T' is consistent.

Let $\varphi_1, \dots, \varphi_k$: formulas, $S'_{\varphi_t} \subseteq S_{\varphi_t}$ for $t = 1, \dots, k$
finite

We will show that $T \cup \bigcup_t S'_{\varphi_t}$ is consistent.

Let $N \neq T$ infinite

LR2/6

Lemma. If $A = \{a_n, n \in \mathbb{N}\} \subseteq \mathbb{N}$ and

$\varphi(x_1, \dots, x_k) \in \mathcal{F}_L$, then $\exists X \subseteq \mathbb{N}$ infinite is order φ -indiscernible
[i.e. (*) holds for φ]

Proof For $Y = \{n_1 < \dots < n_k\} \in [\mathbb{N}]^k$

let $f(Y) = \begin{cases} 0, & \text{when } N \models \neg \varphi(a_{n_1}, \dots, a_{n_k}) \\ 1, & \text{when } N \models \varphi(a_{n_1}, \dots, a_{n_k}) \end{cases}$

$f: [\mathbb{N}]^k \rightarrow \{0, 1\}$.

Let $X \subseteq \mathbb{N}$: homogeneous for f X is good.
infinite □
Lemma.

Applying Lemma a few times we find

$A = \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{N}$ s.t. $\forall 1 \leq t \leq k$ (*) holds for φ_t

Assume S'_{φ_t} ($1 \leq t \leq k$) use only constants c_{i_1}, \dots, c_{i_N} , $i_1 < \dots < i_N$
interpretation $\downarrow \quad \downarrow$
 $a_1, \dots, a_N \in \mathbb{N}$

so: $N \neq T \cup \bigcup_t S'_{\varphi_t}$.

Corollary (Ermenfucht - Mostowski)

Assume T is countable, consistent, with infinite models.

Then $\exists M \neq T$ stable $\text{Aut}(Q, \leq) \hookrightarrow \text{Aut}(M)$,

$T \mapsto T^S$. Let $M^S \models T^S$ abelian, ~~is~~ containing

$$A = \{a_q : q \in \mathbb{Q}\} \text{ order indiscernible}$$

$$\text{Let } N^S = \text{acl}(A) \prec M^S$$

(†) For $f \in \text{Aut}(\mathbb{Q}, \leq)$ let $\hat{f} : N^S \rightarrow N^S$ given by:

$$(a) \hat{f}(a_q) = a_{f(q)}$$

(b) If $t(\bar{x})$: a term of L^S , then

$$\hat{f}(t^{N^S}(a_{q_1}, \dots, a_{q_n})) = t^{N^S}(\hat{f}(a_{q_1}), \dots, \hat{f}(a_{q_n})).$$

$$\hat{f} \in \text{Aut}(N^S) \Rightarrow \hat{f} \in \text{Aut}(N), \text{ where } N = N^S \upharpoonright_L.$$

$$f \longmapsto \hat{f}$$

$$\text{Aut}(\mathbb{Q}, \leq) \hookrightarrow \text{Aut}(N). \quad N \text{ ~~producible~~ } \text{ abelian}, N \models T$$

$N^S \models$ is called an Ehrenfeucht-Mostowski model.

Comments

• the definition of \hat{f} is correct:

- each $a \in N^S$ is of the form $a = t^{N^S}(\bar{a}_{\bar{q}})$, $\bar{a}_{\bar{q}} \subseteq A$

- if $a = t^{N^S}(\bar{a}_{\bar{q}}) = t'^{N^S}(\bar{a}'_{\bar{q}'})$, then

$$N^S \models t(\bar{a}_{\bar{q}}) = t'(\bar{a}'_{\bar{q}'}) \Rightarrow N^S \models t(\bar{a}_{f(\bar{q})}) = t'(\bar{a}'_{f(\bar{q}')})$$

[A : order indiscernible, f preserves \leq]

$\hat{f} \in \text{Aut}(N^s) : f \in \text{Aut}(N^s)$

$N^s \models \varphi(a_{q_1}, \dots, a_{q_n}) \iff \varphi(\hat{f}(a_{q_1}), \dots, \hat{f}(a_{q_n}))$ [by order indiscernibility].

Ultraproducts

(a) Products: $M_i, i \in I$: L-structures.
 $\rightsquigarrow M = \prod_I M_i$ product of structures $M_i, i \in I$.

$|M| = \prod_{i \in I} |M_i| = \{f : I \rightarrow \bigcup_{i \in I} |M_i| : \forall i f(i) \in |M_i|\}$.

L-structure on M:

• P: relational symbol of L.

$P^M (f_1, \dots, f_n) \iff \forall i P^{M_i}(f_1(i), \dots, f_n(i))$

• F: function symbol of L

$F^M (f_1, \dots, f_n) = \langle F^{M_i}(f_1(i), \dots, f_n(i)) : i \in I \rangle \in |M|$.

• constant symbol c of L

$c^M = \langle c^{M_i} : i \in I \rangle$.

Examples products of groups, linear spaces, fields, rings...

Troubles 1. Product $K_1 \times K_2$ of fields is not a field.

2. $\text{Th}(\prod_I M_i)$ unrelated to $\text{Th}(M_i), i \in I$.

Solution ultraproducts (families of "large" sets) LR2/9

Def.: \mathcal{U} is an ultrafilter on $I \Leftrightarrow$

\mathcal{U} is an ultrafilter in algebra $\mathcal{P}(I)$

• An ultrafilter \mathcal{U} on I is principal if $\exists a \in I$

$$\mathcal{U} = \mathcal{U}_a = \{X \subseteq I : a \in X\}.$$

Otherwise: \mathcal{U} : non-principal.

Properties of ultrafilter \mathcal{U} :

$$(1) \forall X \subseteq I (X \in \mathcal{U} \vee X^c \in \mathcal{U})$$

$$(2) \forall X \in \mathcal{U} \forall Y \subseteq I \quad Y \in \mathcal{U} \iff X \subseteq Y$$

(3) Every proper filter \mathcal{F} on I extends to an ultrafilter on I .

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\mathcal{U} : ultrafilter on I

$\{M_i\}_{i \in I}$: a family of L -structures

Def Ultraproduct $M = \prod_I M_i / \mathcal{U}$

(a) \sim : an equivalence relation on $\prod_{i \in I} M_i$

$$f \sim g \Leftrightarrow \{i \in I : f(i) = g(i)\} \in \mathcal{U}$$

(b) Let $M = \prod_I M_i / \sim$. We define an L -structure on M :

1) P -predicate symbol, then

$$P^M([f_1]_{\sim}, \dots, [f_n]_{\sim})$$

$$\uparrow \Leftrightarrow \{i \in I : M_i \models P^{M_i}(f_1(i), \dots, f_n(i))\}$$

2) F : func. symbol of L :

$$F^M([f_1]_{\sim}, \dots, [f_n]_{\sim}) = [g]_{\sim}$$

$$\text{where } g(i) = F^{M_i}(f_1(i), \dots, f_n(i))$$

3) C : constant symbol of L :

$$C^M = [\langle C^{M_i} \rangle_{i \in I}]_{\sim}$$

This definition is correct.

Let's see for 1):

Suppose $f_1 \sim f_1', \dots, f_n \sim f_n' \Rightarrow$

$$A := \{i \in I : f_1(i) = f_1'(i) \wedge \dots \wedge f_n(i) = f_n'(i)\} \in \mathcal{M}$$

$$\text{Let } B = \{i \in I : \mathcal{P}^{M_i}(f_1(i), \dots, f_n(i))\}$$

$$C = \{i \in I : \mathcal{P}^{M_i}(f_1'(i), \dots, f_n'(i))\}$$

$$B \in \mathcal{M} \Leftrightarrow B \cap A \in \mathcal{M}$$

$$C \in \mathcal{M} \stackrel{?}{\Rightarrow} C \cap A \in \mathcal{M}, \text{ but } B \cap A = C \cap A!$$

Let's see for 2):

$$A \subseteq \{i \in I : F^{M_i}(f_1(i), \dots, f_n(i)) = F^{M_i}(f_1'(i), \dots, f_n'(i))\}$$

$$\mathcal{M} \Rightarrow \mathcal{M} \Rightarrow \langle F^{M_i}(f_1(i), \dots, f_n(i)) \rangle_{i \in I}$$

$$\langle F^{M_i}(f_1'(i), \dots, f_n'(i)) \rangle_{i \in I}.$$

Remark

If \mathcal{M} is principal then $\mathcal{M} \cong \mathcal{M}_a$

" $\mathcal{M}_a, a \in I$ "

Proof

$$\mathcal{M} \xrightarrow{\cong} \mathcal{M}_a$$
$$[f]_n \longmapsto f(a) \quad \blacksquare$$

Theorem (Łoś)

For $\varphi(x_1, \dots, x_n) \in \mathcal{F}_L$ and $[f_1]_{\mathcal{M}}, \dots, [f_n]_{\mathcal{M}} \in |M|$,

where $M = \prod_{i \in I} M_i / \mu$:

$$M \models \varphi([f_1]_{\mathcal{M}}, \dots, [f_n]_{\mathcal{M}}) \iff \underbrace{\bigwedge_{i \in I} M_i \models \varphi(f_1(i), \dots, f_n(i))}_{\substack{\text{D} \\ \mu}}$$

An auxiliary fact

If $t(x_1, \dots, x_n) \in \mathcal{T}_L$ then

$$t^M([f_1]_{\mathcal{M}}, \dots, [f_n]_{\mathcal{M}}) = \left[\langle t^{M_i}(f_1(i), \dots, f_n(i)) \rangle_{i \in I} \right]_{\mu}$$

Proof: exc.

Proof (Łoś)

Induction on $|\varphi|$

(a) φ atomic, $\bar{x} = (x_1, \dots, x_n)$

$$(i) \varphi: t(\bar{x}) = t'(\bar{x})$$

$$M \models \varphi([\bar{f}]_{\mathcal{M}}) \iff t^M([\bar{f}]_{\mathcal{M}}) = t'^M([\bar{f}]_{\mathcal{M}})$$

$$\iff \langle t^{M_i}(f_i(i)) : i \in I \rangle \sim \langle t'^{M_i}(f_i(i)) : i \in I \rangle$$

aux. fact

$$\Leftrightarrow \{i \in I : M_i \models t(f(i)) = t'(f(i))\} \in \mathcal{U}$$

$$(ii) \varphi: \underbrace{P(t_1(\bar{x}), \dots, t_n(\bar{x}))}_{\bar{t}(\bar{x})}$$

$$M \models \varphi([\bar{f}]_n) \Leftrightarrow M \models P^M([\bar{t}^M([\bar{f}]_n)]_n)$$

$$\Leftrightarrow \{i \in I : P^{M_i}(\bar{t}^{M_i}(\bar{f}(i)))\} \in \mathcal{U}$$

$$\Leftrightarrow \{i \in I : M_i \models \underbrace{P(\bar{t}(\bar{f}(i)))}_{\varphi(\bar{f}(i))}\}$$

(b) \neg, \wedge

$$(i) \varphi: \neg \psi \quad M \models \varphi([\bar{f}]_n) \Leftrightarrow M \not\models \psi(\bar{f})$$

$$\stackrel{\text{ind.}}{\Leftrightarrow} \{i \in I : M_i \models \psi(\bar{f}(i))\} \notin \mathcal{U}$$

$$\stackrel{\mathcal{U}: \text{ul.}}{\Leftrightarrow} \{i \in I : M_i \not\models \psi(\bar{f}(i))\} \in \mathcal{U}$$

$$\Leftrightarrow \{i \in I : M_i \models \varphi(\bar{f}(i))\} \in \mathcal{U}$$

$$(ii) \varphi: \varphi_1 \wedge \varphi_2 \quad \text{easy}$$

$$(iii) \varphi: \exists y \psi(\bar{x}, y)$$

$$\stackrel{\text{"}\Rightarrow\text{"}}{M \models \varphi([\bar{f}]_n)} \Rightarrow \exists g \in |M| \quad M \models \psi([\bar{f}]_n, [g]_n)$$

$$\stackrel{\text{ind.}}{\Leftrightarrow} \{i \in I : M_i \models \psi(\bar{f}(i), g(i))\} \in \mathcal{U}$$

\cap

$$A := \{i \in I : M_i \models \exists y \psi(F(i), y)\} \in \mathcal{U}$$

$\varphi(\bar{F}(i))$

" \Leftarrow " Assume $A \in \mathcal{U}$

We define a witness $[g]_{\mathcal{U}} \in M$ for y

$$\text{in } \varphi([\bar{F}]_{\mathcal{U}}) = \exists y \psi([\bar{F}]_{\mathcal{U}}, y)$$

- when $i \in A$ $g(i) \in |M_i|$ s.t.

$$M_i \models \psi(F(i), g(i))$$

- when $i \notin A$ then $g(i) \in |M_i|$ arbitrary

So:

$$A \subseteq \{i : M_i \models \psi(F(i), \bar{g}(i))\}$$

$$\overset{\mathcal{U}}{M} \Rightarrow \overset{\mathcal{U}}{M}$$

by the ind. hypothesis for ψ .

$$M \models \psi([\bar{F}]_{\mathcal{U}}, [g]_{\mathcal{U}}) \Rightarrow M \models \varphi([\bar{F}]_{\mathcal{U}}).$$



A special case

$$M_i = N \text{ for all } i \in I$$

$$M = \prod_I M_i / \mu = \underbrace{N^I / \mu}_{\text{Ultrapower}}$$

$$\text{let } f: N \rightarrow N^I / \mu$$

$$f(a) = [f_a]_{\mu}, \quad f_a: I \rightarrow |N|$$

$$f_a \equiv a$$

(a diagonal embedding)

Remark

(a) f is 1-1

(b) $f[N] < N^I / \mu$

(c) $f: N \xrightarrow{\cong} N^I / \mu$

Proof(c) Let $\varphi(\bar{x}) \in \mathcal{F}_L, \bar{a} \in N$.

$$N \models \varphi(\bar{a}) \Leftrightarrow \exists i \in I: N \models \varphi(\underbrace{[f_a(i)]}_{\equiv a}) \in \mu$$

$$\stackrel{\text{Los Thm}}{(\Rightarrow)} N^I / \mu \models \varphi(f(\bar{a}))$$

Compactness thm (new proof)

Assume T is a set of sentences s.t.

every finite $T_0 \subseteq T$ has a model.

Then T has a model.

Proof Let $I = [T]^{<\aleph_0} = \{T_0 \subseteq T : T_0 \text{ is finite}\}$

For $i \in I$ let $M_i \models i$

Let $\varphi \in T$. $\{i \in I : M_i \models \varphi\}$

\cup
 $X_\varphi := \{i \in I : \varphi \in i\}$

$\{X_\varphi : \varphi \in T\} \subseteq \mathcal{P}(I)$

has f. i. p. $\rightarrow \{\varphi_1, \dots, \varphi_n\} \in X_{\varphi_1} \cap \dots \cap X_{\varphi_n} \neq \emptyset$
(finite intersection property)

so $\exists \mathcal{U}$: an uf on I st. $\forall \varphi \in T X_\varphi \in \mathcal{U}$

Let $M = \prod_I M_i / \mathcal{U}$

• $M \models T$

Let $\varphi \in T$. $M \models \varphi \Leftrightarrow \{i \in I : M_i \models \varphi\} \in \mathcal{U}$
 \cup
 $X_\varphi \in \mathcal{U} \quad \checkmark$

Example Let \mathcal{U} : a non-principal
ultrafilter on $\mathbb{N} = \omega$
(a Frechet ultrafilter)

If $\{M_i : i < \omega\}$ s.t. $\|M_i\| = k < \omega$

then $\|\prod_{\omega} M_i / \mathcal{U}\| = k$

Proof Look at φ_k : "there's exactly k -many elements"

Special case Let $\mathcal{R} = (\mathbb{R}, +, \cdot; r, f, P)$
 $r, f, P: \text{cell}$

Let \mathcal{U} : a Frechet ultrafilter on ω .

$$\mathbb{R}^* = \mathbb{N}^{\omega} / \mathcal{U}$$

\downarrow
 \mathbb{R}

Properties of \mathbb{R}^* :

(a) $\|\mathbb{R}^*\| = 2^{\aleph_0}$

(b) Every stable consistent type $p(x)$ over \mathbb{R}^* is
realized in \mathbb{R}^*

Set theory as "meta theory" for mathematics.

• What exists?

• for sure: \emptyset (in style of Descartes)

Set theory:

(1) describes procedures of creating sets

(2) describes properties of sets.

For simplicity: "every individual is a set"

• Language of set theory: $L = \{ \in \}$

↑
relational
binary
symbol

The first try: "META-META SET THEORY"

Informal [If $\varphi(x) \in \mathcal{F}_L$ then
there is $y = \{ x : \varphi(x) \}$

Formally: $\exists y \forall x (x \in y \leftrightarrow \varphi(x))$

Russel antinomy:

Let $\varphi(x) : x \notin x$

$\exists y \forall x (x \in y \leftrightarrow x \notin x)$ is contradictory,
it has no model.

If $(M, E) \models \exists y \forall x (x \in y \leftrightarrow x \notin x)$
 $\begin{matrix} \uparrow & \uparrow \\ a \in M & a \in M \end{matrix}$ $a \in a \leftrightarrow a \notin a$

Therefore: need care when constructing sets.

• General rule: allow as many sets as possible (by Gödel, freedom principle)

Axioms of ZFC (Zermelo-Frenkel set theory with choice axiom)

A1 (Existence) " $\exists \emptyset$ "

$$\exists x \forall y \quad y \notin x$$

A2 (Extensionality) If x, y have the same elements, then $x = y$

A3 (Pair axiom)

$$(\forall x, y) \exists z \forall t (t \in z \leftrightarrow t = x \vee t = y)$$

A4 (Comprehension)

If $\varphi(x, \bar{y})$ is a formula, X : a set then

$$\forall \bar{y} \exists Y = \{x \in X : \varphi(x, \bar{y})\}$$

A5 (Union) $\forall X \cup X$ exists

A6 (Power set) $\forall X \mathcal{P}(X)$ exists

A7 (Replacement)

Axiomatic ZFC Axioms

A1. (~~istnienie~~ ^{existence}): "~~istnieje~~ ^{exists} \emptyset " : $\exists x \forall y y \notin x$

A2. (~~ciast~~ extensionality)

If x, y have the same elements, then $x = y$

A3. (Pair) $\forall x, y \exists z \forall t (t \in z \leftrightarrow t = x \vee t = y)$

A4. Comprehension (~~wyrozniczenia~~)

Assume ~~$\varphi(x, \bar{z})$~~ $\varphi(x, \bar{z}) \in \mathcal{F}_{L_{ZF}}$, X : a set \rightsquigarrow

$$\exists Y = \{x \in X : \varphi(x, \bar{z})\}$$

Formally: a axiom scheme for a formula $\varphi(x, \bar{z})$.

$$\forall x \exists y \forall t (t \in y \leftrightarrow t \in x \wedge \varphi(t, \bar{z}))$$

A5. Union: ~~$\forall x$~~ $\forall x \cup x$ exists

A6. Power set: $\forall x \exists \mathcal{P}(x)$

A7. Replacement (~~zastepowanie~~)

If $\varphi(x, y, \bar{z}) \in \mathcal{F}_{L_{ZF}}$ is a function [i.e. $\forall x \exists! y \varphi(x, y, \bar{z})$]

then $\forall X \exists Y = \{y : (\exists x \in X) \varphi(x, y, \bar{z})\}$

A8. Infinity ("There is an infinite set")

How to write it down?

For example: $\text{ind}(x) = (\forall y \in x) (y \cup \{y\} \in x) \wedge \emptyset \in x$
(inductive)

Axiom: $\exists x \text{ind}(x)$

A9. Regularity $\forall x \exists y \in x y \cap x = \emptyset$

A10. Choice Axiom:

If $\forall a \in A \ a \neq \emptyset$, then $\exists f: A \rightarrow \cup A \ \forall a \in A \ f(a) \in a$

Remark 1. (ZFC) $\neg \exists x \ \forall y \ y \in x$ (There is no set of all sets)

Proof Suppose not. and x is the set (capo di tutti capi) of all sets. Then x violates several axioms:

- Comprehension: let $x' = \{y \in x : y \notin y\} \rightsquigarrow$ Russell antinomy
- regularity + pairing: $x \in x \Rightarrow \underline{\underline{\{x\}}}$

Therefore classes

Let $\varphi(x, \bar{y}) \in \mathcal{F}_{L_{ZF}}$ class

then for every \bar{y} we think of $\varphi(x, \bar{y})$ as $C_{\varphi, \bar{y}} = \{x : \varphi(x, \bar{y})\}$

A class is proper if ~~true~~ it is not a set.

Example: $C_{=} = \{x : x = x\}$ is a proper class.

Natural numbers: $\mathbb{N} = \bigcap \{x : \text{ind}(x)\}$; a set because: let x_0 : inductive (by ∞ -axiom).

5.1 Zaleca się przygotowywanie prac dyplomowych przy użyciu programu Tex, z uwagi na przystosowanie tego programu do profesjonalnego składu tekstów matematycznych. Dopuszczalne jest przygotowywanie prac dyplomowych przy użyciu programu Microsoft Word lub podobnych edytorów tekstu, pod warunkiem zachowania zasad składu tekstów matematycznych.

5.2 Strona tytułowa pracy dyplomowej powinna być zgodna ze wzorem umieszczonym na stronie Instytutu Matematycznego (zakładka Praca dyplomowa).

5.3 Strony pracy dyplomowej powinny być numerowane zaczynając od strony tytułowej.

Then $\mathbb{N} = \{t \in x_0 : \forall x (ind(x) \rightarrow t \in x)\}$
exists by comprehension axiom.

\mathbb{N} : the smallest inductive set.

$$\begin{array}{ccccccc} \emptyset \in \mathbb{N}, & \emptyset \cup \{\emptyset\} \in \mathbb{N}, & 2 \in \mathbb{N}, & \dots, & n = \{0, 1, \dots, n-1\} & \overset{\mathbb{N}}{\cup} & \\ \parallel & \parallel & \parallel & & & & \\ 0 & 1 = \{\emptyset\} & 1 \cup \{1\} = \{0, 1\} & & & & \end{array}$$

Def. $Ord(x) = \left\{ \begin{array}{l} \text{"}(x, \in) \text{ is a well-ordered set"} \\ \text{and} \\ \forall y \in x \ y \subseteq x \end{array} \right\}$
that is: " x is transitive".

Fact. (ZF) $Ord(x) \wedge Ord(y) \rightarrow \left\{ \begin{array}{l} x \in y \vee y \in x \vee x=y \\ \text{and} \\ Ord(x \cup \{x\}) \end{array} \right.$

(2) $Ord(x) \wedge y \in x \Rightarrow Ord(y) \wedge$
 $x \text{ transitive} \Rightarrow y \subseteq x$ " y : an initial segment of x ".

(3) $\neg \exists x \ x = \{y : Ord(y)\}$ (But ~~the~~ $Ord \neq \{y : Ord(y)\}$
is a (proper) class)

Proof (3) Suppose $x = \{y : ord(y)\}$.

Then $ord(x)$, hence $x \in x$.

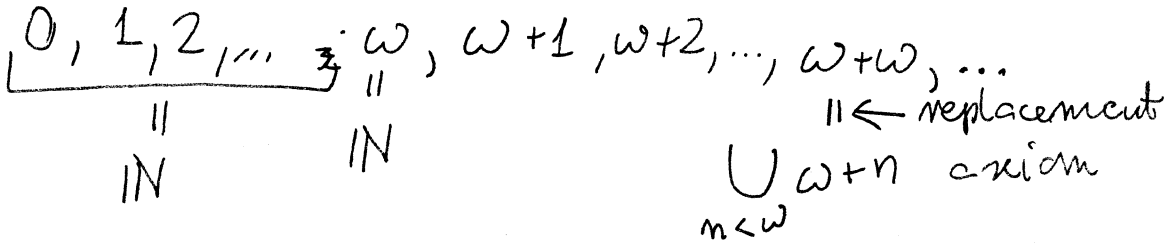
Let $x' = \{x\}$. x' violates regularity axiom

since: $y \in x' \Rightarrow y \cap x' \neq \emptyset$

Remark Ord is a well-ordered (by \in) class.
transitive

$Ord(x) \Rightarrow x \cup \{x\} \in Ord$: a successor of x
 $x+1$

Examples of ordinal numbers: (Ord)



$\omega_1, \omega_1+1, \dots$
 \parallel uncountable

$\{\alpha \in \text{Ord} : \alpha \text{ countable}\}$

von Neumann hierarchy of sets $V_\alpha, \alpha \in \text{Ord}$:

- $V_0 = \emptyset$
 - $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
 - $V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$, when λ : limit ordinal
- transfinite recursion.

How to ~~construct~~ define V_α honestly, by an LZF-formula?

$$x = V_\alpha \iff (\exists f: \text{function}) \left(\text{Dom } f = \alpha+1 \wedge \begin{cases} f(0) = \emptyset \\ (\forall \beta < \alpha) f(\beta+1) = \mathcal{P}(f(\beta)) \\ (\forall \beta < \alpha) (\text{Lim}(\beta) \rightarrow f(\beta) = \bigcup_{\gamma < \beta} f(\gamma)) \end{cases} \right)$$

and $f(\alpha) = x$

f : a constructor function/sequence

5.1 Zaleca się przygotowywanie prac dyplomowych przy użyciu programu TeX, z uwagi na przystosowanie tego programu do profesjonalnego składu tekstów matematycznych. Dopuszczalne jest przygotowywanie prac dyplomowych przy użyciu programu Microsoft Word lub podobnych edytorów tekstu, pod warunkiem zachowania zasad składu tekstów matematycznych.

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5. Wymagania techniczne i edytorskie

Fact, (1) $V_\alpha \subseteq V_\beta$ for $\alpha < \beta$

(2) $\alpha \in V_{\alpha+1} \setminus V_\alpha$

(3) $\forall x \exists \alpha x \in V_\alpha$ hence $\forall x x \in V$

here: $V = \bigcup_{\alpha \in \text{Ord}} V_\alpha$: a class proper class
 $x \in V \Leftrightarrow \exists \alpha (\text{Ord}(\alpha) \wedge x \in V_\alpha)$

Proof, (1), (2) : exercises.

(3) (a.a.) Suppose $x \notin V$.

Claim $\exists x' \notin V \forall y \in x' y \in V$

Proof Case 1 $\forall y \in x y \in V$, Then let $x' = x$.

Case 2: \neg Case 1

Let $x_1 = x \cup \underbrace{Ux \cup UUx \cup UUUx \cup \dots}_{\text{TC}(x)}$

"
 $\text{TC}(x)$, transitive closure of x
 (the smallest set y s.t. $x \subseteq y$) and
 transitive

[Exercise:
 $a \in \text{TC}(x) \Rightarrow a \subseteq x$.

Now: if $x_1 \in V$, then $x \subseteq x_1$, hence $x \in V$ \Downarrow .

So $x_1 \notin V$.

Let $y = \{t \in x_1 : t \notin V\}$. \mathbb{R}

If $y = \emptyset$, then $x_1 \subseteq V$ and we ~~are~~ are done in the Claim.

Otherwise $y \neq \emptyset$. By regularity choose $x' \in y$ with $x' \cap y = \emptyset$.

LR/3 (6)

So $x' \in \mathcal{X}_1 \Rightarrow x' \subseteq \mathcal{X}_1 \Rightarrow$

and by (*) $\forall v \in x' \quad v \in V_{x'}$
 $x' \subseteq V_{x'}$

claim 1

Now we define

$$f: x' \rightarrow \text{Ord} \quad f(v) = \min \{ \alpha : v \in V_\alpha \}$$

By replacement axiom:

$\text{Rng } f$ is a set. Hence $\bigcup \text{Rng } f = \beta \in \text{Ord}$.

$$x' \subseteq V_\beta \Rightarrow x' \in V_{\beta+1} \quad \Downarrow$$

Thm (on transfinite induction)

Let $\varphi(\alpha, \bar{y}) \in \mathcal{F}_{LZF}$.

If $\varphi(0, \bar{y})$ and $(\forall \alpha \in \text{Ord}) (\forall \beta < \alpha \Rightarrow \varphi(\beta, \bar{y}) \rightarrow \varphi(\alpha, \bar{y}))$

then $\forall \alpha \in \text{Ord} \quad \varphi(\alpha, \bar{y})$.

V is very ample (obszerny): ^(practically) interprets all mathematics

Troubles (1) V is too ample (allows existence of pathological objects, like Banach-Tarski paradox)

5. Wymagania techniczne i edytorskie

- 5.1 Zaleca się przygotowywanie prac dyplomowych przy użyciu programu Tex, z uwagi na przystosowanie tego programu do profesjonalnego składu tekstów matematycznych. Dopuszczalne jest przygotowywanie prac dyplomowych przy użyciu programu Microsoft Word lub podobnych edytorów tekstu, pod warunkiem zachowania zasad składu tekstów matematycznych.
- 5.2 Strona tytułowa pracy dyplomowej powinna być zgodna ze wzorem umieszczonym na stronie Instytutu Matematycznego (zaktądka Praca dyplomowa).
- 5.3 Strony pracy dyplomowej powinny być numerowane zaczynając od strony tytułowej.

(2) There are simple L_{ZF} -sentences undecidable in ZFC, e.g. CH: $(\forall X \subseteq \mathbb{R})$
 (X countable or $X \sim \mathbb{R}$)

(7)
LR/3

Positive aspects of V:

V ~~is not~~ interprets all mathematics

$$\mathbb{N} = \{0, 1, \dots\} = \omega \in V_{\omega+1}. \quad \mathbb{R} = \mathcal{P}(\mathbb{N}) \in V_{\omega+2}.$$

$$\mathbb{R}^{\mathbb{R}} \in V_{\omega+5} \text{ (mathematical analysis)}$$

CH:

① Gödel (~1930) $\vdash \text{Con ZFC} \rightarrow \text{Con}(ZFC + CH)$

② Cohen (~1960) $\vdash \text{Con ZFC} \rightarrow \text{Con}(ZFC + \neg CH)$

Ad ①: constructible universe L:
 (Gödel universe)

$$L_0 = \emptyset, \quad L_{\alpha+1} = \text{Def}(L_\alpha, \in) =$$

$$= \{X \subseteq L_\alpha : \exists \varphi(x, \bar{y}) \in \mathcal{F}_{L_{ZF}} \exists \bar{a} \subseteq L_\alpha$$

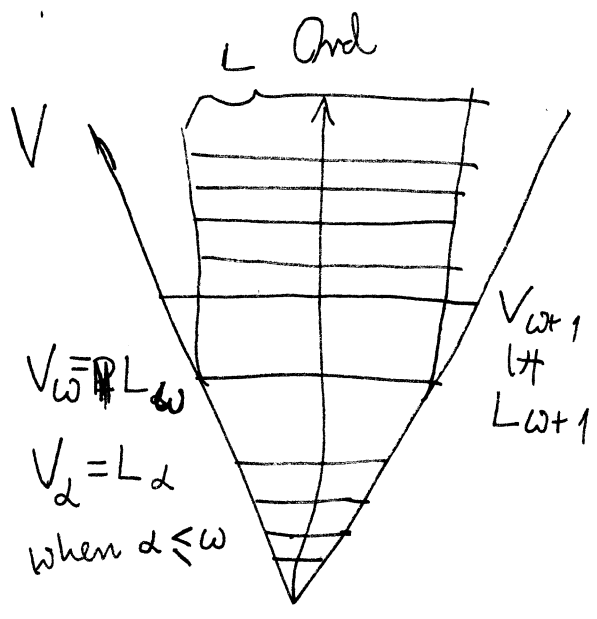
$$X = \{b \in L_\alpha : (L_\alpha, \in) \models \varphi(b, \bar{a})\}$$

how to write it down in L_{ZF} ?

$$L_\delta = \bigcup_{\alpha < \delta} L_\alpha$$

Lim

$$L = \bigcup_{\alpha \in \text{Ord}} L_\alpha : \text{a proper class}$$



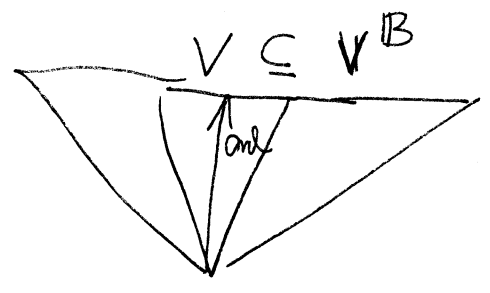
Fact

- (1) $Ord \subseteq L \subseteq V$
- (2) $\alpha \subseteq L_{\alpha} \subseteq V_{\alpha}, \alpha \in L_{\alpha+1}$
- (3) "L ⊨ ZFC + CH"
(but: L is not a set)
- (4) there is a total well-ordering on L
- (5) $L \models (ZFC + V=L)$

Ad 3 Cohen forcing, boolean models:

Idea 1. ^{logical} ~~boolean~~ values of sentences $\in B$: a boolean algebra.
 $\varphi \Rightarrow \llbracket \varphi \rrbracket_B \in B.$

2. Construct a boolean universe $V^B \supseteq V$ consisting of "boolean terms"



$V_0^B = \emptyset$

$V_{\alpha+1}^B = \{ f : f \text{ a function, } \text{Dom } f \subseteq V_{\alpha}^B, \text{Rng } f \subseteq B \}$

$f = \text{a set of pairs } \langle x, b \rangle, \text{ where } x \in V_{\alpha}^B, b \in B$

5.1 Zaleca się przygotowywanie prac dyplomowych przy użyciu programu Tex, z uwagi na przystosowanie tego programu do profesjonalnego składu tekstów matematycznych. Dopuszczalne jest przygotowywanie prac dyplomowych przy użyciu programu Microsoft Word lub podobnych edytorów tekstu, pod warunkiem zachowania zasad składu tekstów matematycznych.
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Idea $\llbracket t \in f \rrbracket_B = b$, where $\langle t, b \rangle \in f$.

(9)
LR/3

B chosen so that:

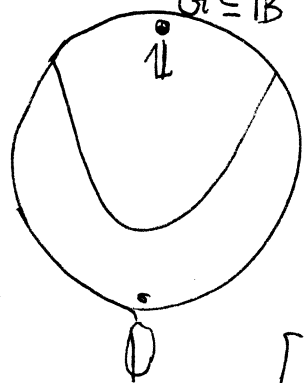
• $\llbracket \varphi \rrbracket_B = \perp_B$ for every $\varphi \in ZFC$

• $\llbracket CH \rrbracket_B \neq \perp_B$

hence: If ZFC is consistent, then $ZFC \not\vdash CH$

3. Additional trick. (Solovay, 1970)

$G \subseteq B$: a filter, even ultrafilter, so $B/G \cong \{0, 1\}$



" $V^{B/G}$ " = $V[G]$: a "real model"

G : a generic ultrafilter in B of $ZFC + \neg CH$

[this trick is possible under some additional assumptions, like that V : countable]

But:

Gödel $\vdash \text{Con}(ZFC) \rightarrow ZFC \not\vdash \text{Con}(ZFC)$

i.e.: "ZFC is consistent".

So if ZFC is consistent, then $ZFC \not\vdash \exists M \models ZFC$.

(noting to say of a countable $M \models ZFC$)

But. Def κ is (strongly) inaccessible

if $\kappa > \aleph_0$, κ regular and $\forall \alpha < \kappa$ $2^\alpha < \kappa$.
(i.e. $f(\kappa) = \kappa$)

Fact κ inaccessible $\Rightarrow (V_\kappa, \in) \models ZFC$.

Corollary:

LR/3

$\underbrace{ZFC + \text{"}\exists x: \text{inaccessible"}}$ $\vdash \text{Con}(ZFC)$

||

call it: ZFC'

But: (Gödel thm). $\text{Con } ZFC' \Rightarrow ZFC' \not\vdash \text{Con}(ZFC')$

How to write down a sentence: "Con ZFC"?

$L_{ZF} \subseteq V_w$ (a natural interpretation or a Gödel numbering)

Remark The relation $\text{Sat}(M, \varphi, \bar{a}) \Leftrightarrow (M, \mathcal{E}) \models \varphi(\bar{a})$

[where $\varphi \in L_{ZF}$, $\bar{a} \in M$]

is definable $\varphi(\bar{x})$ by a formula of L_{ZF} [exercise].

[But (thinking of L_{ZF} as a subset of V_w):

the relation $\text{True}(\varphi) \Leftrightarrow V \models \varphi$ is not definable in V

Warning!

(Tarski thm on non-definability of truth, a special case)

Hence: research on "large cardinal numbers"

For example: $\kappa \in \mathcal{C}N$ is measurable if $\exists \kappa > \aleph_0$ and

5.3 Strony pracy dyplomowej powinny być numerowane zaczynając od strony tytułowej.

5.2 Strona tytułowa pracy dyplomowej powinna być zgodna ze wzorem umieszczonym na stronie Instytutu Matematycznego (zaktądka Praca dyplomowa).

5.1 Zaleca się przygotowywanie prac dyplomowych przy użyciu programu Tex, z uwagi na przystosowanie tego programu do profesjonalnego składu tekstów matematycznych. Dopuszczalne jest przygotowywanie prac dyplomowych przy użyciu programu Microsoft Word lub podobnych edytorów tekstu, pod warunkiem zachowania zasad składu tekstów matematycznych.

and $\exists U \subseteq P(\mathcal{N})$ an ultrafilter, closed
 (non-principal) under
 intersections of
 length $< \aleph$.

[like a \aleph -additive (0-1)-measure on $P(\mathcal{N})$]

Def. An ultrafilter U on I is called \aleph -complete if U is closed under intersections of length $< \aleph$.

Remark ZFC \vdash Peano arithmetic PA is consistent
 (as ZFC \vdash IN \neq PA).

Interpretations in ZFC:

- an ordered pair: $\langle a, b \rangle = \{\{a\}, \{a, b\}\}$
 remembers \uparrow (Kuratowski) \nwarrow may
 of its 1. and 2. Coordinate be defined
 Coordinate in many ways.

• more generally:

$$\langle a_1, \dots, a_n \rangle = \langle a_1, \langle a_2, \dots, \langle a_{n-1}, a_n \rangle \dots \rangle \rangle$$

nested sharp parentheses.

• relation R in a set A :

$$R \subseteq A^k = \{\langle a_1, \dots, a_k \rangle : a_i \in A\}$$

• function $f: A \rightarrow B$ " $=$ " 1-valued relation
 $f \subseteq A \times B$.

• similarly formulas of L_{ZF} may be regarded

as some elements of V_ω (so that $L_{ZF} \subseteq V_\omega$)

• logical connectives, quantifiers as some operators

on $L_{ZF} \subseteq V_\omega$

$\wedge (\varphi, \psi) = \varphi \wedge \psi, \quad \exists x: \varphi \dots$

Assume $M = (A, R)$: an L -structure, where $L = \{ \underline{R} \}$
 \uparrow finite $\neq \emptyset$ / from the real world. \uparrow binary relational symbol

Then $\exists M' \in V_\omega \quad M' \cong M.$

Proof (1) We find $A' \in V_\omega \quad |A'| = |A|.$

(2) We find $R' \subseteq A' \times A', R' \in V_\omega$ s.t.
 $\langle A', R' \rangle \cong \langle A, R \rangle$

(3) $M' = \langle A', R' \rangle \in V_\omega. \quad M' \cong M, \text{ but}$
 ~~$M \cong M$~~ M matters up to \cong !

Alternative set theories:

Example : New Foundations : NF , Quine ~~in~~ ~ 1940.

$L = \{ L_{ZF} = \{ \emptyset \}.$

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Idea: avoid Russell contradiction

(13)
LR/3

$$\{x : \underbrace{x \notin x} \}$$

$$\neg x \in x$$

strange formula

Avoid strange formulas.

Def $\varphi \in L_{ZF}$ is stratifiable if

$$\exists f : \text{Var}(\varphi) \rightarrow \mathbb{N} \quad \forall x, y \in \text{Var}(\varphi)$$

all variables in φ

$$\left\{ \begin{array}{l} \text{If } x=y \text{ appears in } \varphi, \text{ then} \\ f(x) = f(y) \\ \text{If } x \in y \text{ appears in } \varphi, \text{ then} \\ f(y) = f(x) + 1. \end{array} \right.$$

Axioms of NF:

1. Extensionality: $x \forall t (t \in x \leftrightarrow t \in y) \rightarrow x = y$

2. "Stratified" ^{full} comprehension:

Assume $\varphi(x, \bar{y})$ is stratifiable. Then

$$\exists z \forall x (x \in z \leftrightarrow \varphi(x, \bar{y})) \quad (\text{here } z = \{x : \varphi(x, \bar{y})\})$$

[NF is finitely axiomatizable

. In NF: there is a set of all sets.

Troubles with set theory (ZFC) as a "metatheory" for mathematics:

- too many sets \rightarrow byproduct: pathological objects.
- independence of ZFC } of fundamental conjectures.
undecidability in ZFC }

Reaction:

- restrict to objects, whose existence is not problematic:
 - computable objects.

Computability:

Objects: for example natural numbers represented as:

$n \leftrightarrow \underbrace{|||| \dots |}_n$, (n-many sticks, matches)

or $n = \underline{(011001)}$ $n = (1011001)_2$: binary representation.

Generally:

$\emptyset \neq \Sigma$: a finite set of "concrete" objects.. e.g. $\Sigma = \{0, 1\}$.
("alphabet")

$\Sigma^* = \{ \text{finite tuples of elements of } \Sigma \}$ (
words over $\Sigma \leftarrow$ still concrete objects.
(~~computable~~)

For example

$\mathbb{N} \approx \Sigma^*$ for $\Sigma = \{1\}$ or $\mathbb{N} \approx \Sigma^*$ for $\Sigma = \{0, 1\}$.

Other ~~computable~~ concrete objects:

- subsets of Σ^* , but: not all! (like in ZFC)

intention: identify concrete subsets of Σ^*
"Computable."

LR-N4/2

Similarly: $f: \Sigma^* \rightarrow \Sigma^*$ we want to focus on
"concrete" = computable functions.

Computable = ?

Turing Machine TM: an abstract ~~comp~~ computer.

Alan Turing... died ~1956?

[there are many other equivalent formalizations
of computability]

Let Σ : a finite alphabet.

Turing machine M over Σ consists of:

(1) working ^{scanners/writers} heads G_0, \dots, G_n (głowie robocze)

(2) working tapes T_1, \dots, T_n ; input tape T_0 .
taśmę

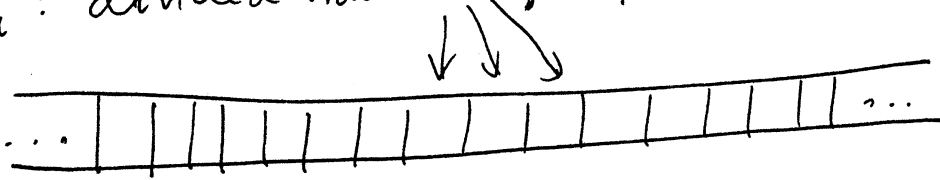
5. Wymagania techniczne i edytorskie

5.1 Zaleca się przygotowywanie prac dyplomowych przy użyciu programu Tex, z uwagi na przystosowanie tego programu do profesjonalnego składu tekstów matematycznych. Dopuszczalne jest przygotowywanie prac dyplomowych przy użyciu programu Microsoft Word lub podobnych edytorów tekstu, pod warunkiem zachowania zasad składu tekstów matematycznych.

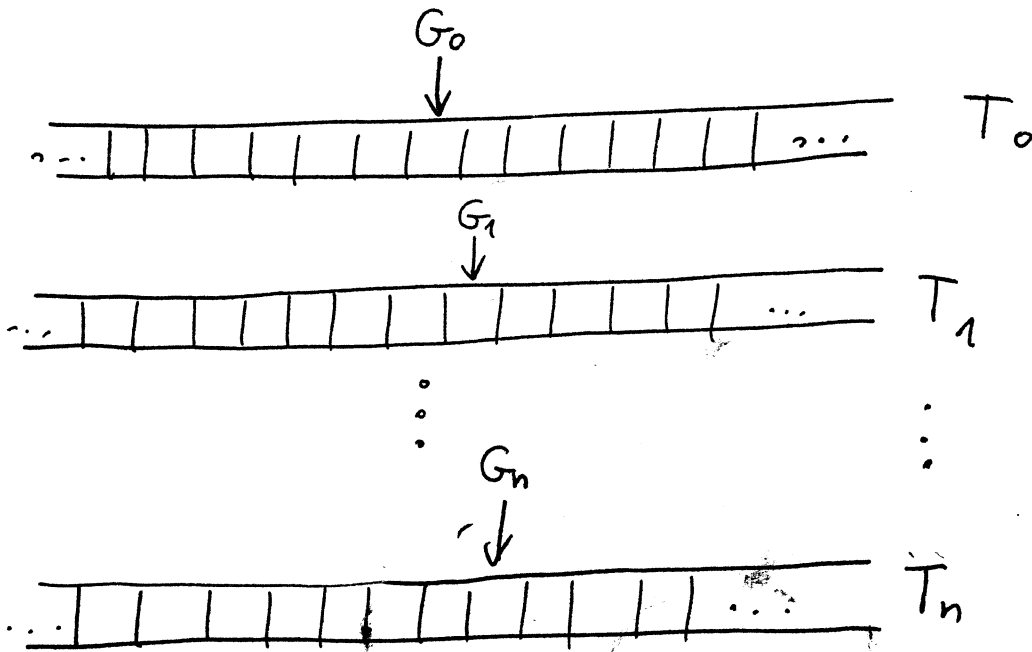
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Tape T_i : divided into cells, left- and right-infinite. LR-N⁴/3



M:



Each head G_i sees a single cell of T_i , each cell contains a letter $\in \Sigma$ or is empty [in any given moment] blanc B

(3). A finite set S of states of M

• A transition function $f: S \times (\Sigma \cup \{B\})^{n+1} \rightarrow S \times (\Sigma \cup \{B\})^{n+1} \times \{L, R, \emptyset\}$

• distinguished states $\in S$:

- s_0 : initial state
- end: final state
- yes, no $\in S$.

Operation of M : in time, divided into moments: $0, 1, 2, 3, \dots$

in steps $t = 1, 2, 3, \dots$

Step t : ~~the~~ operation of M between moment $t-1$ and moment t .

(1) configuration of M in moment t:

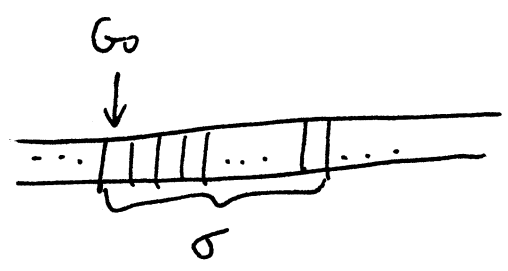
- (a) each cell of each T_i contains a letter $\in \Sigma$ or $\sqrt{13}$ blanc (B)
- (b) each head G_i sees a single cell of T_i with content $c_i \in \Sigma \cup \{B\}$.
- (c) M is in a state $s \in S$.

(2) step t+1 of M (from moment t to moment t+1)

- (a) calculates $f(s, c_0, \dots, c_n) = (s', c'_0, \dots, c'_n, v_0, \dots, v_n)$, $\forall v_i \in \{L, R, 0\}$
- (b) replaces content c_i of the cell of T_i scanned by G_i , by c'_i .
- (c) $\begin{cases} \text{if } v_i = L, \text{ moves } G_i \text{ one cell left} \\ \text{if } v_i = R, \text{ moves } G_i \text{ one cell right} \\ \text{if } v_i = 0, \text{ does not move } G_i. \end{cases}$
- (d) changes the state of M from s to s'.

(3) configuration of M in moment t=0:

- state $s = s_0$
 - on T_0 : an initial word $\sigma \in \Sigma^*$
- G_0 sees the cell with the first letter of σ



for $i > 0$ G_i sees a cell of T_i , \forall all cells of T_i are empty (blanc).

(4) in moment t:

- if $s = \text{end}$, yes or no, then M ends operation. terminates

Spróbuji ponownie

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(a) if $s = \text{end}$, then the word ^{currently} written on T_0 is called the outcome of M on input σ .
 ↑
 initial word.

(b) if $s = \text{yes}$, we say that M accepts σ

(c) if $s = \text{no}$, we say that M rejects σ .

Def Let $L \subseteq \Sigma^*$. M recognizes $L \Leftrightarrow$
 ↑
 "language"
 $(\forall \sigma \in \Sigma^*) \begin{cases} \sigma \in L \Rightarrow M \text{ accepts } \sigma \\ \sigma \notin L \Rightarrow M \text{ rejects } \sigma \end{cases}$

Def. Let $f: \Sigma^* \dashrightarrow \Sigma^*$. M computes $f \Leftrightarrow \forall \sigma \in \Sigma^*$
 ↑
 partial function
 (i.e. $\text{Dom } f \subseteq \Sigma^*$)
 $\begin{cases} f(\sigma) \downarrow \Rightarrow \text{on input } \sigma \text{ } M \text{ terminates} \\ \text{with } \text{output } f(\sigma) \\ f(\sigma) \uparrow \Rightarrow \text{on input } \sigma \text{ } M \\ \text{does not terminate its} \\ \text{operation.} \end{cases}$

where:

$f(\sigma) \downarrow = \text{"}\sigma \in \text{Dom } f\text{"}$, $f(\sigma) \uparrow = \text{"}\sigma \notin \text{Dom } f\text{"}$.

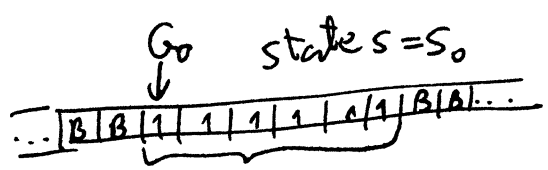
Def

- (1) $L \subseteq \Sigma^*$ is TM-computable $\Leftrightarrow \exists M: \text{TM}$ M recognizes L
- (2) $f: \Sigma^* \dashrightarrow \Sigma^*$ is TM-computable $\Leftrightarrow \exists M: \text{TM}$ M computes f .

Example Let $\Sigma = \{1\}$, $L = \{ \underbrace{1 \dots 1}_n : n \text{ even} \} \approx \{ \text{even numbers} \}$

L is TM-computable:

M with 1 tape only: T_0



transition function: (a) $f(s_0, B) = (\text{yes}, B, 0)$

(b) $f(s_0, 1) = (s_1, 1, R)$

$$(c) f(s_1, B) = (n_0, B, 0)$$

$$(d) f(s_1, 1) = (s_0, 1, R)$$

Representation of \mathbb{N} :

$$(a) \Sigma = \{1\}, \mathbb{N} \approx \Sigma^*$$

$$(b) \Sigma = \{0, 1\}, \mathbb{N} \approx \Sigma^*$$

so natural numbers \approx words over Σ . binary representation.

Def. $L \subseteq \mathbb{N}$ is TM-computable $\Leftrightarrow \exists M: \text{TM}$ M recognizes L

$f: \mathbb{N} \rightarrow \mathbb{N}$ is TM-computable $\Leftrightarrow \exists M: \text{TM}$ M computes f

Different approach to COMPUTABILITY.

Recursive functions $f: \mathbb{N}^n \rightarrow \mathbb{N}$:

(a) basic functions: $S: \mathbb{N} \rightarrow \mathbb{N}$, $S(x) = x + 1$
successor function

$$0: \mathbb{N}^n \rightarrow \mathbb{N}, 0(\bar{x}) = 0$$

$$I: \mathbb{N} \rightarrow \mathbb{N}, I(x) = x, \quad I_j^m: \mathbb{N}^m \rightarrow \mathbb{N}$$

$$I_j^m(x_1, \dots, x_n) = x_j$$

(b) defining schemes:

(a) composition: Given $f(x_1, \dots, x_n)$, $g_1(\bar{y}_1), \dots, g_n(\bar{y}_n)$

$$\text{obtain } h(\bar{y}_1, \dots, \bar{y}_n) = f(g_1(\bar{y}_1), \dots, g_n(\bar{y}_n))$$

(b) simple recursion:

Given $f(\bar{x})$, $g(\bar{x}, y, z)$ obtain $h(\bar{x}, y)$ such that

$$\begin{cases} h(\bar{x}, 0) = f(\bar{x}) \\ h(\bar{x}, n+1) = g(\bar{x}, n, h(\bar{x}, n)) \end{cases}$$

Spróbuj ponownie

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(3) operation minimum:

given $f(\bar{x}, y)$ obtain $h(\bar{x})$ such that

$$h(\bar{x}) = \min \{y : f(\bar{x}, y) = 0\}.$$

Warning to defining schemes (1)-(3):

functions f, g may be partial, then h also may be partial:

$$\text{Ad (1): } h(\bar{y}_1, \dots, \bar{y}_n) \downarrow \Leftrightarrow g(\bar{y}_1) \downarrow, \dots, g(\bar{y}_n) \downarrow \text{ and } f(g(\bar{y}_1), \dots, g(\bar{y}_n)) \downarrow$$

$$\text{Ad (2): } h(\bar{x}, 0) \downarrow \Leftrightarrow f(\bar{x}) \downarrow$$

$$\text{Ad (2): } h(\bar{x}, n+1) \downarrow \Leftrightarrow h(\bar{x}, n) \downarrow \text{ and } g(\bar{x}, n, h(\bar{x}, n)) \downarrow$$

Ad (3):

$$h(\bar{x}) \downarrow \Leftrightarrow \text{there is } y \text{ s.t. } f(\bar{x}, y) = 0 \text{ and}$$

$$\forall y' < y (f(\bar{x}, y') \downarrow \text{ and } f(\bar{x}, y') \neq 0)$$

Def. Rec = the smallest family of functions $f: \mathbb{N}^n \rightarrow \mathbb{N}, n \geq 0$, containing basic functions and closed under defining schemes.

• f is recursive $\Leftrightarrow f \in \text{Rec}$

Def. $A \subseteq \mathbb{N}^n$ is recursive $\Leftrightarrow \chi_A \in \text{Rec}$.

Examples:

$$+ : \begin{cases} x+0 = 0 = 0(x) \\ x+(n+1) = (x+n)+1 = S(x+n) \end{cases}$$

$$\cdot : \begin{cases} x \cdot 0 = 0 \\ x \cdot (n+1) = x \cdot n + x \end{cases}$$

The set \mathbb{P} of prime numbers is recursive.

The function $(n \mapsto p_n = n\text{-th prime number})$ is recursive.

Proof

$$(1) P(x) : \begin{cases} P(0) = 0 \\ P(n+1) = n \end{cases} \quad \begin{array}{l} \text{predecessor function} \\ P \in \text{Rec.} \end{array}$$

$$(2) x \dot{-} y = \begin{cases} x-y, & \text{when } x \geq y \\ 0, & \text{when } x < y \end{cases} \quad \begin{cases} x \dot{-} 0 = x \\ x \dot{-} (n+1) = P(x \dot{-} n) \end{cases}$$

natural subtraction

$$(3) \text{ Let } H(x, y) = (x \dot{-} y) + (y \dot{-} x) : x = y \Leftrightarrow H(x, y) = 0$$

$$(4) f(y) \in \text{Rec} \Rightarrow f'(x) = \prod_{y < x} f(y) \text{ recursive.}$$

$$\begin{cases} \prod_{y < 0} f(y) = 1 = S(0) \\ \prod_{y < n+1} f(y) = f(n) \cdot \prod_{y < n} f(y) \end{cases}$$

$$(5) x \in \mathbb{P} \Leftrightarrow \forall y < x \forall z < x \quad y \cdot z \neq x$$

$$\Leftrightarrow \forall y < x \forall z < x \quad H(x, y \cdot z) \neq 0$$

$$\Leftrightarrow g(x) = \prod_{y < x} \prod_{z < x} H(x, y \cdot z) \neq 0$$

$$(6) \text{ Let } h(x) = \min(g(x), 1) = 1 \dot{-} (1 \dot{-} g(x)), h: \mathbb{N} \rightarrow \{0, 1\}$$

[simulacja $F: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$]

$$x \in \mathbb{P} \Leftrightarrow h(x) = 1$$

$$F(x, y) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$$

$$\text{Rec} \ni h = \chi_{\mathbb{P}} \text{ so } \mathbb{P} \in \text{Rec.}$$

Spróbuj ponownie

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$$(7) p_0 = 2$$

$$p_{n+1} = \min \{ x : x > p_n \text{ and } x \in \mathbb{P} \}$$

$$= \min \{ x : (p_{n+1}) \dot{-} x = 0 \text{ and } 1 \dot{-} h(x) = 0 \}$$

$$= \min \{ x : ((p_{n+1}) \dot{-} x) + (1 \dot{-} h(x)) = 0 \}$$

Thm (1) $A \subseteq \mathbb{N}^n$ is recursive \Leftrightarrow A is TM-computable

(2) $f: \mathbb{N}^n \rightarrow \mathbb{N}$ is recursive \Leftrightarrow f is TM-computable.

Proof. \Rightarrow obvious.

[Basic functions are TM-computable,

TM-computable functions are closed under defining schemes]

\Leftarrow (2), $n=1$. sketch.

Assume M : TM computing $f: \mathbb{N} \rightarrow \mathbb{N}$

(under some representation of natural numbers as words).

Assume M has k tapes, the set of states S , transition function F .

• $\text{conf}(M) = [\text{content of tapes, state}(M), \text{positions of working heads of } M]$

Configuration

\approx coded as a natural number n .

for example $n = p_0^{\epsilon_0} p_1^{\epsilon_1} \dots p_k^{\epsilon_k}$ codes $\langle \epsilon_0, \epsilon_1, \dots, \epsilon_k \rangle$.

We define $g: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$

• $g(t, n) = \text{configuration of } M \text{ on input } n, \text{ in moment } t$.

• $g \in \text{Rec}$

Let $h(n) = \min \{ t : M \text{ stops in moment } t, \text{ on input } n \}$

[state s recovered from $g(t, n)$
is end]

~~Therefore~~

Let $f(n) =$ content of the input tape in moment $h(n)$ [R-N4/10]

(may be recursively recovered from $g(n, h(n))$).

Therefore $f \in \text{Rec}$.

Def $A \subseteq \mathbb{N}$ is recursively enumerable \Leftrightarrow

$$A = \emptyset \text{ or } \exists f: \mathbb{N} \rightarrow \mathbb{N} \text{ recursive } A = \text{Rng}(f)$$

Church thesis, Assume $A \subseteq \mathbb{N}$.

Then A is recursive $\Leftrightarrow A$ is computable
(i.e. there is an algorithm determining, for $n \in \mathbb{N}$, if $n \in A$)

Fact Assume $A \subseteq \mathbb{N}$. If both A and $\mathbb{N} \setminus A$ are recursively enumerable.

Proof wlog $A \neq \emptyset \neq \mathbb{N} \setminus A$. Choose total recursive f, g with $A = \text{Rng } f, \mathbb{N} \setminus A = \text{Rng } g$.

An algorithm determining for $n \in \mathbb{N}$: if $n \in A$

1. ~~total~~ Compute $f(0), g(0), f(1), g(1), \dots$

2. When in sequence $f(0), g(0), f(1), g(1), \dots$, n appears
then answer if $n = f(i)$, then $n \in A$
if $n = g(i)$, then $n \notin A$

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~~the~~ We list those $x \leq i$ such that in this stage $f(x, x)$ is computed and equals 0.

In this way we create an ^(infinite) recursive list $\{ \}$ of natural numbers, enumerating A ,

[in each stage we add finitely many members to the list]

(3) A is not recursive.

proof (a. a.) Suppose $\chi_A \in \text{Rec}$. Then $\chi_A(\cdot) = f(n, \cdot) \neq 0$ for some $n \in \mathbb{N}$

and $f(n, \cdot)$ is total

Then $f(n, n) = 0 \Leftrightarrow n \in A \Leftrightarrow \chi_A(n) \neq 0 \Leftrightarrow f(n, n) \neq 0 \quad \Downarrow$

Thm: There is a set $A \subseteq \mathbb{N}$, recursively enumerable, but not recursive

Proof: (1) $\exists f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ recursive $(\forall g) \mathbb{N} \rightarrow \mathbb{N}$ (r.e.) $\exists n$ $f(n, \cdot) = g(\cdot)$

f - a universal recursive function - for every partial recursive function there is a section equal to g
(effectively)

- We enumerate the recipes for recursive functions:
 $\alpha_0, \alpha_1, \dots$ (by algorithm)

$$f(n, m) = [\alpha_n \text{ applied to } m]$$

(2) Let $A = \{x \in \mathbb{N} : f(x, x) = 0\}$: rec. enumerable ^{recursive}

Proof: An algorithm, acts in steps, generates a list of nat numbers
[step i] we render i steps in calc. ~~of~~ of $f(x, x)$ for all x .
whenever $f(x, x)$ is calculated, we add it to the list

We have enumerated A .

(3) A is not recursive:

Suppose $\chi_A \in \text{Rec}$, then $\chi_A(\cdot) = f(n, \cdot)$ for some n

$f(n, \cdot)$ is total

$f(n, n) = 0 \Leftrightarrow n \in A \Leftrightarrow \chi_A(n) = 1 \Leftrightarrow f(n, n) \neq 0$, a contradiction

Languages, coding, decidability

Let $L = \{P_i, f_j, c_k\}_{i,j,k} -$ a finite language.

Formulas of L are computable objects -
having a string of symbols we can decide whether it is
indeed a formula

$$\mathcal{F}_L = \left(L \cup \{ (,), \forall, \exists, \wedge, \neg, \top \} \right)^*$$

\uparrow
 $x_2 = x_{11}$

We will think of formulas as of numbers (Gödel coding)

$$\begin{matrix} (,), \forall, \exists, \wedge, \neg, \top, \forall, P, f, c \\ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \end{matrix} \quad \leftarrow P_2 = P_{11}$$

Ex: $\forall x_3 P_1(x_3) \quad \leftarrow \varphi$

$\forall x_{111} P_1(x_{111}) \quad \leftarrow \varphi$ in our language



$\langle 9, 3, 4, 4, 4, 10, 4, 1, 3, 4, 4, 4, 2 \rangle \rightsquigarrow$ the Gödel code

$$2^9 \cdot 3^3 \cdot 5^4 \cdot 7^4 \cdot 11^4 \cdot 13^{10} \cdot 17^4 \cdot 23^1 \cdot 29^3 \cdot 31^4 \cdot 37^4 \cdot 41^4 \cdot 43^2 = \ulcorner \varphi \urcorner$$

(computable)

Def. (1) $A \subseteq \mathcal{F}_L$ is recursive $\Leftrightarrow \{ \ulcorner \varphi \urcorner : \varphi \in A \}$ recursive $\Leftrightarrow A$ is TM-computable

(2) $A \subseteq \mathcal{F}_L$ is recursively enumerable $\Leftrightarrow \{ \ulcorner \varphi \urcorner : \varphi \in A \}$ is r.e. $\Leftrightarrow A$ is TM-computable enumerable

$\Leftrightarrow A = \emptyset$ or there is a total
TM-computable $f: \mathbb{N} \rightarrow \mathcal{F}_L$ with

$$A = f[\mathbb{N}]$$

Example: $\{\varphi \in \mathcal{F}_L : \vdash \varphi\}$ is r.e.

We have a semi-algorithm

Given φ :

we write down all formal proofs (in KRL, L)

we look at their conclusions

if φ is a conclusion, we stop and answer yes

Def. $T \subseteq \mathcal{F}_L$ is decidable if T is recursive

Thm. If T is r.e. and complete, then T is decidable.

Peano arithmetic

TA-true arithmetic

Language: $L_{PA} = \{+, \cdot, 0, S, <\}$ $PA \subseteq Th(\mathbb{N}, +, \cdot, 0, S, <)$

Classically: primitive notions: 0 constant S successor

Axioms: 1) $0 \neq Sx$

2) $Sx = Sy \Rightarrow x = y$

induction scheme 3) $\varphi(x, \dots)$ a formula in our lang. $[\varphi(0, \dots) \wedge \forall x (\varphi(x, \dots) \rightarrow \varphi(Sx, \dots))] \rightarrow \forall x \varphi(x, \dots)$

+ a rule for introducing new function symbols.

Suppose f, g - function symbols of suitable arities

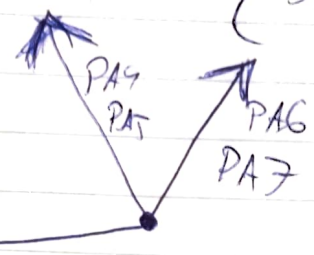
Then we introduce a function symbol h and new axioms for it

$$\begin{cases} h(0, \bar{x}) = f(\bar{x}) \\ h(S_n, \bar{x}) = g(\bar{x}, n, h(n, \bar{x})) \end{cases}$$

(simple) recursion

Example: $+$: $\begin{cases} 0+y=x \\ S_n+x = S(n+x) \end{cases} \therefore \begin{cases} 0 \cdot x = 0 \\ S_n \cdot x = n \cdot x + x \end{cases}$

PA1: $0 \neq Sx$
 PA2: $Sx = Sy \rightarrow x = y$
 PA3: the induction scheme
 PA4 - PA7



- PA is undecidable (Rosser)
- PA is incomplete (Gödel)

Example: $PA \vdash \cancel{\varphi(x)} \quad x+0=x$

- $\varphi(0)$ (by PA4) for $x=0$
- Suppose $\varphi(x)$. We will show $\varphi(Sx)$
 $x+0=x \qquad Sx+0=Sx$

From PA5 we have $S_n+0 = S(x+0) = Sx$
 Now we apply PA3 (φ).

Similarly: $PA \vdash +, \cdot$ are associative, commutative, distributive and " \ll " $\left[x \ll y \stackrel{df}{\iff} \exists z (x+z=y) \right]$ is correct

For $n \in \mathbb{N}$ let $\underline{n} = \underbrace{S \dots S}_n 0 \in \mathcal{F}_{L_{PA}}$. It is called a numeral

Lemma (representability of recursive sets and functions in PA)

- (1) Assume $A \subseteq \mathbb{N}$ is recursive. Then there is a formula $\varphi_A(x) \in \mathcal{F}_{L_{PA}}$ s.t.
 $(\forall n \in \mathbb{N}) \quad n \in A \Rightarrow PA \vdash \varphi_A(\underline{n})$ and $n \notin A \Rightarrow PA \vdash \neg \varphi_A(\underline{n})$
- (2) Assume $f: \mathbb{N}^k \rightarrow \mathbb{N}$. Then there is a formula $\varphi_f(\bar{x}, y) \in \mathcal{F}_{L_{PA}}$ s.t.
 $PA \vdash (\forall \bar{x}) \left(\exists^{rec} y \right) \varphi_f(\bar{x}, y)$ and $\forall \bar{n} \in \mathbb{N}^k$ (if $f(\bar{n})$ then $PA \vdash \varphi_f(\underline{\bar{n}}, f(\bar{n}))$)

consider λ_4

Proof: Enough to prove (2).

Induction on the length of def of f

1° the basic functions: obvious

2° composition scheme: obvious

minimum operation: $f(\bar{n}) = \min \{y : g(\bar{n}, y) = 0\}$

$\varphi_f(\bar{x}, y)$ represented by $\varphi_g(\bar{x}, y, z)$ ind. hyp.

$$\varphi_g(\bar{x}, y, 0) \wedge (\forall y' < y) (\exists z \varphi_g(\bar{x}, y, z) \wedge \neg \varphi_g(\bar{x}, y', 0))$$

$\varphi_f(\bar{x}, y)$ represents f .

(a) Assume $f(\bar{n}) \downarrow = k$ then $g(\bar{n}, k) = 0$ and $(\forall k' < k) g(\bar{n}, k') \downarrow \neq 0$

$$PA \vdash \varphi_g(\bar{n}, k, 0) \wedge (\forall y' < k) (\exists z \varphi_g(\bar{n}, y', z) \wedge \neg \varphi_g(\bar{n}, y', 0))$$

Fact: $PA \vdash x \leq k \Leftrightarrow (x = 0 \vee \dots \vee x = k-1)$

(b) $PA \vdash (\exists^{< \omega} y) \varphi_f(\bar{x}, y) \Leftrightarrow \text{ev}$

(d) recursion scheme:

$$\begin{cases} h(0, \bar{x}) = f(\bar{x}) \\ h(S_n, \bar{x}) = g(\bar{x}, n, h(n, \bar{x})) \end{cases}$$

We have φ_f, φ_g rep. f, g
We want φ_h rep. h .

Trick: coding sequences

Idea: $h(n, \bar{x}) = m \Leftrightarrow (\exists \langle e_0, \dots, e_n \rangle \left[\begin{cases} a_0 = f(\bar{x}) \\ (\forall i < n) [e_{S_i} = g(\bar{x}, i, e_i)] \\ e_n = m \end{cases} \right]$

We use the Chinese remainder theorem to make it quantify over 1 thing

Notes 5.

Thm (Church) PA is undecidable.

Proof (a.a.) Suppose PA: decidable, i.e.

the set $\{\ulcorner \varphi \urcorner : PA \vdash \varphi\}$ recursive.

Let $\{\varphi_0(x), \varphi_1(x), \dots\}$: a recursive enumeration of all formulas of L_{PA} with free variable x .

Let $A = \{n \in \mathbb{N} : PA \vdash \neg \varphi_n(\underline{n})\}$.

PA: decidable \Rightarrow A recursive.

By representability lemma there is a formula $\varphi_A(x)$ representing A.
" $\varphi_n(x)$ for some n .

So: $PA \vdash \varphi_A(\underline{n}) \stackrel{\text{Lemma}}{\Leftrightarrow} n \in A \stackrel{\text{def. A}}{\Leftrightarrow} PA \vdash \neg \varphi_n(\underline{n}) \quad \checkmark$
because PA: consistent (in ZFC)

Corollary (1) (Rosser) If $T \subseteq \overset{\mathcal{F}}{\underset{PA}{L_{PA}}}$ consistent theory, $PA \subseteq T$, then T is not decidable.

(2) (Gödel 1st incompleteness thm).

If $T \subseteq \mathcal{F}_{L_{PA}}$, T recursively enumerable, PA \subseteq T, then
theory consistent

T is incomplete.

Proof (1) the same ~~and~~ proof as Church thm.

(2) follows from (1).

Corollary (Turing, 1936)

LR.N5/2

There is no algorithm deciding if $\models \varphi$, for $\varphi \in \mathcal{F}_L$, $L \geq \{+, \cdot\}$

Proof Representability lemma holds also for a finite

$PA_0 \subseteq PA$ in place of PA .

(PA_0 needed to prove the Chinese remainder theorem...)

Therefore: PA_0 undecidable.

Suppose (a.a.) that $\{\varphi \in \mathcal{F}_L : \models \varphi\}$ is recursive.
the set

Then for $\varphi \in \mathcal{F}_{LPA}$ $PA_0 \vdash \varphi \iff \vdash \wedge PA_0 \rightarrow \varphi$
deduction
turn \Downarrow
 $\models \wedge PA_0 \rightarrow \varphi$
decidable,

so PA_0 : decidable, a contradiction.

Corollary. If ZFC is consistent, then ZFC is undecidable and incomplete.

Proof PA is interpretable in ZFC.

~~Corollary Assume $PA \subseteq T \subseteq TA$. Then there / later.
rec. enumerable
theory~~

Diagonal lemma.

For every formula $G(x) \in \mathcal{F}_{LPA}(x)$ there is a sentence

$F \in \mathcal{F}_{LPA}$ such that $PA \vdash F \leftrightarrow G(\ulcorner F \urcorner)$.

Proof

1. There is a recursive function $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$

s.t.

$$(\forall \varphi(x) \in \mathcal{F}_{LPA}(x)) (\forall n \in \omega) f(\ulcorner \varphi(x) \urcorner, n) = \ulcorner \varphi(n) \urcorner$$

(via TM-computability).

Idea:

Let $H(x) = "G(f(x, x))"$ (diagonal argument.)

Let $F = H(\ulcorner H(x) \urcorner)$.

Notice $f(\ulcorner H(x) \urcorner, \ulcorner H(x) \urcorner) = \ulcorner F \urcorner$

$$\begin{aligned} \text{So } G(\ulcorner F \urcorner) &\stackrel{\Leftrightarrow}{=} G(f(\ulcorner H(x) \urcorner, \ulcorner H(x) \urcorner)) = \\ &= H(\ulcorner H(x) \urcorner) \Leftrightarrow H(\ulcorner H(x) \urcorner) = F \end{aligned}$$

"provably" in PA

Formally.

represents $\otimes f(x, x)$

By Representability Lemma: $\varphi_f(x, y) \in \mathcal{F}_{LPA}$ s.t.

$$PA \vdash \forall x \exists^{<1} y \varphi_f(x, y) \text{ and}$$

for all $n, m \in \mathbb{N}$

$$f(n, n) = m \Rightarrow PA \vdash \varphi_f(\underline{n}, \underline{m})$$

$$H(x) = \exists y (\varphi_f(x, y) \wedge G(y))$$

$$\text{Let } F = H(\ulcorner H(x) \urcorner) = \exists y (\varphi_f(\ulcorner H(x) \urcorner, y) \wedge G(y))$$

$$\ulcorner \ulcorner H(x) \urcorner, \ulcorner H(x) \urcorner \urcorner = \ulcorner H(\ulcorner H(x) \urcorner) \urcorner$$

\Downarrow

$$PA \vdash \varphi_f(\ulcorner H(x) \urcorner, \ulcorner H(\ulcorner H(x) \urcorner) \urcorner)$$

$$\parallel \ulcorner F \urcorner \leftarrow F = H(\ulcorner H(x) \urcorner)$$

$$\text{unique } y \leftarrow PA \vdash \exists^{< y} \varphi_f(\ulcorner H(x) \urcorner, y)$$

$$PA \vdash F \iff G(\ulcorner F \urcorner):$$

In a model of PA (semantically):

\rightarrow Assume F , i.e.: $\underbrace{H(\ulcorner H(x) \urcorner)} = \ulcorner y \urcorner$

$$\exists y (\varphi_f(\ulcorner H(x) \urcorner, y) \wedge G(y))$$

$$PA \vdash \varphi_f(\ulcorner H(x) \urcorner, \ulcorner F \urcorner)$$

$$PA \vdash \exists^{< 1} y \varphi_f(\ulcorner H(x) \urcorner, y)$$

the only witness, so $G(\ulcorner F \urcorner)$.

\leftarrow : Assume $G(\ulcorner F \urcorner)$ holds.

But also $\varphi_f(\ulcorner H(x) \urcorner, \ulcorner F \urcorner)$ holds.

so: $\exists y (\varphi_f(\ulcorner H(x) \urcorner, y) \wedge G(y))$ holds.

so $PA \vdash F \iff G(\ulcorner F \urcorner)$.

Def A formula $G(x) \in \tilde{\mathcal{F}}_{LPA}(x)$

is a truth definition in ~~$(\mathbb{N}, +, \cdot, S, 0)$~~ $(\mathbb{N}, +, \cdot, S)$

iff $(\forall \varphi \in \tilde{\mathcal{F}}_{LPA})$ $\mathbb{N} \models G(\underline{\varphi}) \Leftrightarrow \mathbb{N} \models \varphi$.

sentence truth

Corollary (Tarski, undefinability of ~~truth~~).

There is no truth definition in $(\mathbb{N}, +, \cdot, S, 0)$

Proof Suppose (a.a.) $G(x)$ is a truth definition in \mathbb{N} . Apply the diagonal lemma for $\neg G(x)$.

Get $F \in \tilde{\mathcal{F}}_{LPA}$ s.t. $PA \vdash F \Leftrightarrow \neg G(\underline{F})$

sentence

hence ~~$\mathbb{N} \models F$~~ $\mathbb{N} \models F \Leftrightarrow \neg G(\underline{F})$

F says: "I am false" (Liar's paradox, self reference)

Hence $\mathbb{N} \models F \Leftrightarrow \mathbb{N} \not\models G(\underline{F}) \Leftrightarrow \mathbb{N} \not\models F$

\uparrow
 G : a truth definition. \square

Similarly in ZFC:

If ZFC is consistent, then there is no formula

$G(x) \in \tilde{\mathcal{F}}_{LZF}$ s.t. for every sentence $F \in \tilde{\mathcal{F}}_{LZF}$

$ZFC \vdash F \Leftrightarrow G(\underline{F})$.

Assume $PA \subseteq T \subseteq \tilde{T}_{LPA}$

↑
consistent, recursively enumerable set of sentences
~~theory~~

- there is a formula $Prov_T(x)$ s.t. for every sentence $\varphi \in \tilde{T}_{LPA}$

$$T \vdash \varphi \Leftrightarrow \mathbb{N} \models Prov_T(\ulcorner \varphi \urcorner)$$

More: $T \vdash \varphi \Leftrightarrow PA \vdash Prov_T(\ulcorner \varphi \urcorner)$
Stronger: ↑

Idea: unary relation on \mathbb{N}

$Prov_T(\ulcorner \varphi \urcorner) \Leftrightarrow \exists y \exists k$ (y is a proof of φ in T ,
of length k , using only the first k elements
of T , the first k axioms of $KRL\dots$)

$$\Leftrightarrow \exists y \exists k \left(\underbrace{\langle (y)_0, \dots, (y)_k \rangle, \ulcorner \varphi \urcorner}_{\text{recursive relation}} \right)$$

recursive relation

$$R(y, k, \ulcorner \varphi \urcorner)$$

↓
represented in PA by a formula

$$\varphi_R(y, k, z)$$

$$Prov_T(z) = \exists z \varphi \exists y \exists k \varphi_R(y, k, z).$$

Now let F be the sentence from the diagonal lemma for $G(x) = \neg \text{Prov}_T(x)$.

so $PA \vdash F \leftrightarrow \neg \text{Prov}_T(\ulcorner F \urcorner)$

F says (according to PA): ~~According~~

"I can not be proved in T "

$\text{Con}(T) = \neg \text{Prov}_T(\ulcorner 0=1 \urcorner)$: " T is consistent"

Fact (1) $T \not\vdash F$

(2) If $IN \neq T$, then $T \not\vdash \neg F$
(i.e. $T \subseteq TA$)

(3) $PA \vdash \text{Con}(T) \leftrightarrow F$

Proof (1). Suppose $T \vdash F$. Then $PA \vdash \text{Prov}_T(\ulcorner F \urcorner)$

$T \vdash \neg F \iff PA \vdash \neg F$
 $PA \subseteq T$

but T consistent \Downarrow

(2) Suppose $T \vdash \neg F \Rightarrow$ ~~$IN \neq T$~~

$\Downarrow T \supseteq PA$

$T \vdash \text{Prov}_T(\ulcorner F \urcorner) \Rightarrow IN \neq \text{Prov}_T(\ulcorner F \urcorner)$

\Downarrow
 $T \vdash F \Downarrow$

$$(3) \quad PA \vdash F \leftrightarrow \neg \text{Prov}_T(\ulcorner F \urcorner)$$

$$\Downarrow$$

$$PA \vdash \text{Prov}_T(\ulcorner F \leftrightarrow \neg \text{Prov}_T(\ulcorner F \urcorner) \urcorner)$$

$$\Downarrow \text{properties of } \text{Prov}_T$$

$$PA \vdash \text{Prov}_T(\ulcorner F \urcorner) \leftrightarrow \text{Prov}_T(\ulcorner \neg \text{Prov}_T(\ulcorner F \urcorner) \urcorner)$$

$$\text{But: } PA \vdash \text{Prov}_T(\ulcorner F \urcorner) \rightarrow \text{Prov}_T(\ulcorner \text{Prov}_T(\ulcorner F \urcorner) \urcorner)$$

$$(*)$$

$$\text{Hence: } PA \vdash \text{Prov}_T(\ulcorner F \urcorner) \rightarrow \neg \text{Con}(T)$$

$$\text{and } PA \vdash \text{Con}(T) \rightarrow \neg \text{Prov}_T(\ulcorner F \urcorner)$$

$$PA \vdash \text{Con}(T) \rightarrow F.$$

$$\leftarrow: \quad PA \vdash \neg \text{Con}(T) \rightarrow \text{Prov}_T(\ulcorner F \urcorner)$$

$$PA \vdash \neg \text{Con}(T) \rightarrow \neg F$$

$$PA \vdash F \rightarrow \text{Con}(T).$$

Corollary (Gödel's 2nd incompleteness theorem)

(1) If T is consistent, then $T \cup \{\neg \text{Con}(T)\}$ is consistent.

(2) If $\mathcal{N} \models T$, then $T \cup \{\text{Con}(T)\}$ is consistent.
(i.e. $T \models \text{Con}(T)$)

Proof (1)

$T \vdash F \leftrightarrow \text{Con}(T)$ and $T \not\vdash F$, so $T \not\vdash \text{Con}(T)$
 \Downarrow
 $T \cup \{\neg \text{Con}(T)\}$
 consistent

(2) Since T is consistent, $\mathbb{N} \models \text{Con}(T)$
 so if $\mathbb{N} \models T$ then $\mathbb{N} \models \underbrace{T \cup \{\text{Con}(T)\}}$,
 consistent.

Corollary Assume $\mathbb{N} \models T$.

Let $A(x) = "x \text{ is a proof of } \overset{0=1}{\cancel{0 \neq 1}} \text{ in } T"$.

so $\text{PA} \vdash \neg A(\underline{n})$ for every $n \in \mathbb{N}$

but $\text{PA}, T \not\vdash \forall x \neg A(x)$, because

$T \cup \{\neg \text{Con}(T)\}$ consistent,

If $M \models T \cup \{\neg \text{Con}(T)\}$ then $M \models \exists x \neg A(x)$
 \uparrow
 non-standard "proof"
 of $0=1$ in T .

Corollary

Similarly if ZFC is consistent,

then $\text{ZFC} \cup \{\neg \text{Con}(\text{ZFC})\}$ is consistent.

On (*):

(LR.N5/10)

Σ_1 -formulas in L_{PA} :

of the form $\exists \bar{x} \psi$, where in ψ only bounded quantifiers:

$$\exists x \leq y, \forall x \leq y.$$

Lemma Assume $D(x_1, \dots, x_n) \in \mathcal{F}_{L_{PA}}$ is Σ_1 -formula.

Then $PA \vdash D(x_1, \dots, x_n) \rightarrow \text{Prov}_{PA}(\ulcorner D(x_1, \dots, x_n) \urcorner)$.

Explanation:

For $x_1, \dots, x_n \in \mathbb{N}$

$$(x_1, \dots, x_n) \xrightarrow[\text{recursive}]{f_D} \ulcorner D(x_1, \dots, x_n) \urcorner \in \mathbb{N}$$

\Downarrow
represented by $\exists y (\varphi_{f_D}(\bar{x}, y))$

$$\text{Prov}_{PA}(\ulcorner D(x_1, \dots, x_n) \urcorner) = \exists y (\varphi_{f_D}(\bar{x}, y) \wedge \text{Prov}_{PA}(y))$$

Corollary. Let $n \in \mathbb{N}$

$$PA \vdash D(\underline{n}) \rightarrow \text{Prov}_{PA}(\ulcorner D(\underline{n}) \urcorner).$$

Apply this to $D(x) = \text{Prov}_T(x)$: a Σ_1 -formula.