6.10.2021

Formalizacja matematyki: struktury, języki, spełnianie.

Uproszczony model rzeczywistości matematycznej:

struktura (I rzędu)

Def. 1.1 $\mathcal{M} = (A, f_1, f_2, \dots, f_k, P_1, \dots, P_n, c_1, \dots, c_l)$

model, struktura

zbiór $\neq \emptyset$ uniwersum struktury $\mathcal{M}$

„arność"

$f_1, \dots, f_k$ – funkcje, $f_i : A^{n_i} \longrightarrow A$

$P_1, \dots, P_k$ – predykaty w A, $P_i \subseteq A^{m_i}$
(relacje)

$c_1, \dots, c_l$ – stałe, $c_i \in A$

Przykłady
- gdy $n = 0$, $\mathcal{M}$ nazywamy strukturą algebraiczną (grupy, ciała)
- V, rodzina zbiorów, $(V, \in)$

relacja binarna

Zdania o $\mathcal{M}$:

w pewnym języku: f_i, P_j, c_t to symbole oznaczające funkcje, relacje, stałe.

Odróżnienie między symbolem, a jego znaczeniem.

$\underline{f_i}, \underline{P_j}, \underline{c_l}$ ← ta dolna kreska ma zaznaczać, że mamy na myśli **symbol**,

f_i, P_j, c_l ← ich znaczenia w strukturze $\mathcal{M}$

Def. 1.2 **Język** $L = \{\underline{f_1}, \ldots, \underline{f_k}, \underline{P_1}, \ldots, \underline{P_n}, \underline{c_1}, \ldots, \underline{c_t}\}$ (struktury $\mathcal{M}$) wraz z arnościami tych funkcji i relacji

Inaczej: typ podobieństwa struktury $\mathcal{M}$, sygnatura $\mathcal{M}$

f_i, P_j, c_l to **interpretacje** $\underline{f_i}, \underline{P_j}, \underline{c_l}$

Mówimy, że $\mathcal{M}$ jest **modelem** dla L.

Def. 1.3 $\mathcal{M}$: **model** dla L

$$\Big(\underbrace{M}_{\substack{\text{uniwersum}\\ \text{(oznaczane}\\ \text{też } |M|)}}, \underbrace{f_1^M, \ldots, f_k^M}_{\text{funkcje}}, \underbrace{\underline{P}_1^M, \ldots, \underline{P}_n^M}_{\text{relacje}}, \underbrace{\underline{c}_1^M, \ldots, \underline{c}_t^M}_{\text{stałe}}\Big)$$

w M, interpretacje symboli $f_i, \underline{P}_j, \underline{c}_l$ odpowiednich arności

Jak mówić w L? Do tego służą:

⇒, ⇔ to META SYMBOLE

- symbole języka,
- symbole logiczne: $\wedge, \vee, \neg, \Rightarrow, \Leftrightarrow$, $\forall, \exists, \underset{\text{zmienne}}{x_i} \ (i \in \mathbb{N})$
- symbol równości: $=$
- symbole pomocnicze (nawiasy, przecinki..)

Def. 1.4 Wyrażenia języka L:

1. wyrażenia nazwowe (**termy**): zbiór termów T_L języka L, def. rekurencyjna.
   - zmienna, symbol stałej $\in T_L$ (termy **atomowe**)

- $\tau_1, \dots, \tau_n \in T_L$, f: symbol funkcji z L, n-arny,
  wtedy $\underbrace{f(\tau_1, \dots, \tau_n)}_{\text{term złożony}} \in T_L$

term **stały** = term bez zmiennych

2. formuły języka L: F_L

formuły atomowe

- $(\tau_1 = \tau_2) \in F_L$

  termy

- P_j – n-arny symbol relacyjny

  $P_j(\tau_1, \dots, \tau_n) \in F_L$

formuły złożone

- $\varphi, \psi \in F_L \Rightarrow (\neg\varphi), (\varphi \wedge \psi) \in F_L$
- v : zmienna, $\varphi \in F_L$, wtedy

  $\underbrace{\exists v \varphi}_{\text{"zasięg" } \exists v \text{ w } \exists v \varphi}, \forall v \varphi \in F_L$

Przykład

$$\forall x \exists y (x \in y \wedge \forall x (x \in y \Rightarrow x = y))$$

w $L = \{\in\}$ (binarny symbol rel.)

Def. 1.5 Niech $\varphi \in F_L$, v: zmienna występująca w φ.

- Jeśli to wystąpienie v w φ jest w zasięgu pewnego kwantyfikatora Q w v, to spośród wszystkich takich wystąpień Qv w φ, w których v jest zasięgu, wybieramy to najbardziej na prawo i mówimy, że to Qv **wiąże** to wystąpienie v w φ.
- Jeżeli takiego Qv nie ma, to mówimy, że v w φ jest **wolne**.
- v jest **wolne** w φ, gdy istnieje jej wolne wystąpienie

Konwencja: zapis $\varphi(v_1, v_2, \ldots, v_n)$ oznacza, że wszystkie zmienne wolne w φ są wśród $v_1, \ldots, v_n$.

Def. 1.6 **Zdaniem** w L nazywamy formułę bez zmiennych wolnych.

Co to znaczy, że formuła jest prawdziwa w strukturze M dla L?

(św. Tomasz: prawdziwy sąd to taki, który jest zgodny z rzeczywistością)

Def. 1.7 Model M dla L.

$$\begin{array}{cc} \parallel & '' \\ \left(M, f_1^M, \dots \underline{P}_1^M \dots \underline{c}_1^M\right) & \{f_1 \dots f_k, \underline{P}_1, \dots, \underline{P}_m, \underline{c}_1 \dots \underline{c}_t\} \end{array}$$

Niech $A = \{\underline{a} : a \in M\}$, zbiór NOWYCH symboli stałych.

Tworzymy nowy język $L(M) = L \cup A$. **Interpretacje** termów stałych σ w $L(M)$ w M:

- gdy $\tau = \underline{c}_i \rightsquigarrow \tau^M = \underline{c}_i^M$

  meta-symbol

- gdy $\underset{a \in M}{\tau = \underline{a}} \rightsquigarrow \tau^M = a$

termy atomowe

- gdy $\tau \underset{\wr}{=} f_i(\tau_1, \dots, \tau_n)$

$$\tau^M = f_i^M(\tau_1^M, \dots, \tau_n^M)$$

<u>Def.</u> 1.8 **Spełnianie** zdań φ języku $L(M)$ w M:

$M \models \varphi$: "φ jest **prawdziwe** w M"
spełnione

1. Zdania atomowe — meta symbol

- $M \models \tau_1 = \tau_2 \iff \tau_1^M = \tau_2^M$
- $M \models \underline{P_j}(\tau_1, \dots \tau_n) \iff \langle \tau_1^M, \dots, \tau_n^M \rangle \in \underline{P_j}^M$

2. Zdania złożone

- $M \models \varphi \wedge \psi \iff M \models \varphi$ oraz $M \models \psi$
- $M \models \neg\varphi \iff$ nieprawda, że $M \models \varphi$.
- $M \models \exists v\, \varphi \iff$ istnieje $a \in M$ t. że $M \models \varphi(\underline{a})$, gdzie $\varphi(\underline{a})$ powstaje z φ przez zastąpienie każdego wolnego v przez symbol $\underline{a}$.

Def. 1.9 Jeżeli zdanie φ nie jest prawdziwe w M, to mówimy, że jest **fałszywe** w M.

Konwencje: spełnienie w M formuły $\varphi(x_1, \dots, x_n)$: $\langle a_1, \dots, a_n \rangle$ spełnia $\varphi(x_1, \dots, x_n)$, gdy $M \models \varphi(\underline{a}_1, \dots \underline{a}_n)$

Def. 1.10 **Uniwersalne domknięcie** formuły φ: $\overline{\varphi} = \forall x_1 \forall x_2 \dots \forall x_n \varphi$

Mówimy, że $M \models \varphi \iff M \models \overline{\varphi}$

[to znamy dobrze, np. kiedy mówimy o łączności działania w grupie, to zwykle piszemy $(xy)z = x(yz)$, a nie $\forall x \forall y \forall z \left((xy)z = x(yz)\right)$.]

11.10.2021

Def. 2.1 $\varphi \in \mathcal{F}_L$ jest **tautologią** (klasycznego rachunku logicznego), gdy $\forall \underset{\text{model dla } L}{M}\ M \models \varphi$.

Piszemy $\models \varphi$.

Jak rozpoznać, że $\models \varphi$? NIEROZSTRZYGALNE

Niekiedy jest łatwo, np. $x = x$, „$=$ jest relacją równoważności"

Def. 2.2 **Formuły zdaniowe**:

Z: zbiór zmiennych zdaniowych

↳ S – zbiór formuł zdaniowych nad Z.

Definicja rekurencyjna:

- $v \in Z \implies v \in S$
- $\alpha, \beta \in S \implies \neg\alpha, (\alpha \wedge \beta) \in S$

„Jak to wszystko sprecyzować, to może humaniści by stracili pracę"

NIE MÓWIĆ „TAUTOLOGIA TO ZDANIE ZAWSZE PRAWDZIWE"

Potocznie tautologia to zdanie, którego prawdziwość wynika z jego struktury.

Def. 2.3 **Wartościowanie logiczne**: dowolna funkcja $v: S \to \{0,1\}$ t. że $v(\neg\alpha) = 1 - v(\alpha)$, $v(\alpha \wedge \beta) = \min\{v(\alpha), v(\beta)\}$ dla dowolnych $\alpha, \beta \in S$.

Def. 2.4 $\alpha \in S$ jest **tautologią** klasycznego rachunku zdań, gdy $\underset{\text{wartościowanie}}{\forall v} : S \to \{0,1\}$ $v(\alpha) = 1$ $(\models \alpha)$

Przykład. $\models \neg(\alpha \wedge \neg\alpha)$ dla każdego $v \in S$.

Istnieje algorytm rozstrzygający, czy $\models \alpha$.

Def. 2.5 Zał. że $\underset{\alpha(p_1,\dots,p_n)}{\alpha} \in S$ jest zbudowana ze zmiennych zdaniowych $p_1,\dots,p_n$ oraz spójników logicznych. Niech $\varphi_1, \dots, \varphi_n \in F_L$. Zał., że $\varphi \in F_L$ powstaje z α przez podstawienie wszędzie φ_i za p_i. Mówimy wtedy, że φ jest **przykładem** formuły α.

Tw. 2.6 Jeśli $\models \alpha$ oraz φ : przykład α, to $\models \varphi$.

Ćw.

Def. 2.7 **Reguły wnioskowania:**

$$\frac{\varphi_1, \dots, \varphi_n}{\varphi} \quad \begin{matrix} \leftarrow \text{przesłanki (premises)} \\ \leftarrow \text{konkluzja, teza} \end{matrix} \quad \varphi_i, \varphi \in \mathcal{F}_L$$

Podobnie $\frac{\alpha_1, \dots, \alpha_n}{\alpha}$, $\alpha_i, \alpha \in S$.

Def. 2.8 Reguła $\frac{\varphi_1 \dots \varphi_n}{\varphi}$ jest **poprawna (sound)**,

gdy $\forall M$ dla L $(M \models \varphi_1, \dots, M \models \varphi_n \Rightarrow M \models \varphi)$.

Dla formuł zdaniowych reguła $\frac{\alpha_1 \dots \alpha_n}{\alpha}$ jest poprawna, gdy

$$\forall v: S \to \{0,1\} \left(v(\alpha_1) = \dots = v(\alpha_n) = 1 \Rightarrow v(\alpha) = 1 \right)$$

Przykłady

- modus ponens (reguła odrywania) cut rule $\qquad \frac{\alpha, \alpha \to \beta}{\beta}$, $\frac{\varphi, \varphi \to \psi}{\psi}$

- reguła generalizacji $\frac{\varphi}{\forall v \varphi}$ v: zmienna

$\forall$ - rule ("A" rule)

Aksjomatyczne ujęcie KRL

A0. przykłady tautologii KRZ

A1. $[\forall v (\varphi \to \psi)] \to (\varphi \to \forall v \psi)$ (gdy v nie jest zm. wolną φ)

A2. $\forall v \varphi \longrightarrow \varphi(v/t)$

pod warunkiem, że żadne z takich wystąpień v w φ nie jest w zasięgu kwantyfikatora wiążącego jakąś zmienną w t

[tu: $t \in T_L$, którego podstawiamy w φ za każde wolne wystąpienie zmiennej]

Istotne zastrzeżenie:

$\varphi: \exists y\ x \neq y$, $t = y$, $\varphi(x/t): \exists y\ y \neq y$

$\exists y (x \neq y) \to \exists y (y \neq y)$ nie jest tautologią.

Aksjomaty równości $\left(\begin{array}{l} v, v_1, v_2 \text{ – zmienne} \\ t \text{ – term} \\ \varphi \text{ – formuła} \end{array} \right)$

- $v = v$

- $v_1 = v_2 \longrightarrow t(\ldots v_1 \ldots) = t(\ldots v_2 \ldots)$

- $v_1 = v_2 \longrightarrow \left(\varphi(\ldots v_1 \ldots) \to \varphi(\ldots {}^{v_1}\!/\!{}_{v_2} \ldots) \right)$

Pod warunkiem, że to $\underbrace{\text{wystąpienie}}$ (wolne) v_1 pod które podstawiamy v_2 nie jest w zasięgu kwantyfikatora wiążącego v_2.

Def. 2.9 Zał. że $X \subseteq \mathcal{F}_L$, $\varphi \in \mathcal{F}_L$. Mówimy, że

$X \vdash \varphi \overset{\text{def.}}{\iff}$ istnieje ciąg formuł

„dowodzi"

$\underbrace{\alpha_1, \ldots, \alpha_n}_{\text{dowód formalny}} = \varphi$ t. że $\forall i \leq n$:

1° $\alpha_i \in X$ lub α_i jest aksjomatem KRL

2° α_i wynika z $\{\alpha_1, \ldots, \alpha_{i-1}\}$ na mocy modus ponens lub $\forall$-reguły, tzn.

$\exists j, t < i \quad \alpha_t = (\alpha_j \to \alpha_i)$

(wtedy $\dfrac{\alpha_j, \alpha_t}{\alpha_i}$ przykład MP)

lub $\exists j < i \quad \alpha_i = \forall v \alpha_j$

Def. 2.10 $\vdash \varphi$, gdy $\emptyset \vdash \varphi$.
(φ jest tezą KRL)

Uwaga: $X \vdash \varphi \Longleftrightarrow \exists \underset{\uparrow}{X_0} \subseteq X \;\; X_0 \vdash \varphi$
skończone

Przykład $\vdash \underbrace{\forall x \varphi \to \exists x \varphi}$

$\beta : \forall x \varphi \to \neg \forall x \neg \varphi$

Gdyby struktura była pusta, to nie działa. Czemu?

$\alpha_1 : \forall x \varphi \to \varphi({}^{x}\!/_{y})$
$\alpha_2 : \forall x \neg \varphi \to \neg \varphi({}^{x}\!/_{y})$] tu y: zmienna spoza φ

$\alpha_3 : \alpha_2 \to (\varphi({}^{x}\!/_{y}) \to \neg \forall x \neg \varphi(x))$ } AO, prawo transpozycji

$\alpha_4 : \varphi({}^{x}\!/_{y}) \to \neg \forall x \neg \varphi(x)$ } MP, α_2, α_3

$\alpha_5 : \alpha_1 \to (\alpha_4 \to \beta)$ [AO] $((\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma)))$

$\alpha_6 : \beta$ $(2 \times \text{MP}, \alpha_1, \alpha_4, \alpha_5)$

Uwaga $\vdash \varphi \Rightarrow \models \varphi$, tzn. KRL jest poprawny (sound)

TW. 2.11 (Gödla o pełności KRL)
completeness theorem

$$\models \varphi \Longrightarrow \vdash \varphi$$

Def. 2.12 $X \subseteq \mathcal{F}_L$. $Cn(X) = \{\varphi \in \mathcal{F}_L : X \vdash \varphi\}$.

konsekwencje

X jest **teorią**, gdy $X = Cn(X)$.

Fakt 2.13 $\varphi \vdash \overline{\varphi}$ i $\overline{\varphi} \vdash \varphi$

d-d ∀-reguła

$\boxed{\vdash \overline{\varphi} \to \varphi}$

Def. 2.14 $X, Y \subseteq \mathcal{F}_L$ są **równoważne**, gdy

$$Cn(X) = Cn(Y)$$

Wniosek 2.15 X i $\{\overline{\varphi} : \varphi \in X\}$ są równoważne.

Def. 2.16 X jest **sprzeczny**, gdy istnieje zdanie φ t. że $X \vdash \varphi \wedge \neg\varphi$. W przeciwnym razie mówimy, że X jest **niesprzeczny**

Def. 2.17 (1) $M \models X \iff \forall \varphi \in X \; M \models \varphi$

M jest modelem X

(2) X jest **zupełny** $\iff$ $\forall$ zdania φ ($X \vdash \varphi$ lub $X \vdash \neg\varphi$)

(3) X jest **rozstrzygalny** $\iff$ istnieje algorytm rozstrzygający, czy $X \vdash \varphi$ dla $\varphi \in \mathcal{F}_L$

Przykład (1) $Th(M) = \{\varphi \in F_L : M \models \varphi\}$: niesprzeczna, zupełna
(teoria str. M)

(2) $\emptyset$ jest niesprzeczny

TW. 2.18 (Gödel, o istnieniu modelu) Jeżeli S jest niesprzecznym zbiorem zdań, to S ma model (mocy $\leq \aleph_0 + |S|$)

12.10.2021

D-d. TW. Gödla o modelu (Leon Henkin) gdy L i S przeliczalne.

Niech $L' = L \cup \{c_n : n < \omega\}$ (zbiór nowych symboli stałych)

Niech $\{\varphi_n(x) : n < \omega\}$: numeracja formuł z $F_{L'}$ ze zmienną wolną x.

Pomocnicza funkcja $f : \omega \to \omega$ rosnąca i t. że $c_{f(n)}$ nie występuje w formułach $\varphi_0(x), \dots, \varphi_n(x)$ dla $n = 0, 1, \dots$

Niech $S_n = S \cup \{\underbrace{\exists x\, \varphi_i(x) \longrightarrow \varphi_i(c_{f(i)})}_{\text{Aksjomat Henkina}} : i < n\}$

$S = S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots$

Niech $S_\omega = \bigcup_n S_n$.

Fakt S_ω jest niesprzeczny.

D-d. nie wprost

Skoro S_ω sprzeczne, to pewne S_n sprzeczne.

Niech n to najmniejsze takie że S_{n+1} spr.

($n \geq 0$ bo $S_0 = S$ niesprzeczne)

(tw. o dedukcji: φ zdanie, φ formuła, $X \subseteq F_L$,
$X \cup \{\varphi\} \vdash \psi \Leftrightarrow X \vdash \varphi \to \psi$)

$$S_n \vdash (\exists x\, \varphi_n(x) \to \varphi_n(c_{f(n)})) \longrightarrow \alpha \wedge \neg\alpha$$

$$\Downarrow \text{A0, MP} \qquad (p \to (q \wedge \neg q)) \to \neg p \quad (\alpha : \text{zd.})$$

$$S_n \vdash \neg(\exists x\, \varphi_n(x) \longrightarrow \varphi_n(c_{f(n)}))$$

$$\Updownarrow \text{A0, MP}$$

$$S_n \vdash \exists x\, \varphi_n(x) \wedge \neg\varphi_n(c_{f(n)})$$

$$\Downarrow$$

$S_n \vdash \exists x\, \varphi_n(x)$ oraz $S_n \vdash \neg\varphi_n(c_{f(n)})$

Z wybor f symbol $c_{f(n)}$ nie występuje w zdaniach z S_n. W dowodzie $\neg\varphi_n(c_{f(n)})$ z S_n możemy zastąpić każde wystąpienie $c_{f(n)}$ przez nową zmienną y nie występującą w formułach dowodu. Dostajemy dowód $\neg\varphi_n(y)$ z S_n. Stąd $S_n \vdash \neg\varphi_n(y)$.

Na mocy $\forall$-reguły $S_n \vdash \forall y\, \neg\varphi_n(y)$

Na mocy A2, MP $S_n \vdash \forall x \neg \varphi_n(x)$

Ale $S_n \vdash \exists x\, \varphi_n(x)$, stąd S_n sprzeczne.

Ale n miało być minimalne.

FAKT $\square$

Z tw. Lindenbauma niech $S' \supseteq S_\omega$ będzie niesprzecznym, zupełnym zb. zdań.

$C = \{c_n : n < \omega\}$. ← tu def. relację równoważności:

$$c_n \sim c_m \iff S' \vdash c_n = c_m.$$

(Ćw. sprawdzić że to rel. równoważności)

Budujemy model M z klas abstrakcji:

$|M| = C/_\sim$. Musimy zinterpretować symbole języka L' w M:

(1) $\underline{R}_i^M\left([c_{i_1}]_\sim, [c_{i_2}]_\sim, \dots, [c_{i_n}]_\sim\right)$

$$\overset{def}{\iff} S' \vdash R_i(c_{i_1}, \dots, c_{i_k})$$

(2) $f_j^M\left([c_{i_1}]_\sim, \dots, [c_{i_k}]_\sim\right) = [c_{i_{k+1}}]_\sim$

$$\overset{def}{\iff} S' \vdash f_j(c_{i_1}, \dots, c_{i_k}) = c_{i_{k+1}}$$

(3) $\underline{c}_n^M = [c_n]_\sim, \; {}_{L}{\ni}\, \underline{c}^M = [c_n]_\sim$ gdy $S' \vdash \underline{c} = c_n$

Poprawność (2):

$$S' \vdash \exists x\, f_j(\overbrace{c_{i_1}, \ldots, c_{i_k}}^{\bar{c}}) = x \quad \text{bo z A2}$$

$$\forall x\, (x \neq f_j(\bar{c})) \to f_j(\bar{c}) \neq f_j(\bar{c})$$

$$\Downarrow \text{A0}$$

$$f_j(\bar{c}) = f_j(\bar{c}) \to \exists x\, (x = f_j(\bar{c}))$$

$$\vdash f_j(\bar{c}) = f_j(\bar{c})$$

$$\Downarrow \text{MP}$$

$$\vdash \exists x\, \underbrace{f_j(\bar{c}) = x}_{\shortparallel\, \varphi_n(x) \text{ dla pewnego } n}$$

Ponadto $S' \vdash \exists x\, \varphi_n(x) \to \varphi_n(c_{f(n)})$

Zatem z MP $S' \vdash \varphi_n(c_{f(n)})$.

(3) Analogicznie

<u>Lemat.</u> $\forall$ zdania $\alpha \in \mathcal{F}_{L'}$ $(M \models \alpha \iff S' \vdash \alpha)$

<u>D-d.</u> Ind. względem dł. α.

1° α: atomowe

(a) $\alpha : t_1 = t_2$

$M \models t_1 = t_2$

$\iff t_1^M = t_2^M$

$\iff [c_n]_\sim = [c_m]_\sim$

$\iff S' \vdash c_n = c_m \overset{\text{ćw.}}{\iff} S' \vdash t_1 = t_2$

ćw. istnieją c_n, c_m t. że

$S' \vdash t_1 = c_n, \quad S' \vdash t_2 = c_m$

$\Downarrow$ ind. wzg. dł. t_1

$t_1^M = [c_n]_\sim$

(b) $\alpha : R(t_1, \dots, t_k)$

$$M \models R_i(t_1, \dots, t_k) \iff R_i^M([c_{i_1}]_\sim, \dots, [c_{i_k}]_\sim)$$

$\Uparrow$

$S' \vdash t_l = c_{i_l}$

$\Downarrow$

$t_l^M = [c_{i_l}]_\sim$

$\overset{\text{def. } M}{\iff} S' \vdash R_i(c_{i_1}, \dots, c_{i_k})$

$\overset{\text{aks. } =}{\iff} S' \vdash R_i(t_1, \dots, t_k)$

2° spójniki

(a) negacja: $M \models \neg\alpha \iff \neg(M \models \alpha)$

$\underset{\text{z zał. ind.}}{\iff} \neg(S' \vdash \alpha) \Leftarrow$ S': niesprzeczny zupełny

$\Downarrow$

$S' \vdash \neg\alpha$

(b) koniunkcje: podobnie

3° kwantyfikatory

$\exists$: $\alpha: \exists x\, \varphi_n(x)$. $M \models \exists x \varphi_n(x) \Longrightarrow M \models \varphi_n([c_k]_n)$

$\underset{\text{z zał. ind.}}{\Longleftrightarrow} S' \vdash \varphi_n(c_k) \underset{A2}{\Longrightarrow} S' \vdash \exists x\, \varphi_n(x)$

$S' \vdash \exists x\, \varphi_n(x) \Longrightarrow S' \vdash \varphi_n(c_{f(n)}) \overset{\text{zał. ind.}}{\Longrightarrow} M \models \varphi_n(c_{f(n)})$

$$\Downarrow$$

$M \models \exists x\, \varphi_n(x)$

Lemat ■

Z tego skoro $M \models S'$, to $M \models S$

Tw. ■

Wn. 2.13 tw. Gödla o pełności KRL

$$\models \varphi \Rightarrow \vdash \varphi$$

D-d. BSO φ: zdanie (patrz $\overline{\varphi}$)

Zał. nie wprost że $\nvdash \varphi$. Wtedy z tw. o dedukcji $\{\neg\varphi\}$ niesprzeczny. Z tw. o istnieniu modelu $\exists M \models \neg\varphi$ więc $\not\models \varphi$. ■

Tw. 2.20 (o zwartości)

Jeżeli każdy skończony podzbiór $X \subseteq \mathcal{F}_L$ ma model, to X ma model.

Wn. 2.21 $X \vdash \varphi \iff (\forall M \models X)\, M \models \varphi$ ■

Ustalamy (przeliczalny) L, rozważamy tylko struktury dla L.

Def. 3.1 $M \equiv N$ (elementarnie równoważne)

$$\Big\Updownarrow \text{def.}$$

$$Th(M) = Th(N) \iff \forall \varphi \in \mathcal{F}_L \left(M \models \varphi \iff N \models \varphi \right)$$

Def. 3.2 $M \underset{\substack{\uparrow \\ \text{podstruktura}}}{\subseteq} N \iff |M| \subseteq |N|$ oraz

$$\forall i,j,t \quad f_i^M = f_i^N \restriction_{|M|},$$

$$R_j^M = R_j^N \restriction_{|M|},$$

$$c_t^M = c_t^N$$

Def. 3.3 $g: M \xrightarrow{\cong} N \iff g: |M| \xrightarrow[\text{na}]{1-1} |N|$

izomorfizm

oraz $\forall \bar{a} \in |M|$

$\forall b \in |M|$

$\boxed{\begin{array}{l} \bar{a} = (a_1, \ldots, a_n) \\ g(\bar{a}) = (g(a_1), \ldots, g^{(a_n} \end{array}}$

$$\forall i, j, t \begin{cases} M \models R_i(\bar{a}) \iff N \models R_i(g(\bar{a})) \\ M \models f_j(\bar{a}) = b \iff N \models f_j(g(\bar{a})) = g(b) \\ M \models c_t = b \iff N \models c_t = g(b) \end{cases}$$

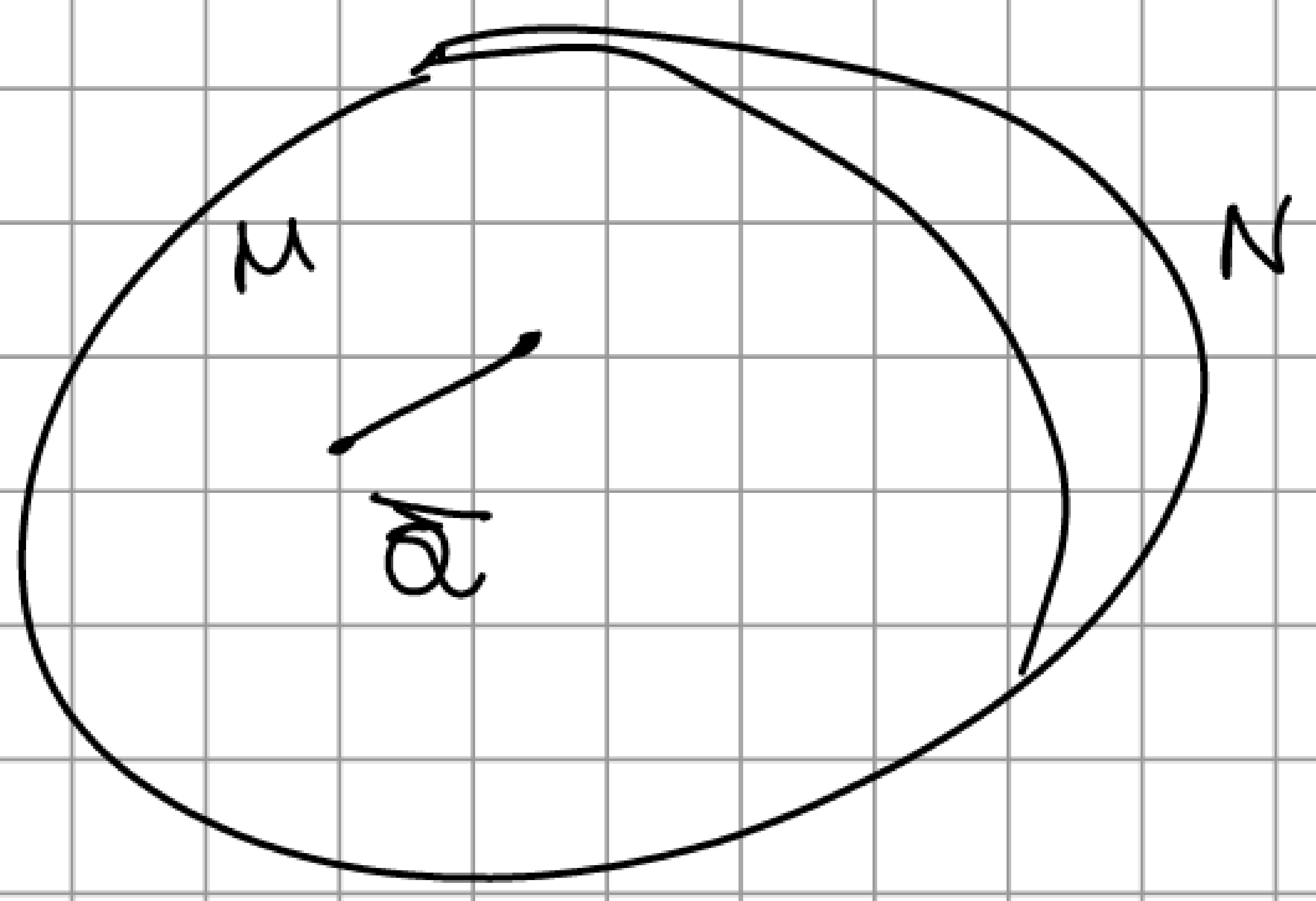
Def. 3.4 $M \cong N \iff \exists g: M \xrightarrow{\cong} N$

izomorficzne

Def. 3.5 $M \prec N$ gdy $M \subseteq N$ oraz

elementarna podstruktura

$$\forall \bar{a} \subseteq M \quad \forall \varphi(\bar{x}) \in \mathcal{F}_L \quad (M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{a}))$$



Def. 3.6 $\underset{\text{elementarne}}{f : M \xrightarrow{\equiv} N} \iff \forall \bar{a} \in M \ \forall \varphi(\bar{x}) \in F_L$

$$M \models \varphi(\bar{a}) \iff N \models \varphi(f(\bar{a}))$$

Test Tarskiego-Vaughta Załóżmy, że $A \subseteq |M|$.

Wtedy A jest uniwersum elementarnej podstruktury M wtedy, gdy spełnia warunek:

⊛ Dla każdej formuły $\varphi(x, \bar{y}) \in F_L$, $\bar{a} \in A$ jeśli $M \models \exists x\, \varphi(x, \bar{a})$, to dla pewnego $b \in A$, $M \models \varphi(b, \bar{a})$

D-d. "$\Rightarrow$" $A = |N|$, $N \prec M$, $\bar{a} \subseteq A$

$$M \models \exists x\, \varphi(x, \bar{a}) \Rightarrow N \models \exists x\, \varphi(x, \bar{a})$$

$\iff$ dla pewnego $b \in A$ $N \models \varphi(b, \bar{a})$

$$\Downarrow$$

$$M \models \varphi(b, \bar{a})$$

„$\Leftarrow$"

(a) $A = |N|$ dla pewnego $N \subseteq M$.

tzn. że $f_i^M[A] \subseteq A$. Tak jest, bo:

dla $\bar{a} \subseteq A$ $M \models \exists x\, f_i(\bar{a}) = x$

Z (*) $M \models f_i(\bar{a}) = b$ dla $b \in A$, tzn. $f_i^M(\bar{a}) = b$

(b) Dla każdej formuły $\varphi(\bar{y}) \in F_L$ i $\bar{a} \subseteq A$

$M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{a})$ (to chcemy pokazać)

indukcja względem dł. φ:

- φ: atomowa: OK, bo $N \subseteq M$
- przejście przez spójniki: łatwe
- przejście przez kwantyfikatory

$\exists$: $\varphi(\bar{y}) = \exists x \psi(x, \bar{y})$

$M \models \exists x \psi(x, \bar{a}) \overset{(*)}{\iff} M \models \psi(b, \bar{a})$ dla pewn. $b \in A$ $\overset{\text{zał. ind.}}{\iff} N \models \psi(b, \bar{a})$ dla pewn. $b \in A$

$\iff N \models \exists x \psi(x, \bar{a})$ ■

Wn. 3.7 (tw. Löwenheima - Skolema, Malceva)

(a) Jeśli $A \subseteq |M|$, to $(\exists N \prec M)(A \subseteq |N|,$
(Dolne)
$\|N\| \leq |A| + |L|)$

$|L| \overset{\text{def.}}{=} |F_L|$

(b) Jeśli M: nieskończony, to $\exists N \succ M$
(Górne)

$\uparrow$
dowolnie dużej mocy

<u>D-d.</u> (a) $\kappa = |A| + |L|$. Konstruujemy ciąg

$$a_\alpha \in M,\ \alpha < \kappa \quad (a_{<\alpha} = \{a_\beta : \beta < \alpha\})$$

Rekurencyjnie

- Na kroku α rozważamy formułę
$\varphi_\alpha(x) \in \mathcal{F}_L(Aa_{<\alpha})$ t. że $M \models \exists x\, \varphi_\alpha(x)$

Jako a_α bierzemy dowolny świadek tego

∎

20.10.2021

Proof (upward L-S, Malcev thm)

Consider theory $T = Th(M, \underline{a}^M)_{a \in |M|}$ in Language $L(M)$.

$\uparrow$ elemental diagram of M

Let $\varkappa$ be any cardinal number.

Let $\{c_\alpha : \alpha < \varkappa\}$ be a set of new constant symbols. Let $T' = T \cup \{c_\alpha \neq c_\beta : \alpha < \beta < \varkappa\}$ in $L' = L(M) \cup \{c_\alpha : \alpha < \varkappa\}$. We'll prove that T' is consistent.

Let $X \subseteq T'$ be finite. We expand $(M, a)_{a \in |M|}$ interpreting finitely many c_α's appearing in X as distinct elements of M

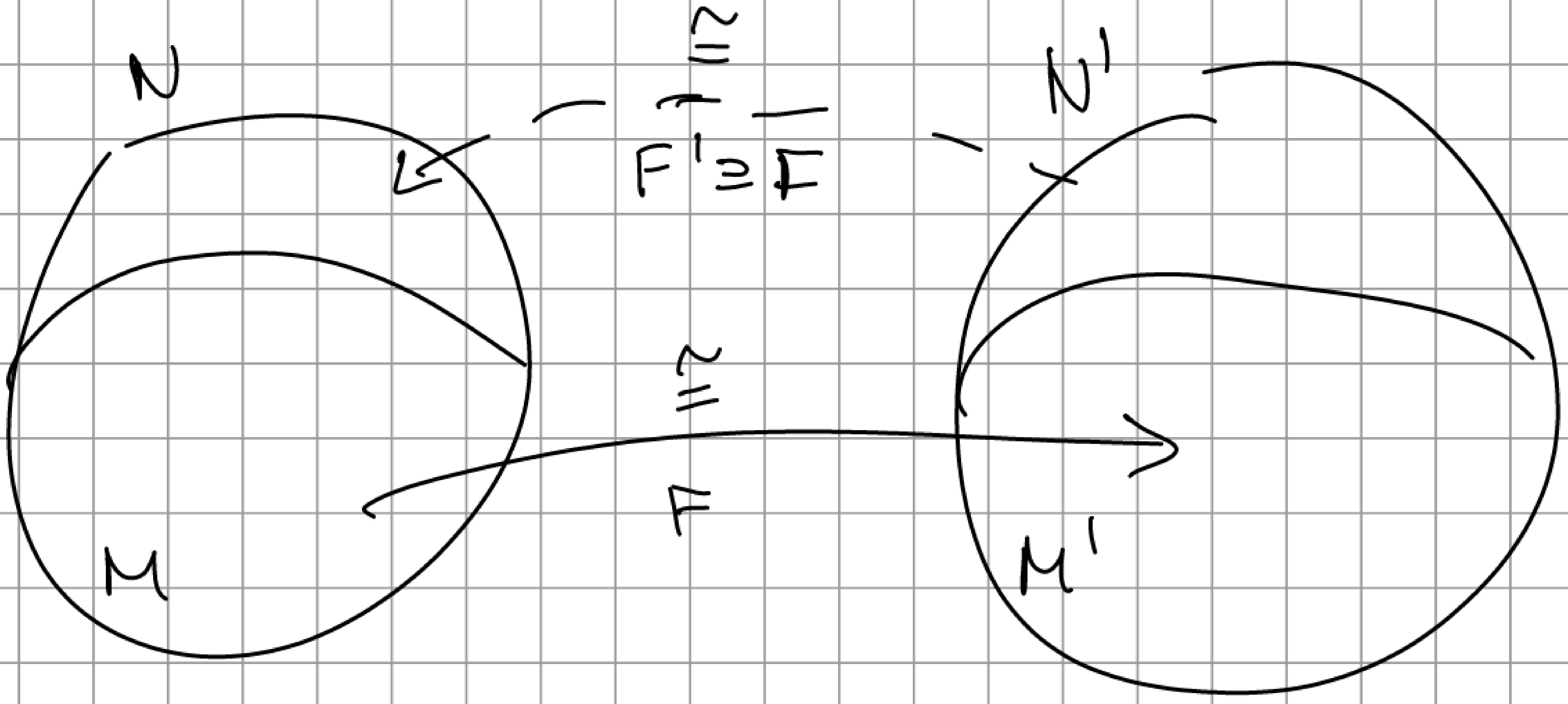
$$\leadsto$$

We get a model of X, so X is consistent, so T' is (compactness thm)

Let $\underset{(N', \underline{a}^N)_{a\in|M|}}{\underset{\|}{N'}} \models T' \supseteq T$. Also $N' \models c_\alpha \neq c_\beta$ for all $\alpha < \beta < \varkappa$. Hence $\|N'\| \geq \varkappa$

Exercise: • $\{\underline{a}^N : a \in |M|\} = |M'|$ for some $M' \prec N'$

• $\underset{a \mapsto \underline{a}^N}{M \cong M'}$



• we can extend F so that N is isomorphic to N'.

Löwenheim-Skolem paradox

"We cannot express uncountability"

My interpretation of this paradox:

Let $(V, \in) \models ZFC$ be some "big" model that contains $\mathbb{N}$ and is transitive

$\forall A, B \in V$
$A \subseteq B \Leftrightarrow A \in B$

Take subset of ZFC (call it $ZFC^{\star}$ s.t. it's finite and $ZFC^{\star} \models$ "For every X we have $|\mathcal{P}(X)| \neq |X|$"

(This is Cantor's theorem, it is true in ZFC and we can take some finite number of ZFC axioms that prove this).

Take some $\mathbb{N} \subseteq \underset{\uparrow}{\overset{\overset{V}{\rotatebox{90}{$\subseteq$}}}{M}} \models ZFC^{\star}$ (L-S thm).

countable

We also need M to be transitive (just add some axioms to ZFC*).

Now, $M \models$ "$\mathcal{P}(\mathbb{N}) \neq \mathbb{N}$" which can be written as

$$M \models \quad \forall X \subseteq \mathbb{N} (X \in A) \longrightarrow |A| \neq |\mathbb{N}|$$

We now that $A \in M (\Longleftrightarrow A \subseteq M)$, M is contable, but proves that A is uncountable! (which is not true in V).

Problem How to determine if $M \equiv N$ or $M \cong N$?

↓

Ehrenfeucht – Fraissé games

Assume, that $\mathcal{L}$ if a finite relational Language. We consider $\mathcal{L}$-structures M, N.

<u>Def. 4.1</u> $\Gamma_n(M,N)$ is a game consisting of n moves (each move is one move for a player).

n-move game

We call the players

↳ I (spoiler) ↳ II (prover)

At each move players pick one element of M and N (one from each). On move i $(1 \leq i \leq n)$ the players have already picked elements $a_1, \dots, a_{i-1} \in M$, $b_1, \dots, b_{i-1} \in N$. Now I picks one of M, N and then one element from it. Player II responds with picking an element from the other model: $a_i \in M$, $b_i \in N$.

After n moves: $\{a_1, \dots, a_n\} \subseteq M$, $\{b_1, \dots, b_n\} \subseteq N$.

$$f : a_i \longmapsto b_i$$

Player II wins if f is an isomorphism of the induced substructures.

Example When $M \cong N$, then II obviously win. Just tak $F : M \xrightarrow{\cong} N$ and pick according to it.

Example $$(\mathbb{Z}, \leq) \underset{\not\cong}{\equiv} (\underset{\substack{\wr\wr \\ \mathbb{Z}}}{\mathbb{Z}_1}, \leq) \sqcup (\underset{\substack{\wr\wr \\ \mathbb{Z}}}{\mathbb{Z}_2}, \leq)$$

$$(\underset{\mathbb{Z}_1}{\cdots\cdots}) < (\underset{\mathbb{Z}_2}{\cdots\cdot})$$

For instance: $n=5$

$$\text{I}: 1 \longrightarrow \text{II}: 1_1$$

$$\text{II}: 32 \longleftarrow \text{I}: 1_2$$

$$\vdots$$

Arbitrary number greater then 5 wins!

Proof excersice. ∎

Thm 4.2 (Ehrenfreut – Fraïssé)

$$M \equiv N \iff \forall n \text{ II has a winning strategy in } \Gamma_n(M,N)$$

Def. 4.3 X is **a set of axioms of theory** T when $Cn(X)=T$.

Examples of theories

Let $L=\{\leq\}$.

1. LO : axioms "$\leq$ is a linear order".
   Linear order
2. DLO : axioms of LO and:
   dense LO
   $$(\forall x,y)(x<y \to \exists z \; x<z<y)$$
3. DLO_0 : axioms of DLO and:
   DLO without endpoints
   $$\forall x \exists y,z \; y<x<z$$

DLO_0 is consistent, because it has a model: $\mathbb{Q}$. Moreover, it is complete and decidable.

Def. 4.3 Theory T is **$\varkappa$-categorical** if for every $M, N \models T$ s.t. $\|M\| = \|N\| = \varkappa$ we have $M \cong N$.

Remark 4.4 DLO_0 is $\aleph_0$-categorical

Proof (Back-and-forth argument)

Assume $M = \{a_n, n < \omega\} \models DLO_0$

$N = \{b_n, n < \omega\} \models DLO_0$

We construct functions f_n and sets $A_n \subseteq M$, $B_n \subseteq M$ such that

(a) $f_n \subseteq f_{n+1}$ (which implies $A_n \subseteq A_{n+1}$, $B_n \subseteq B_{n+1}$)

(b) $f_n : A_n \xrightarrow{\cong} B_n$ of induced structures

(c) $a_n \in A_{2n+1}$, $b_n \in B_{2n+2}$

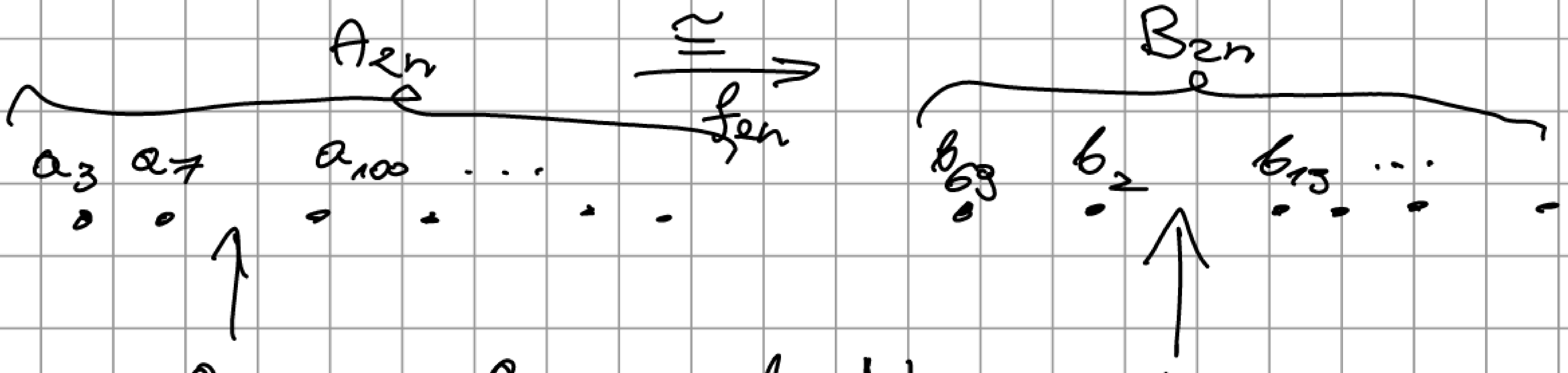
Construction We will construct those objects recursively:

- $f_0 = A_0 = B_0 = \emptyset$
- recursive step: $2n \longrightarrow 2n+1 \longrightarrow 2n+2$

We have $f_{2n} : A_{2n} \longrightarrow B_{2n}$

Look at a_n. If $a_n \in A_{2n}$, then $f_{2n+1} = f_{2n}$.

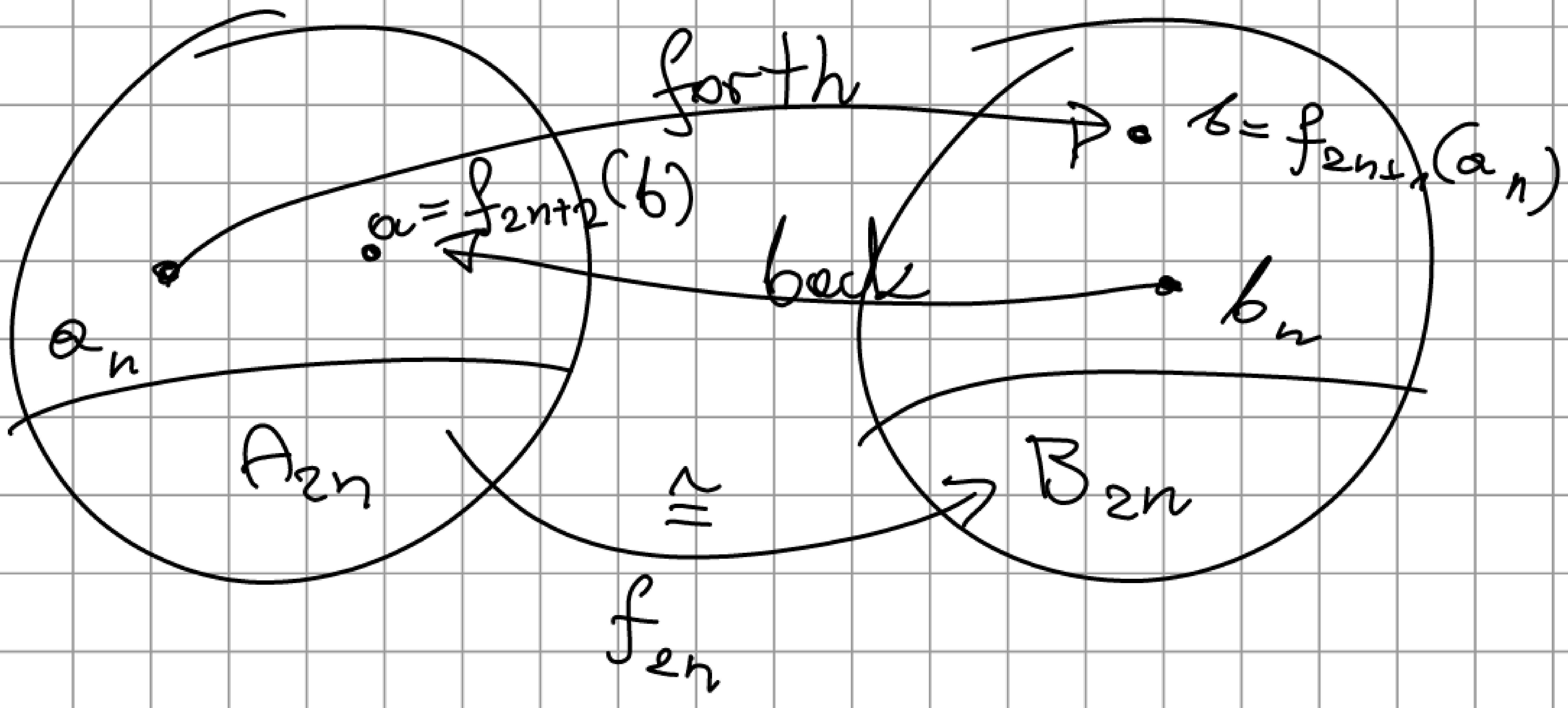
Else then $A_{2n+1} = A_{2n} \cup \{a_n\}$. Now:



a_n find any $b \in N$ s.t. b

Now $B_{2n+1} = B_{2n} \cup \{b\}$, $f_{2n+1} = f_{2n} \cup \langle a_n, b \rangle$

Similarly we chose a for b_n and construct $f_{2n+2}, A_{2n+2}, B_{2n+2}$.



Now $f = \bigcup_n f_n$, by (c) $\text{Dom} f = M$, $\text{Rng} f = N$ and by (b) f is an isomorphism.

Corollary 4.5 DLO_0 is complete

Proof Let $\varphi \in \mathcal{F}_L$ be a sentence. Suppose (ad absurdum) that $DLO_0 \nvdash \varphi$ and $DLO_0 \nvdash \neg\varphi$.

Hence, by "$\vdash \Leftrightarrow \models$" there is $M, N \models DLO_0$

s.t. $M \models \varphi$, $N \models \neg\varphi$

$$\begin{array}{ccc} M & & N \\ \curlyvee & & \curlyvee \\ M_0 & \cong & N_0 \end{array} \quad \text{, but} \quad \begin{array}{l} M_0 \models \varphi \\ N_0 \models \neg\varphi \end{array} \quad ↯$$

(downward L-S)

countable

27.10.2021

Henkin's proof of Gödel's model existance thm.

Why Henkin's axiom? Recall:

S - a consistent set of formulas

$\cap$

S' in $\mathcal{L}' = \mathcal{L} \cup \{c_n : n \in \mathbb{N}\}$

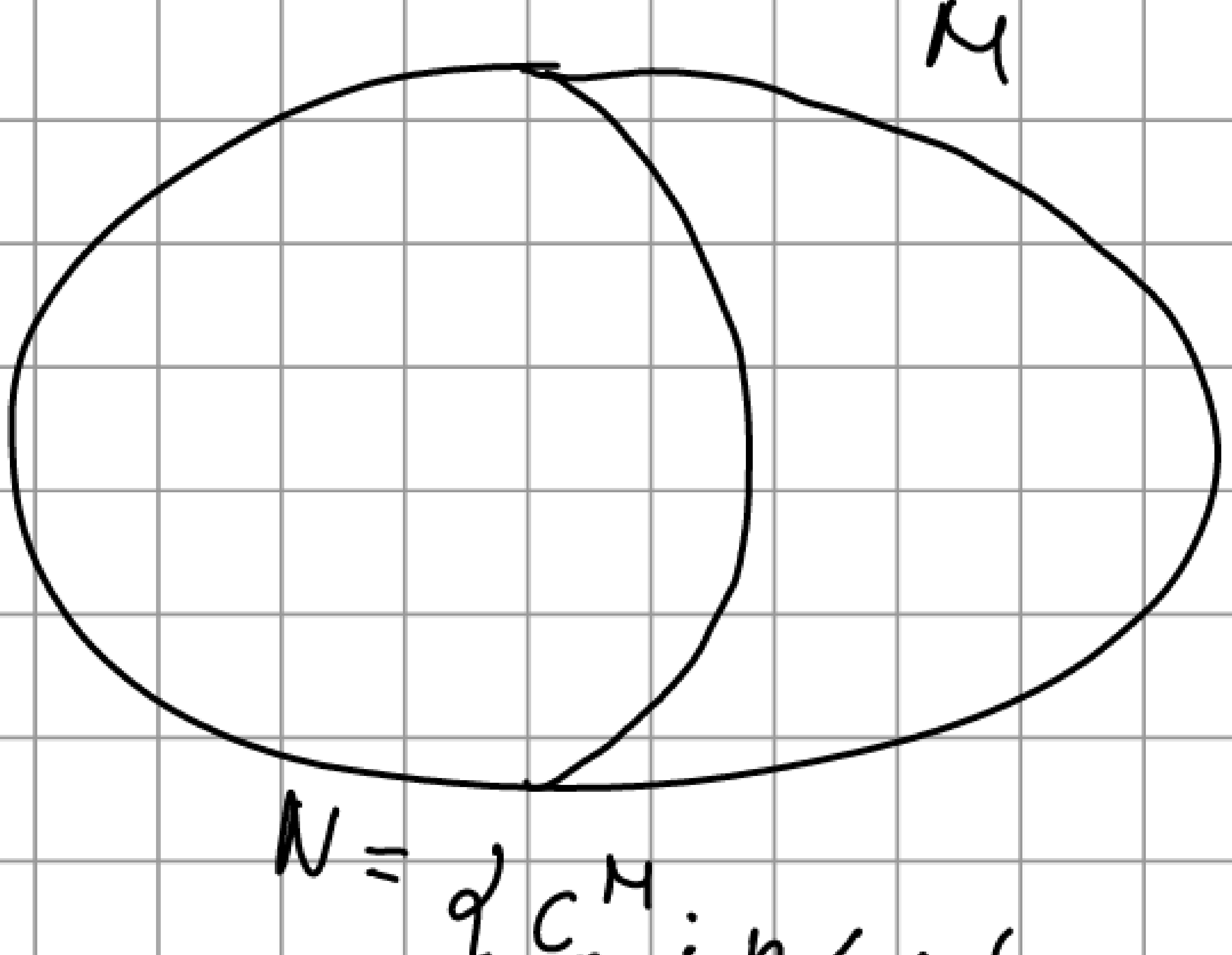
consistent
complete

Contains $H_n : \exists x \, \varphi_n(x) \longrightarrow \varphi_n(c_{f(n)})$

Enumeration of all formulas in $\mathcal{F}_{\mathcal{L}'}$

Suppose $M \models S'$

Henkin's axioms + completness of S' ensures that N satisfies T-V test



$N = \{c_n^M : n < \omega\}$

$M \models \exists x \underbrace{\varphi(x, \bar{c})}_{}$

$\varphi_n(x)$ for some n

S' is complete, so

$S' \vdash \exists x \, \varphi_n(x)$

Also $S' \vdash H_n \overset{MP}{\Rightarrow} S' \vdash \varphi_n(c_{f(n)}) \Rightarrow M \models \varphi_n(c_{f(n)})$

the witness from N

So $N \preccurlyeq M$, $N \models S'$

S' contains the "atomic diagram" of N

- $P_i(\bar{c})$ is decided is S'
- $f_j(\bar{c}) = c'$ — " —
- $c_n = c_m$ — " —

: this is enough to determine structure of N

DLO$_0$: $\aleph_0$-categorical $\Rightarrow$ complete

Corollary 4.6. DLO$_0$ is decidable

Proof(a) Let $\varphi \in \mathcal{F}_L$: a sentence

Algorithm: generate proofs in DLO$_0$: $\bar{\alpha}_0, \bar{\alpha}_1, \dots$
(in steps $0, 1, \dots$)

cntbly many

It exists because set of axioms is finite

In step i, $\bar{\alpha}_i$ is generated.

We verify if the conclusion of $\bar{\alpha}_i$ happens to be φ or $\neg\varphi$.

If YES, then we get either $DLO_0 \vdash \varphi$ or $DLO_0 \nvdash \varphi$, STOP.

If NOT, proceed to step $i+1$.

There is some proof of φ or $\neg\varphi$, because DLO_0 is complete. ■

Remark 4.7 If T is a complete theory with **recursively enumerable** set of axioms, then T is decidable.

There is an effective way of generating these axioms in a list

Proof (b) (more practical)

Fact 4.8 DLO_0 is **quantifier eliminable** (q.e.)

that is: $\forall \varphi \in \mathcal{F}_L \; \exists \psi \in \mathcal{F}_L \;\; DLO_0 \vdash \varphi \leftrightarrow \psi$

with no quantifiers (quantifiers free)

Proof (Fact 4.8) Let $\varphi(\bar{x}) \in \mathcal{F}_L$.

There are finitely many configurations $C_1(\bar{x}), \dots, C_n(\bar{x})$

A **configuration** $C(\bar{x})$: a conjunction of atomic, negated atomic formulas about $\bar{x}$ completely describing ordering $<$ on $\bar{x}$

$$\mathrm{DLO}_0 \vdash \dot{\bigvee_{i=1}^{n}} c_i(\bar{x})$$

exclusive alternative

$$\Updownarrow$$

$$(\mathbb{Q}, \leq) \models \dot{\bigvee_{i=1}^{n}} c_i(\bar{x})$$

<u>Exersice</u> T is complete consistent theory, $M \models T$, $\varphi \in \mathcal{F}_L$, then $T \vdash \varphi \iff M \models \varphi$

Also $\forall i \left[\mathrm{DLO}_0 \vdash c_i(\bar{x}) \to \varphi(\bar{x}) \text{ or } \mathrm{DLO}_0 \vdash c_i(\bar{x}) \to \neg\varphi(\bar{x})\right]$

because: if $\bar{a}, \bar{b} \in \mathbb{Q}$, both satisfing $c(\bar{x})$, then

$\exists f \in \mathrm{Aut}(\mathbb{Q}, \leq)\ f(\bar{a}) = \bar{b}$ fact $\blacksquare$

Let $I = \{i \in \{1, \dots, n\} : \mathrm{DLO}_0 \vdash c_i(\bar{x}) \to \varphi(\bar{x})\}$

<u>excersice</u>: $\mathrm{DLO}_0 \vdash \varphi(\bar{x}) \leftrightarrow \underbrace{\bigvee_{i \in I} c_i(\bar{x})}_{\psi:\ \text{quantifiers free}}$

$$\Downarrow$$

DLO_0 is decidable

(We trasform φ to ψ and look if it holds in $(\mathbb{Q}, \leq)$) $\blacksquare$

Examples of theories with q.e.

(1) ACF_p = the theory of algebraically closed fields of char. $p \geqslant 0$

(2) RCF = the theory of real closed fields

$$Th(\mathbb{R}, +, \cdot, 0, 1, \leqslant)$$

(3) Many easy theories:

e.g. the theory T of independent unary predicates $P_n(x)$, $n<\omega$

<u>axioms</u>: $A_{I,J} : \exists x \left(\bigwedge_{n\in I} P_n(x) \wedge \bigwedge_{n \in J} \neg P_n(x) \right)$

$I, J \subseteq \omega$ finite, disjoint

A model M of T:

$$M = (2^\omega, P_n)_{n<\omega}, \quad P_n(f) \Longleftrightarrow f(n)=0$$

(4) Abelian groups, modules over a fixed ring

They have only reduction of quantifiers to the level of $\underbrace{\text{pp}}_{\text{positive primitive}}$-formulas

$$\exists \bar{y}\, \underset{\substack{\uparrow \\ \text{matrix}}}{A} \bar{y} = \bar{x}$$

(5) $(\mathbb{R}, +, \cdot, <, \exp,$ analytic functions with compact supports$)$

$$\Downarrow$$

"almost" q.e. <u>o-minimal</u>

<u>Corollary 4.9</u> (weak Hilbert nullstellensatz)

If $\underset{\mathrm{ACF}_p}{\underset{\shortparallel \quad \shortparallel}{K \subseteq L}}$ and $F(\bar{x})$: a system of equations and non-equations with parameters in K, solvable in L, then $F(\bar{x})$ has a solution in K.

<u>Proof</u> In ACF_p: $F(\bar{x}) = F(\bar{x}, \underset{K}{\underset{\cap}{\bar{a}}}) \rightsquigarrow F(\bar{x}, \bar{y})$

by q.e. $\mathrm{ACF}_p \vdash \exists \bar{x}\, F(\bar{x}, \bar{y}) \longleftrightarrow \underset{\text{q.f.}}{\psi(\bar{y})}$

So $L \models \exists \bar{x}\, F(\bar{x}, \bar{a}) \iff L \models \psi(\bar{a})$

$$\begin{tikzcd} L \models \psi(\bar{a}) \arrow[d, Leftrightarrow] \\ K \models \psi(\bar{a}) \end{tikzcd}$$

$\cup$ because ψ has no quant.

So $\mathrm{ACF}_0 \mathrel{=\!|} K \models \exists \bar{x}\, F(\bar{x}, \bar{a})$ $\blacksquare$

Def 5.1 An algebra of sets: $\mathcal{C} \subseteq \mathcal{P}(X)$ st.

(1) $\emptyset, X \in \mathcal{C}$

(2) $\mathcal{C}$ closed under Boolean operations

$$^c, \cup, \cap, \triangle, \dots$$

$$\left(\mathcal{C}, \cup, \cap, {}^c, \underset{\underset{\emptyset}{\parallel}}{\mathbb{0}^{\mathcal{C}}}, \underset{\underset{X}{\parallel}}{\mathbb{1}^{\mathcal{C}}}\right)$$: a Boolean algebra

Generally $L_{BA} = \{\vee, \wedge, ', \mathbb{0}, \mathbb{1}\}$ BA: the theory of Boolean algebras

Axioms of BA

(1) $\vee, \wedge$: associative, commutative, distributive,

(2) $\mathbb{0}' = \mathbb{1}$, $\mathbb{1}' = \mathbb{0}$, $x'' = x$, $(x \vee y)' = x' \wedge y'$
$(x \wedge y)' = x' \vee y'$

(3) $x \vee (x \wedge y) = x$, $x \wedge (x \vee y) = x$

Thm (Stone) Every B. Algebras is $\cong$ an algebra of sets

Let $\mathbb{A} = (A, \wedge, \vee, ', \mathbb{0}, \mathbb{1})$: a BA.

- For $x, y \in \mathbb{A}$ $\underset{\substack{\uparrow \\ \text{a partial} \\ \text{ordering}}}{x \leq y} \iff x \vee y = y$
  $\iff x \wedge y = x$

- $\overset{\neq \mathbb{0}}{a} \in \mathbb{A}$ is an atom if $\forall x \in \mathbb{A}$ $x \leq a \to x = \mathbb{0}$
  $\vee\, x = a)$
- $\mathbb{A}$ is atomless if it has no atoms.

<u>Example</u> the algebra of sets $\mathcal{C} \subseteq \mathcal{P}(\mathbb{Q})$ generated by open intervals with rational endpoints

is atomless BA.

<u>Def. 5.2</u> $\emptyset \neq \mathcal{U} \subseteq \mathbb{A}$ is a **filter** if

(1) $x \leq y$ and $x \in \mathcal{U} \Rightarrow y \in \mathcal{U}$

(2) $x, y \in \mathcal{U} \Longrightarrow x \wedge y \in \mathcal{U}$

<u>Def. 5.3</u> $\mathcal{U}$ is a **proper filter** if $\mathbb{0} \notin \mathcal{U}$

<u>Def. 5.4</u> $\mathcal{U}$ is an **ultrafilter** in $\mathbb{A}$ if

it is a maximal proper filter in $\mathbb{A}$

Remark 5.5 Every proper filter extends to an ultrafilter.

Def. 5.6 $S(\mathbb{A}) = \{\text{ultrafilters in } \mathbb{A}\}$: a topological space

↑ **Stone space of $\mathbb{A}$**

the basis of topology: $[a] = \{\mathcal{U} \in S(\mathbb{A}) : a \in \mathcal{U}\}$
(for $a \in \mathbb{A}$)

(it is clopen)

Proof of Stone representation thm

$\mathcal{C} = \{[a] : a \in \mathbb{A}\} \subseteq P(S(\mathbb{A}))$

$\mathbb{A} \ni a \xrightarrow{\widehat{=}} [a]$ ■

3.11.2021

Corollary BA = the theory of algebras of sets

$= \{\varphi \in \mathcal{F}_{L_{BA}} : \varphi$ holds in every algebra of sets $\}$

Proof: ex.

Types, definable sets. L: fixed (cntbl), T: complete consistent theory in L.

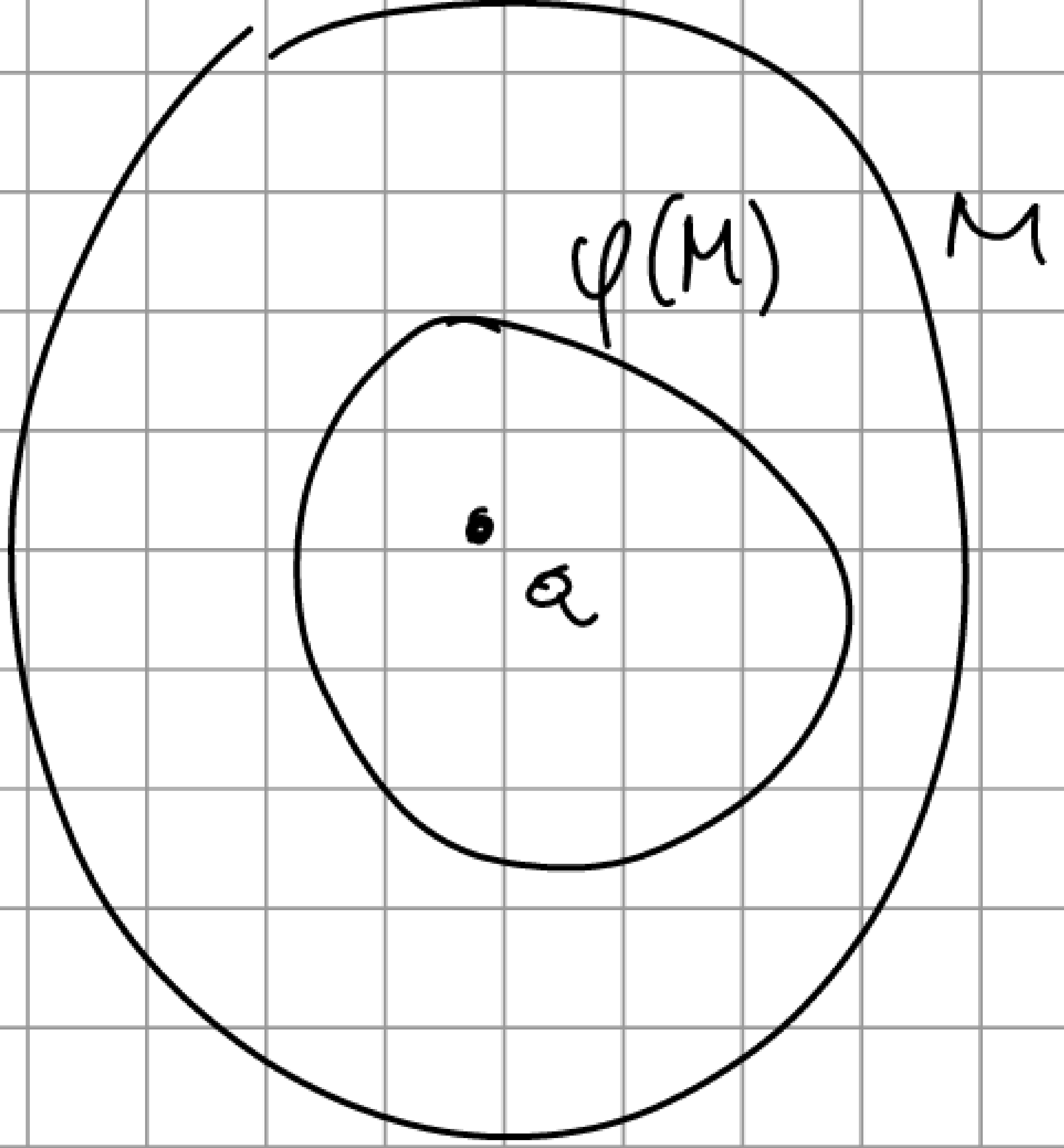
$M \models T \quad a \in M$

$\varphi(x) \in \mathcal{F}_L$

$\varphi(M) = \{b \in M : M \models \varphi(b)\}$

$\nearrow$

definable (sub)set of M



Def. $tp^M(a) = \{\varphi(x) \in \mathcal{F}_L : M \models \varphi(a)\}$

$\uparrow$

type of a in M

Def. $\mathcal{F}_L(x) = \{$formulas of $\mathcal{F}_L$ of the form $\varphi(x)\}$

Likewise: $\mathcal{F}_L(\overline{x})$

$x_1, \dots, x_n$

Def. $\varphi(x) \in \mathcal{F}_L(x)$ is T-consistent

$$\Updownarrow$$

$$T \vdash \exists x\, \varphi(x)$$

$$\Updownarrow$$

$$M \models \exists x\, \varphi(x)$$

$$\Updownarrow$$

$$\varphi(M) \neq \emptyset$$

Def. $p(x) \subseteq \mathcal{F}_L(x)$ is T-consistent

$$\Longleftrightarrow \forall \underset{\text{finite}}{p_0(x)} \subseteq p(x) \left(\bigwedge_i p_0(x)_i \text{ is } T\text{-consistent} \right) \Longleftrightarrow T \vdash \exists x \bigwedge_i p_0(x)_i$$

$$\underbrace{p(x) \text{ is a type in variable } x}_{\text{a 1-type}}$$

Likewise: $p(\bar{x}) \subseteq \mathcal{F}_L(\bar{x})$: a k-type in variables $\bar{x}$

Def. A type $p(x)$ is <u>complete</u> $\Leftrightarrow$ p is consistent and $\forall \varphi(x) \in \mathcal{F}_L(x)\, (\varphi \in p(x) \vee \neg\varphi \in p(x))$

Example $tp^M(a)$ is a complete type.

Remark $a \in M \prec N \Rightarrow tp^M(a) = tp^N(a)$

A generalization: $\underset{\text{a set of parameters}}{A \subseteq |M|}$, $L(A) = L \cup \{\underset{\substack{\uparrow \\ \text{new constant symbols}}}{\underline{a}} : a \in A\}$

$T(A) := Th(M, \underset{\substack{\shortparallel \\ a}}{\underline{a}^M})_{a \in A}$ – a structure in $L(A)$

A type over A (in T) = a type in $T(A)$

<u>Def.</u> A type $p(x)$ is realized in M by a, if

$$\forall \varphi(x) \in p(x)\ M \models \varphi(a) \iff p(x) \in tp^M(a)$$

<u>Remark</u> Assume $p(x)$ is a consistent type over A.

Then there is an elementary extension $N \succ M$ s.t. $p(x)$ is realised N.

<u>Proof</u> Let $T' = Th(M, m)_{m \in M} \cup \{\varphi(\underset{\substack{\uparrow \\ \text{a new constant symbol}}}{c}) : \varphi \in p(x)\}$

T' is consistent (compactness theorem)

⋮

Corollary Every consistent type extends to a complete type.

Proof $\underset{\text{over } A \text{ in } M}{p(x)}$ — $p(x)$ is realised in $\underset{\substack{\cup\!\!\!\!\shortmid \\ c^N}}{N} \succ M$ by c^N

$$\Longrightarrow p(x) \subseteq \underbrace{tp^N(c^N/A)}_{\text{a complete type over } A \text{ in } T} := tp^N(c^N) \text{ in } (N,a)_{a\in N}$$

$$= \{\varphi(x) \in \mathcal{F}_{L(A)}(x) : M \models \varphi(a)\}$$

Def. $Def(M) = \{$definable subsets of $M\}$

Remark

(1) $Def(M)$ is BA over subsets of M

(2) For every $a \in M$ $\{\varphi(M) : \varphi(x) \in tp^M(a)\}$ is an ultrafilter in $Def(M)$

$\sim$ on $\mathcal{F}_L(x)$: $\varphi \sim \psi \Leftrightarrow T \vdash \varphi(x) \leftrightarrow \psi(x)$

$$\Leftrightarrow \varphi(M) = \psi(M)$$

Proof "$\Rightarrow$" is clear. "$\Leftarrow$" (ad absurdum)

suppose $T \nvdash \varphi(x) \leftrightarrow \psi(x) \Leftrightarrow T \nvdash \forall x\,(\varphi(x) \leftrightarrow \psi(x))$

$\underset{T \text{ is complete}}{\Leftrightarrow} T \vdash \exists x\,(\varphi(x) \triangle \psi(x)) \Leftrightarrow M \models \exists x\,(\varphi(x) \triangle \psi(x))$ ↯

$\underbrace{\mathcal{L}_1(\emptyset)}_{\text{Lindenbaum algebra}} = \left(\mathcal{F}_L(x)/_{\sim}, \wedge, \vee, ', \mathbb{0}, \mathbb{1} \right)$

- $[\varphi]_\sim \wedge [\psi]_\sim = [\varphi \wedge \psi]_\sim$
- $[\varphi]_\sim \vee [\psi]_\sim = [\varphi \vee \psi]_\sim$
- $[\varphi]_\sim' = [\neg \varphi]_\sim$
- $\mathbb{0} = [x \neq x]_\sim$
- $\mathbb{1} = [x = x]_\sim$

Remark

(1) $[\varphi(x)]_\sim \xmapsto{F} \varphi(M)$

gives $F : \mathcal{L}_1(\emptyset) \xrightarrow{\cong} \mathrm{Def}(M)$

(2) $tp^M(a)/_\sim$: an ultrafilter in $\mathcal{L}_1(\emptyset)$

(3) $p(x)/_\sim$ is an ultrafilter in $\mathcal{L}_1(\emptyset)$ for every complete type $p(x)$ in T.

(4) $\{\text{complete types in } T, \text{ in } x\} \underset{\text{onto}}{\overset{1-1}{\longleftrightarrow}} S(\mathcal{L}_1(\emptyset))$

$\{\text{consistent types in } T, \text{ in } x\}$ "$\longleftrightarrow$" filters in $\mathcal{L}_1(\varphi)$

Ad 4 Assume that $p(x)$ is a complete type in T. Then $p(x)$ is closed under $\sim$

$$\Longrightarrow p(x)/_{\sim} \in S(L_1(\emptyset))$$

Vice versa, if $\mathcal{U} \in S(L_1(\emptyset))$, then

$$p_{\mathcal{U}} = \{\varphi(x) \in \mathcal{F}_L(x) : [\varphi]_{\sim} \in \mathcal{U}\}$$ is a

complete consistent theory in T, in X.

For $A \subseteq M \rightsquigarrow \mathrm{Def}_A(M) = \{\varphi(M) : \varphi(x) \in \mathcal{F}_{L(A)}\}$

$$\varphi(\bar{x}, \underset{M}{\overset{\bar{a}}{\cap}}) \in \mathcal{F}_{L(A)}(\bar{x}) \rightsquigarrow \underbrace{\varphi(M) = \{\bar{b} \in M^k}_{} : M \models \varphi(\bar{b}, \bar{a})\}$$

where $\varphi(\bar{x}, \bar{y}) \in \mathcal{F}_L$

[set definible in M over paremeters A]

[A-definible in M]

$\mathrm{Def}_A(M)$ – the algebra of A-definable subsets of M.

$$\begin{tikzcd}
L \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & L(A) \arrow[d, rightsquigarrow] \\
L_1(\emptyset) \arrow[d, rightsquigarrow] & L_1(A) \arrow[d, rightsquigarrow] \\
S(L_1(\emptyset)) & S(L_1(A))
\end{tikzcd}$$

the corresponding Lindenbaum algebra of formulas $\varphi(x) \in \mathcal{F}_L(x)$

$S_1(A)$ = the space of all complete 1-types over A in T.

$$S_1(A) \longleftrightarrow \underbrace{S(\mathcal{L}_1(A))}_{\text{Stone space topology}}$$

<u>Recall</u> assume $\mathbb{A}$: BA

$$\begin{tikzcd} \mathbb{A} \arrow[d, squiggly] \\ S(\mathbb{A}) \end{tikzcd}$$

$$\mathbb{A} \ni a \rightsquigarrow [a] = \{\mathcal{U} \in S(\mathbb{A}) : a \in \mathcal{U}\}$$

$\{[a] : a \in \mathbb{A}\}$ is a basis of Stone topology in $S(\mathbb{A})$. It is compact, Hausdorff, 0-dimensional, $[a]$ - clopen

Topology on $S_1(A)$:

Basic open sets: $[\underset{\mathcal{F}_{L(A)}(x)}{\underset{\cap}{\varphi}}] = \{p(x) \in S_1(A) : \varphi(x, \bar{a}) \in p(x)\}$

Likewise: $L_k(A)$, $S_k(A)$ k-types over A in T

in variables $\overline{x}$.

Def Let κ be any cardinal number. We say that M is κ-saturated if $\underset{|A|<\kappa}{(\forall A \subseteq |M|)}$ $(\forall p \in S_1(A))$ p is realised in M.

M is saturated if it is $\|M\|$-saturated

Corollary (1) $\forall M \forall \kappa \exists N \succ M$ N is κ-saturated

(2) If $\underbrace{\kappa \text{ is regular and } 2^{<\kappa} = \kappa}$

$$\Updownarrow$$

$$\kappa^{<\kappa} = \kappa$$

$$2^{<\kappa} = \sup_{\mu<\kappa} 2^{\mu}$$

$(I, \leq)$ a linear ordering.

$\underset{\text{cofinality}}{\mathrm{cf}(I)} = \min\{|J| : \underset{\substack{\text{cofinal} \\ \| \\ \text{unbounded}}}{J \subseteq I}\}$

$\underset{\substack{\cap \\ \mathrm{CN}}}{\kappa} \in \mathrm{Ord}$ $\kappa = \{\alpha \in \mathrm{Ord} : \alpha < \kappa\}$

$(\kappa, <)$ well-ordered

Lecture 10. 11. 2021

Recall: $\kappa \in CN$, $\kappa \geq \aleph_0$, L: ctble, T: complete consistent theory in L

Def. $M \models T$

(1) M is κ-saturated, if $\forall A \subseteq M$, $|A|<\kappa$ $\forall p \in S_1(A)$ p is realized in M

(2) M is saturated if M is $\|M\|$-saturated.

Corollary (1) $\forall \kappa\ \forall M\ \exists N \succ M$ N is κ-saturated

(2) If $\kappa > \aleph_0$ is $\underbrace{\text{regular and } 2^{<\kappa}=\kappa}$, then $\forall M$ $\exists N \succ M$ of power $\leq \kappa$

$\Updownarrow$ ← exercise

$\kappa^{<\kappa} = \kappa$

($\|N\| = \kappa$ and N = saturated).

Proof

Remark. For $A \subseteq M$, $\underset{\|}{\overset{S_1(A)}{}}|S(A)| \leq 2^{|A|+\aleph_0}$

Proof. $|S(A)| \leq |\mathcal{P}(L_1(A))| \leq 2^{|L_1(A)|}$, $|L_1(A)| = |A| + \aleph_0$.

Idea of the proof of (1).

Let $\mu = 2^\kappa$. Then $cf(\mu) > \kappa$

We construct a elementary chain of models N_α, $\alpha<\mu$, of power $\leq \mu$ each such that:

(1) $M = N_0 \prec N_1 \prec \ldots \prec N_\alpha \prec N_\beta \prec \ldots$ for $\alpha < \beta < \mu$

Recursively:

(2) At step $\alpha = \beta+1$: we have a model N_β.

We choose $N_\alpha \succ N_\beta$ so that:

$(\forall B \subseteq N_\beta, |B|<\kappa)\ \forall p \in S_1(B)$ p is realized in N_α.

and $\|N_\alpha\| \leq \mu$.

- there are $\leq \mu$ types p to consider
- there is a model $N' \succ N_\beta$ realizing all of them
- by downward Löwenheim–Skolem, can assume $\|N'\| \leq \mu$.

(3) Limit step: assume $\alpha \in \mathrm{Lim}$ (a limit ordinal) and N_β already are chosen for all $\beta < \alpha$.

Then $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$.

Then let $N = \bigcup_{\alpha<\mu} N_\alpha$

- $M = N_0 \prec N$
- N is κ-saturated.

Proof. Assume $A \subseteq N$, $|A| < \kappa$, $p \in S_1^N(A)$

$cf(\mu) > \kappa \Rightarrow (\exists \alpha < \mu)\ A \subseteq N_\alpha$

because: Say $A = \{a_\gamma : \gamma < \nu\}$, where $|A| = \nu < \kappa$.

For $\gamma < \nu$ let $\alpha_\gamma < \mu$ s.t. $a_\gamma \in N_{\alpha_\gamma}$.

~~$\{\alpha_\gamma : \gamma < \nu\} \subseteq$~~

$$\left.\begin{array}{l}\underbrace{\{\alpha_\gamma : \gamma < \nu\}}_{\text{of power } \leq \nu < \kappa < cf\,\mu} \subseteq \mu\end{array}\right] \Rightarrow \exists \alpha < \mu\ \ \{\alpha_\gamma : \gamma < \nu\} \subseteq \alpha.$$

that is: $(\forall \gamma < \nu)\ a_\gamma \in N_{\alpha_\gamma} \prec N_\alpha$

$a_\gamma \in N_\alpha$

So: $A \subseteq N_\alpha$.

So: $A \subseteq N_\alpha \prec N \Rightarrow S_1^{N}(A) = S_1^{N_\alpha}(A)$

So $p \in S_1^{N_\alpha}(A)$,

By the successor step: p is realized in $N_{\alpha+1}$ by some c.

$N_{\alpha+1} \prec N$, hence c realizes p also in N. □

(2) A similar proof.

Auxiliary lemma on $\prec$:

(1) $\prec$ is transitive [exercise]

(2) If γ is a limit ordinal and $(N_\alpha : \alpha < \gamma)$ is an elementary chain of structures, then there

[i.e. for $\alpha < \beta < \gamma$ $N_\alpha \prec N_\beta$]

is a structure N_γ [called the union of the chain]

s.t.

(a) $|N_\gamma| = \bigcup_{\alpha<\gamma} |N_\alpha|$

(b) $N_\alpha \prec N_\gamma$ for all $\alpha < \gamma$. [Tarski]

Proof

Definition of N_γ: • $|N_\gamma| = \bigcup_{\alpha<\gamma} |N_\alpha|$

• Interpretations of L-symbols in N_γ:

(i) P: relational symbol:

Let $a_1, \dots, a_n \in |N_\gamma|$.

$P^{N_\gamma}(a_1, \dots, a_n) \iff P^{N_\alpha}(a_1, \dots, a_n)$ holds for some [≡ every] $\alpha < \gamma$ with $a_1,\dots,a_n \in |N_\alpha|$

[does not depend on the choice of α,
because $(N_\alpha)_{\alpha<\gamma}$: elementary].

- f : a function symbol of L:

$a_1,\dots,a_n, a_{n+1} \in |N_\gamma|$

$f^{N_\gamma}(a_1,\dots,a_n) = a_{n+1} \Leftrightarrow$ for $\begin{array}{c}\text{some}\\ \equiv \\ \text{every}\end{array}$ $\alpha<\gamma$ with $a_1,\dots,a_{n+1} \in |N_\alpha|$

$f^{N_\alpha}(a_1,\dots,a_n) = a_{n+1}$

- c : a constant symbol of L.

$c^{N_\gamma} = c^{N_\alpha}$ for $\begin{array}{c}\text{any}\\ \equiv \\ \text{every}\end{array}$ $\alpha < \gamma$.

(b): $N_\alpha \prec N_\gamma$ for every $\alpha<\gamma$

Inductive statement:

$$(\ \forall \underset{\varphi(\bar x)}{\underset{\shortparallel}{\varphi}} \in \mathcal{F}_L \ \underbrace{\forall \alpha<\gamma \ \forall \bar a \subseteq N_\alpha \ [N_\alpha \models \varphi(\bar a) \Leftrightarrow N_\gamma \models \varphi(\bar a)]}_{(*)}$$

Proof by induction on $|\varphi|$:

1. φ quantifier-free.

Then $(*)$ true, because for every $\alpha<\gamma$, $N_\alpha \underset{\substack{\uparrow \\ \text{substructure.}}}{\subseteq} N_\gamma$

2. Induction step:

Assume $\varphi(\bar{x}) = \exists y\, \underbrace{\psi(\bar{x}, y)}_{\text{shorter, so } (*) \text{ holds for } \psi}$.

$$\overset{\overset{\bar a}{\cap}}{N_\alpha} \models \exists y\, \psi(\bar a, y) \iff N_\alpha \models \psi(\bar a, b) \text{ for some } b \in N_\alpha$$

$$\Downarrow \text{ induction hypothesis}$$

$$N_\gamma \models \psi(\bar a, b)$$

$$\Downarrow$$

$$N_\gamma \models \exists y\, \psi(\bar a, y)$$

$$N_\gamma \models \exists y\, \psi(\bar a, y) \iff N_\gamma \models \psi(\bar a, b) \text{ for some } b \in N_\gamma$$

Choose $\underset{\alpha <}{\beta} < \gamma$ s.t. $b \in N_\beta$.

$$\Downarrow \text{ induction assumption}$$

$$N_\beta \models \psi(\bar a, b)$$

$$\Downarrow$$

$$N_\beta \models \exists y\, \psi(\bar a, y)$$

$$\Downarrow\ N_\alpha \prec N_\beta$$

$$N_\alpha \models \exists y\, \psi(\bar a, y)$$

Example.

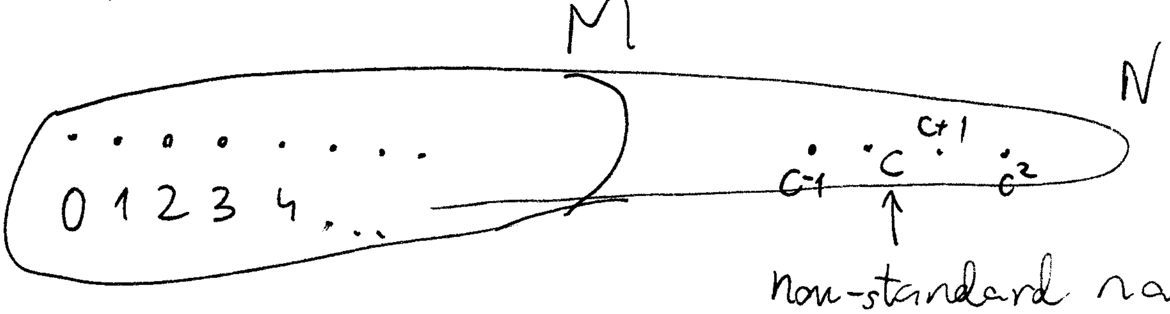
$M = (\mathbb{N}, \overbrace{+, \cdot, 0, 1, <}^{L})$ $TA = Th(M)$ true arithmetic.

For $n \in \mathbb{N}$ let $\underline{n} = \overbrace{\underline{1} + \ldots + \underline{1}}^{n}$ (an L-term) not a number
(numeral n)

Let $p(x) = \{x > \underline{n} : n \in \mathbb{N}\}$. : a consistent type in TA.

Let $N \succ M$ countable s.t. p is realized in N, by some $c \in N$.

$$\underset{\uparrow}{N} \succ M$$



non-standard natural number, infinitely large

$M \models (\forall x \neq 0)\, \exists y\ \ y+1=x$

$\Downarrow$

~~N ⊨ "istnieje poprzednik c"~~ ← też

$N \models$ "there is a predecessor of c" ← also infinitely large

Similarly: $c < c+1 < c+2 < \ldots < c^2 \ldots$

Example 2. Non-standard analysis.

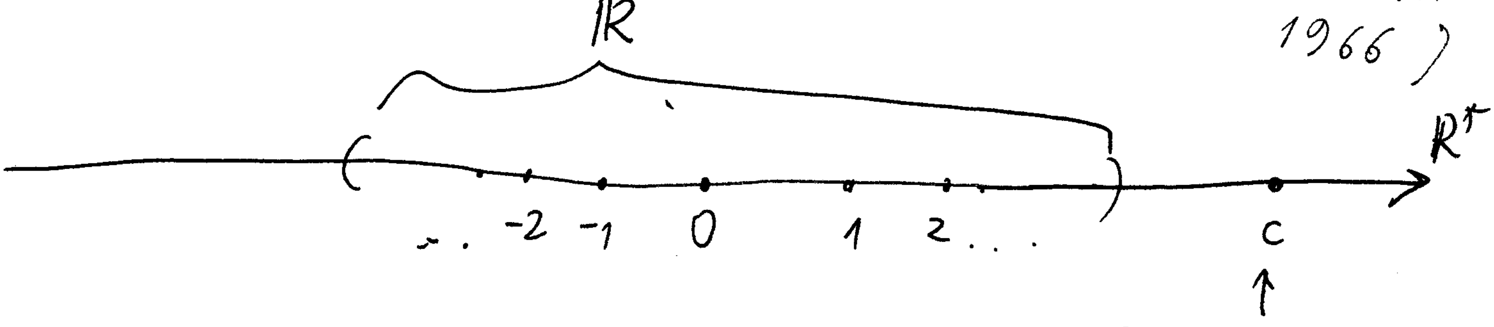
$\mathcal{R} = (\mathbb{R}, \overbrace{+, \cdot, 0, 1, f, r, R}^{L})$

f : all functions : $\mathbb{R}^k \to \mathbb{R}$
R : all relations $\subseteq \mathbb{R}^m$
r : all reals

analysis
$A = Th(\mathcal{R})$

Let $\overset{\mathbb{R}^*}{\overset{\parallel}{M}} \models A$ $\aleph_1$-saturated. Wlog $\mathcal{R} \prec M$

analysis in M: "non-standard analysis" (Abraham Robinson 1966)



infinitely large

$$\cdot\ \frac{1}{c} \in \mathbb{R}^* : \text{ in } \mathbb{R}^* : \left.\begin{array}{l} \cdot \text{ positive}, \neq 0 \\ \cdot < \frac{1}{n} \text{ for every } n \in \mathbb{N} \end{array}\right] \text{infinitesimal}$$

Omitting types

Assume T: a complete theory, $M \models T$, $p(x)$ a type in T (consistent) (over $\emptyset$)

Def M omits p $\Leftrightarrow \neg \exists a \in M$ a realizes p

Def $p(x)$ is isolated, if $\exists \varphi(x) \in F_L$

$$T \vdash \exists x \varphi(x)$$

and

Notation: $\varphi(x) \vdash p(x) \rightsquigarrow$ $\left[T \vdash \varphi(x) \longrightarrow \psi(x) \text{ for every } \psi(x) \in p(x) \right.$

[T in the background]

Remark (1). Assume $p(x)$ is a complete type.

Then p is isolated $\Leftrightarrow$ $\{p\}$ is open in $S(\emptyset)$
[i.e. p is isolated in the topological sense].

(2) p is ~~is~~ isolated $\Leftrightarrow$ the set $\{q(x) \in S(\emptyset) : p(x) \subseteq q(x)\}$ has non-empty interior in $S(\emptyset)$
[Stone topology]

Proof (1). $\varphi \vdash p \Leftrightarrow [\varphi] = \{p\}$ in $S(\emptyset)$, ~~for~~
(If $T \vdash \exists x \varphi(x)$)

(2): exercise.

Remark If $p(x)$ is isolated, then $\forall M \models T$ p is realized in M [i.e. $p(x)$ can not be omitted].

Pf. $\varphi(x) \vdash p(x)$ $\quad M \models T \quad M \models \exists x\, \varphi(x)$

$\overset{\in}{c}$ ~~within~~ witness.

$\Leftarrow$

c realizes $p(x)$.

Thm (A. Ehrenfeucht)

If T is countable, complete, consistent, ~~p~~ $p(x)$ is non-isolated, then $\exists M \models T$ M omits p.

Proof (similar to Henkin)

Let $\{c_n : n \in \omega\}$: a set of new constant symbols, $L' = L \cup \underbrace{\quad\quad}$.

- $\{\varphi_n(x) : n < \omega\}$: enumeration of $\mathcal{F}_{L'}(x)$.
- $h : \omega \to \omega$ increasing, s.t $c_{h(n)}$ does not appear in $\varphi_0, \dots, \varphi_n$.
- Let $H_i = \{\exists x\, \varphi_i \to \varphi_i(c_{h(i)})\}$; Henkin's axiom

We define an increasing sequence of consistent sets $T = T_0 \subseteq T_1 \subseteq \dots \subseteq \mathcal{F}_{L'}$ s.t.

(a) $T_{2i+1} = T_{2i} \cup \{H_i\}$

(b) $T_{2i+2} = T_{2i+1} \cup \{\neg\psi(c_i)\}$ for some $\psi(x) \in p(x)$.

Recursive construction:

- Suppose we have T_{2i} : consistent.

  then $T_{2i+1} = T_{2i} \cup \{H_i\}$ consistent (as in Henkin's proof)

- Assume $T_{2in} = T \cup \{\psi_j(\underbrace{\bar{c}, c_i}_{\text{new constants symbols from among } c_n,\ n<\omega}) : j<k\}$

Suppose we can not pick $\psi(x) \in p(x)$ so that

$T_{2i+1} \cup \{\neg\psi(c_i)\}$ is consitent.

Hence $T_{2i+1} \vdash \psi(c_i)$ for every $\psi(x) \in p(x)$

Let $\Psi = \bigwedge_{j<k} \psi_j(\bar{c}, c_i)$

by deduction thm: $T \vdash \Psi(\bar{c}, c_i) \to \psi(c_i)$ for every $\psi(x) \in p(x)$

$\bar{c}, c_i$: new constant symbols

$$\Downarrow$$

~~$T \vdash \psi(\bar{y}, x) \to \psi(x)$~~

$$T \vdash \Psi(\bar{y}, x) \to \psi(x)$$ [if needed: $\bar{y}, x$: new variables]

But: $\models [\Psi(\bar{y}, x) \to \psi(x)] \to (\exists \bar{y}\, \Psi(\bar{y}, x) \to \psi(x))$

So: $T \vdash (\exists \bar{y}\, \Psi(\bar{y}, x)) \to \psi(x)$ for every $\psi(x) \in p(x)$

But: $T \vdash \exists x (\exists \bar{y}\, \Psi(\bar{y}, x))$ [as: $T \cup \{\Psi(\bar{c}, c_i)\}$ consistent]

hence: the formula

$\exists \bar{y}\, \Psi(\bar{y}, x)$ isolates $p(x)$ ↯

<u>Therefore</u> can find T_{2i+2} !

Now as in Henkin's proof

let $T_\infty = \bigcup_i T_i$ and $\underset{\uparrow}{S} \supseteq T_\infty$

consistent complete theory.

We construct a model M of S from constant symbols $c_n, n<\omega$.

By case ~~A~~ (b) $(T_{2i+1} \rightsquigarrow T_{2i+2})$:

$M \models \neg\psi(c_i)$ for some $\psi \in p$, so

no $\overset{\Downarrow}{c_i}$ realizes $p(x)$ and M omits p.

<u>Remark</u>.

It is much harder to omit types in uncountable theories.

Def. ($\kappa \in CN$, $\kappa \geqslant \aleph_0$)

T is κ-categorical, if $(\forall M, N \models T)(\|M\| = \|N\| = \kappa \Rightarrow M \cong N)$

[categorical in κ]

Corollary (Ryll-Nardzewski, Svenonius, Engeler) $\circlearrowright$:

(1) T is $\aleph_0$-categorical

(2) $\forall n$ $S_n(\emptyset)$ is finite

~~(3)~~ <u>Proof</u> (1) $\Longrightarrow$ (2) Suppose for some n, $S_n(\emptyset)$ is infinite.

Hence there is a non-isolated $p(\bar{x}) \in S_n(\emptyset)$.

omitting types thm $\Longrightarrow$ $\exists M_1 \models T$ omitting $p(\bar{x})$

realizing types thm $\Longrightarrow$ $\exists M_2 \models T$ realizing $p(\bar{x})$ $\quad M_1 \not\cong M_2$ ↯

(2) $\Rightarrow$ (1)

Suppose $M = \{a_n : n < \omega\}$, $N = \{b_n : n < \omega\}$ countable models of T.

We will show : $M \cong N$.

We construct a sequence of finite functions

$$\emptyset = f_{-1} \subseteq f_0 \subseteq f_1 \subseteq \dots \subseteq f_n \subseteq \dots \quad n < \omega \text{ s.t.}$$

(a) $\mathrm{Dom}\, f_i \subseteq M$, $\mathrm{Rng}\, f_i \subseteq N$

(b) $a_i \in \mathrm{Dom}\, f_i$, $b_i \in \mathrm{Rng}\, f_i$

(c) f_i is <u>elementary</u>, i.e. $tp^M(d_1,\dots,d_k) = tp^N(f(d_1),\dots,f(d_k))$,

where $\mathrm{Dom}\, f_i = \{d_1,\dots,d_k\}$.

Recursively:

Suppose we have f_i. We will find f_{i+1} $(i \geq -1)$

<u>Step 1</u> (forth) $a_{i+1} \hookrightarrow \mathrm{Dom}\, f_{i+1}$:

If $a_{i+1} \in \mathrm{Dom}\, f_i$, then do nothing

If $a_{i+1} \notin \mathrm{Dom}\, f_i$, then let

$p = tp^M(\langle d_1,\dots,d_k, a_i \rangle)$

$p \in S_{k+1}(\emptyset) \leftarrow$ finite $\Rightarrow$ p isolated :

$\varphi(x_1,\dots,x_k, y) \vdash p(\overset{d_1}{x_1},\dots,\overset{d_k}{x_k}, \overset{a_i}{y})$

so $\exists y\, \varphi(x_1,\dots,x_k,y) \in tp^M(d_1,\dots,d_k)$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ||$

$N \models \exists y\, \varphi(\overset{(f_i(d_1)\dots f_i(d_k),y)}{\cancel{d_1,\dots,d_k}, y}) \Leftarrow tp^N(f_i(d_1),\dots,f_i(d_k))$

$\Downarrow$

Let b be a witness: ~~$N \models \varphi(d_1,\dots,d_k, b)$.~~

$N \models \varphi(f_i(d_1),\dots,f_i(d_k), b)$

φ izolates type in $S_{k+1}(\emptyset)$

so $\underbrace{tp^M(d_1,\dots,d_k,a_i)}_{\substack{\Psi\\ \varphi}} = \underbrace{tp^N(\cancel{d}\, f_i(d_1),\dots,f_i(d_k),b)}_{\substack{\Psi\\ \varphi}}$

Let $f' = f_i \cup \{\langle a_i, b\rangle\}$. f' elementary.

Step 2 (back). Replace the roles of M, N. Find ~~b~~ $a \in M$ s.t. $f_{i+1} = f' \cup \{\langle a, b_i\rangle\}$ elementary.

Let $f = \bigcup_i f_i$ $f: M \xrightarrow{\equiv} N$ (by (c)), $\mathrm{Dom}\, f = M$, $\mathrm{Rng}\, f = N$ $\Rightarrow f: M \xrightarrow{\cong} N$.

17.11.2021

Assume M: a ctbl structure $\Rightarrow$ $Aut(M)$: the group of authomorphisms of M.

- $Aut(M) \curvearrowright M$: $\underset{\overset{\uparrow}{Aut(M)}}{\sigma} \cdot \underset{\overset{\uparrow}{M}}{a} = \sigma(a)$

also on M^n:

$\sigma \cdot \langle a_1, \dots, a_n \rangle = \langle \sigma(a_1), \dots, \sigma(a_n) \rangle$

Corollary (Ryll-Nardzewski) $Th(M)$ is $\aleph_0$-categorical iff $\forall n$ on M^n there are only finetely many orbits of $Aut(M)$.

Proof "$\Rightarrow$" Fix n. $S_n(\emptyset) = \underbrace{\{p_1, \dots, p_k\}}_{\text{all isolated}}$.

Let $\varphi_i(\bar{x}) \vdash p_i(\bar{x})$, $i = 1, \dots, k$

Clearly: $\bigcup \underbrace{[\varphi_i]}_{\text{clopen}} = S_n(\emptyset) \Rightarrow$ in $\mathcal{L}_n(\emptyset)$

$\overset{\shortparallel}{[\varphi_1 \vee \varphi_2 \vee \dots \vee \varphi_k]}$ $\quad [\varphi_1 \vee \dots \vee \varphi_n]_\sim = \mathbb{1}$

Hence $T \vdash \varphi_1(\bar{x}) \vee \ldots \vee \varphi_k(\bar{x})$

More: $[\varphi_i] \cap [\varphi_j] = \emptyset$ for $i \neq j$, so

$T \vdash \varphi_1(\bar{x}) \dot{\vee} \varphi_2(\bar{x}) \dot{\vee} \ldots \dot{\vee} \varphi_k(\bar{x})$ $\qquad \{\bar{a} \in M^n : M \models \varphi_k(\bar{a})\}$

Hence $M^n = \varphi_1(M) \dot{\cup} \ldots \dot{\cup} \varphi_k(M)$

i.e. $T \vdash \forall \bar{x} \left(\bigvee_i \varphi_i(\bar{x}) \wedge \bigwedge_{1 \le i < j \le k} \neg(\varphi_i(\bar{x}) \wedge \varphi_j(\bar{x}))\right)$

- $\varphi_1(M), \ldots, \varphi_k(M)$: orbits of the $Aut(M)$ on M^n:

- let $\bar{a} \in \varphi_i(M)$, $\bar{b} \in \varphi_j(M)$. If $i \neq j$ then

  $\neg \exists \sigma \in Aut(M)\ \sigma(\bar{a}) = \bar{b}$.

  If $i = j$, then $\exists \sigma$ .... because

  then $tp(\bar{a}) = tp(\bar{b})$

  $$\overset{?}{\Downarrow}$$

  construct σ

  (back-and-forth arg)

"$\Leftarrow$" Fix n. Notice that $\mathcal{L}_n(\emptyset)$ finite.

If not, then there are pair-wise T-contradictory consistent formulas $\underset{\mathcal{F}_L(\bar{x})}{\underset{\cap}{\varphi_i(\bar{x})}}$, $i < \omega$

This gives infinitely many orbits on M^n.

$\varphi_i(M)$: a union of orbits. ↯

$\lambda_n(\emptyset)$: finite $\underset{\forall n}{\Longrightarrow}$ $S_n(\emptyset)$ is finite $\Big) \Rightarrow T$ is $\aleph_0$-categorical

■

Remark Assume $M = \{a_n : n < \omega\}$, $N = \{b_n : n < \omega\}$: L-structures. s.t. $\forall n$ $tp^M(a_0, \dots, a_{n-1}) = tp^N(b_0, \dots, b_{n-1})$

Then $M \cong N$
$a_i \mapsto b_i$

"Another" proof of Gödels model existence thm:

T : consistent complete $\Rightarrow T$ has a model

We construct sequences: 1) of types $p_n(x)$
2) of new constant symbols c_n
3) of theories T_n s.t.:

a) $T = T_0 \subseteq T_1 \subseteq \dots$

(b) $p_n(x) \in S_1^{T_n}(\emptyset)$

(c) $T_{n+1} = Cn\left(T_n \cup \{\varphi(c_n) : \varphi(x) \in p_n(x)\}\right)$

in language $L_{n+1} = L \cup \{c_i : i \leq n\}$

T_n is complete and consistent: (ind. on n)

- T_0 is from assumptions
- consistency: $p_n(x)$ compl. type in T_n, so for $\underset{\text{finite}}{p' \subseteq p_n}$

$$T_n \vdash \exists x \bigwedge p'(x) \Rightarrow \underset{\text{consistent}}{T_n \cup \{\bigwedge p'(c_{n+1})\}}$$

- completeness

Let $\sigma(c_n)$: a sentence in L_{n+1}

1° $\underset{\text{in } L_n}{\sigma(x)} \in p_n(x)$ or $(\neg\sigma(x)) \in p_n(x)$

$$\Downarrow \qquad\qquad \Downarrow$$

$$T_{n+1} \vdash \sigma(c_n) \qquad T_{n+1} \vdash \neg\sigma(c_n)$$

Let $T' = \bigcup_n T_n$: consistent, complete in $\mathcal{L}' = \mathcal{L} \cup \{c_n : n < \omega\}$. We can ensure in the construction that $\forall \varphi(x) \in \mathcal{F}_{\mathcal{L}'}$ (if $T' \vdash \exists x\, \varphi(x)$, then $\underbrace{T' \vdash \varphi(c_n)}$ for some n)

$$\Updownarrow$$

$$\varphi(x) \in p_n(x)$$

(how.? choose $\varphi(x) \in \mathcal{F}_{\mathcal{L}'}$ s.t. $T' \vdash \exists x \varphi(x)$.
Then $\varphi \in \mathcal{F}_{\mathcal{L}_k}$ for some k, $T_k \vdash \exists x\, \varphi(x)$
for some $n \geqslant k$ we ensure $p_n(x) \ni \varphi(x)$
$\Rightarrow T_{n+1} \vdash \varphi(c_n) \Rightarrow T' \vdash \varphi(c_n)$)

Therefore $\{c_n : n < \omega\}$ form a model of T

Problem of model theory:

- how to: construct models?
- how to: describe models of T?

Stability hierarchy of theories: Assume T complete, consistent with infinite models.

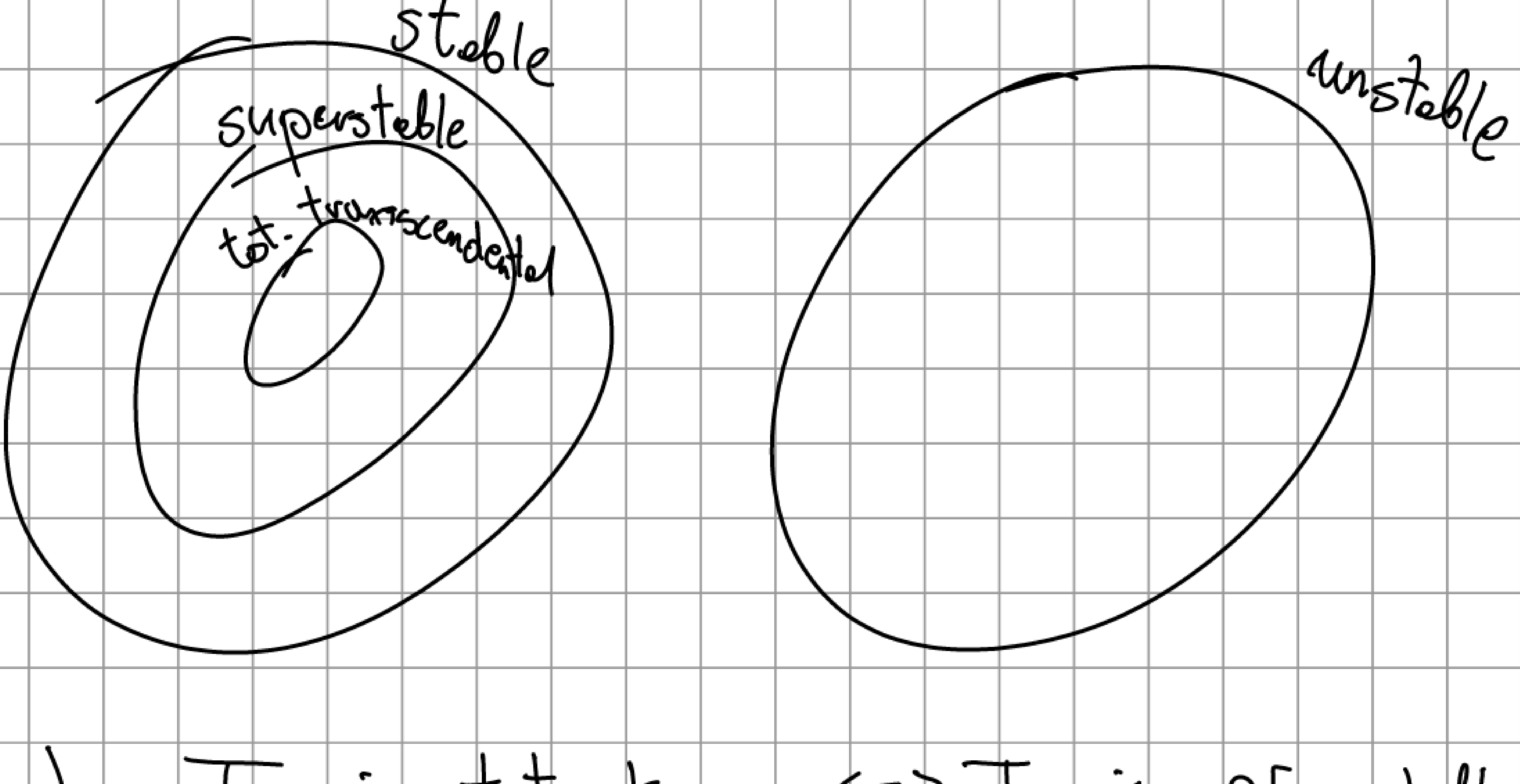
Let $\underset{\overset{\cap}{Cn}}{\kappa} \geqslant \aleph_0$.

(1) T is κ-stable $\Leftrightarrow \forall M \models T \; \forall \underset{|A| \leq \kappa}{A \subseteq |M|} \; |S_1(A)| \leq \kappa$

(2) T is stable $\Leftrightarrow \exists \kappa (T \; \kappa\text{-stable})$

(3) T is superstable $\Leftrightarrow \exists \mu \, \forall \kappa \geqslant \mu$ T is κ-stable

(4) T Totally transcendental $\Leftrightarrow \forall \kappa$ T κ-stable



<u>Remark</u> T is tot. trans. $\Leftrightarrow$ T is $\aleph_0$-stable

<u>Proof</u> „$\Rightarrow$" obv.

„$\Leftarrow$" Suppose T is $\aleph_0$-stable but not κ-stable for some κ

$\underset{\substack{\cup| \\ A}}{M} \models T$ s.t. $|S(A)| > \overset{\substack{\kappa \\ \|}}{|A|} \geq \aleph_0$

We shall find $A_0 \underset{\text{ctbl.}}{\subseteq} A$ with $|S(A_0)| > |A_0| + \aleph_0$

Let $\varphi(x) \in \mathcal{F}_{L(A)}(x)$.

Def. φ is big iff $|S(A) \cap [\varphi]| > |A|$.

$\underset{\substack{\cap \\ S(A)}}{p}$ is big iff $\forall \varphi(x) \in p(x)$

φ is big.

Lemma If $\varphi(x)$ is big, then there is $\psi(x) \underset{}{\overset{}{}}\in \mathcal{F}_{L(A)}(x)$

s.t. both $(\varphi \wedge \psi)(x)$ and $(\varphi \wedge \neg\psi)(x)$

are big.

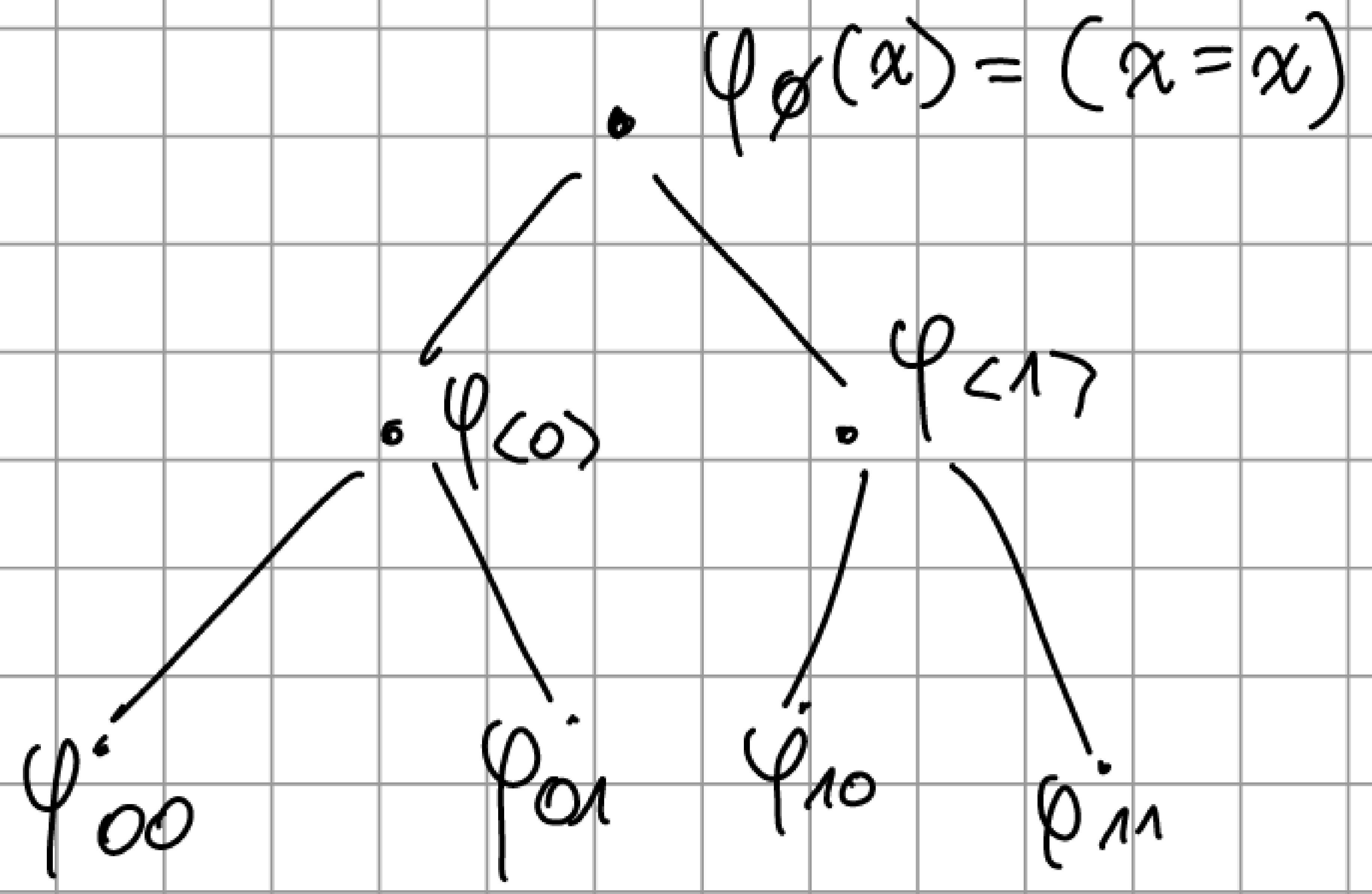
Proof (o.a.) If not, then there is

exactly one big type $\underset{\substack{\| \\ \{\psi(x) \in \mathcal{F}_{L(A)}(x) \,:\, (\varphi\wedge\psi)(x) \text{ is big}\}}}{p}(x) \in S(A) \cap [\varphi]$

big, $\in S(A) \cap [\varphi]$

$\vdots$

From the lemma we construct a tree of big formulas $\varphi_\eta(x) \in \mathcal{F}_{L(A)}(x)$, $\eta \in 2^{<\omega}$



$$T \vdash \varphi_\eta(a) \leftrightarrow \varphi_{\eta_0}(x) \dot{\vee} \varphi_{\eta_1}(x)$$

Let $\underset{\text{ctbl}}{A_0} \subseteq A$ including parameters of all $\varphi_\eta(x)$

For $\eta \in 2^\omega$ $\underbrace{\{\varphi_{\eta|n}(x) : n < \omega\}}_{\text{a consistent type over } A_0 \text{ in } T.} \subseteq p_\eta(x) \in S(A_0)$

$\eta \neq \eta' \Rightarrow p_\eta \neq p_{\eta'}$ so: $|S(A_0)| \geq 2^{\aleph_0} > |A_0| + \aleph_0$

and T is not $\aleph_0$-stable

Examples

(1) $Th(\mathbb{C},+,\cdot) = ACF_0$: $\aleph_0$-stable

(2) $Th(\mathbb{Z},+)$: superstable, but not tot. trans.

(3) $SCF_{p,l}$: the theory of separably closed fields of char. p and Ersov invariant l is stable, but not superstable

(4) $Th(\mathbb{Q},\leq)$, $Th(\mathbb{Z},+,\cdot)$, BA_0, $Th(\mathbb{R},\cdot,+,\leq)$ are unstable

(5) R: a ring with 1, M: R-module $\Rightarrow Th(M)$ is stable

24.11.2021

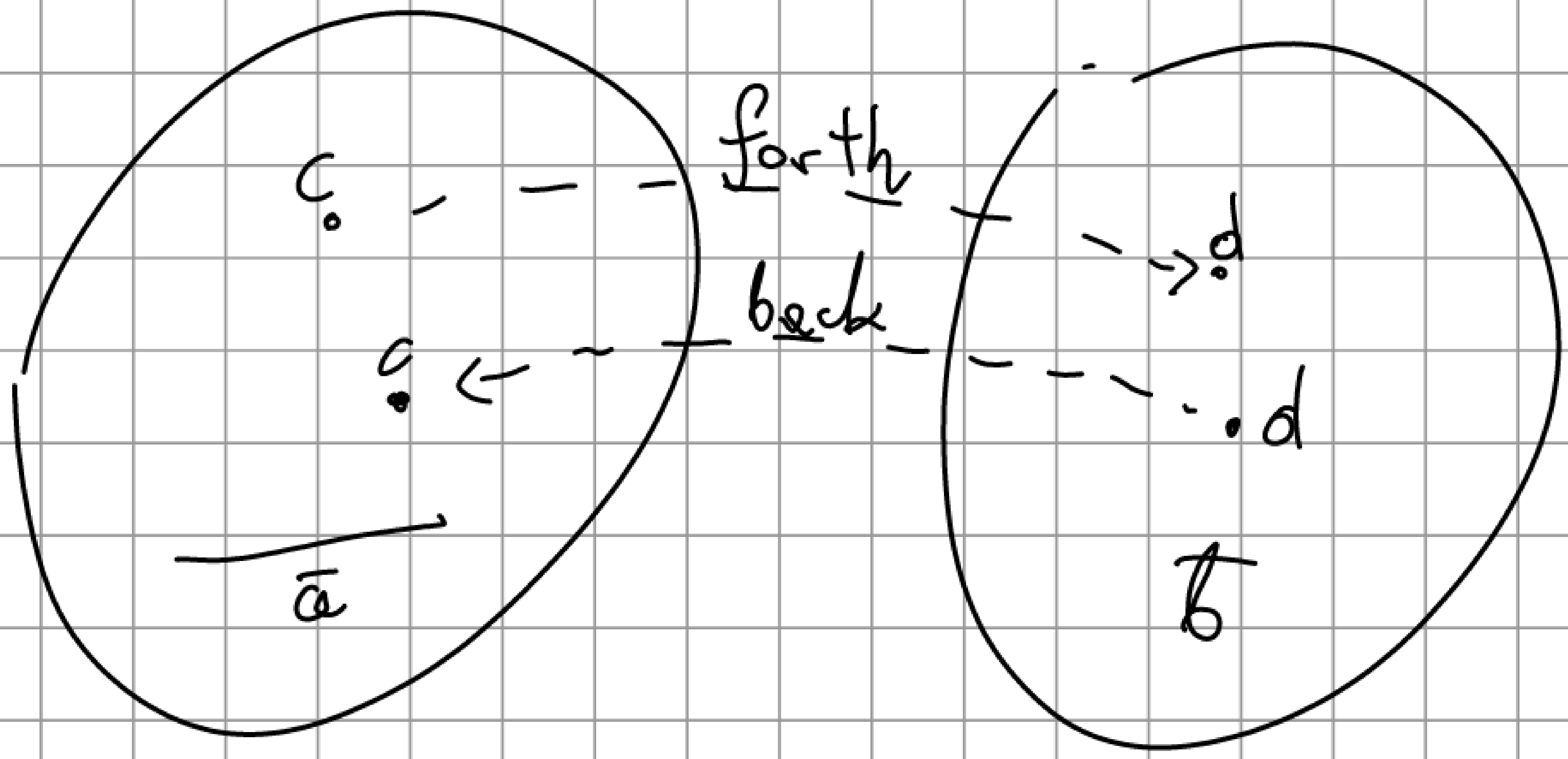
First we define equivalence rel. $\cdot \equiv_\alpha \cdot$ $(\alpha \in Ord)$ on sets M^n, $n < \omega$ (M: fixed)

1) $\bar{a} \equiv_0 \bar{b} \iff tp^M(\bar{a}) = tp^M(\bar{b})$

2) $\delta \in Lim$ $\quad \bar{a} \equiv_\delta \bar{b} \iff \forall \alpha < \delta \ \bar{a} \equiv_\alpha \bar{b}$

3) $\bar{a} \equiv_{\alpha+1} \bar{b} \iff [\forall c \in M \exists d \in M \ \bar{a}c \equiv_\alpha \bar{b}d] \land$

$[\forall d \in M \exists c \in M \ \bar{a}c \equiv_\alpha \bar{b}d]$



Fact $\equiv_\alpha$ is an equivalence relation on M^n

Proof Induction on α

1. $\alpha = 0$ obvious

2. Limit step. $\delta \in Lim$. We assume that $\forall \alpha < \delta$ $\equiv_\alpha$ is an equiv. rel.

$$\Rightarrow \cdot \equiv_{\delta} \cdot = \bigcap_{\alpha<\delta} \cdot \equiv_{\alpha} \cdot$$

3. Successor step. Assume that $\equiv_\alpha$ is an equivalence relation. We prove that $\equiv_{\alpha+1}$ is an eq. rel.. Reflexive, symmetric is trivial. Transitive:

$\bar{a} \equiv_{\alpha+1} \bar{a}' \equiv_{\alpha+1} \bar{a}''$. We want to show that $\bar{a} \equiv_{\alpha+1} \bar{a}''$.

"forth": choose any $c \in M$. Since $\bar{a} \equiv_{\alpha+1} \bar{a}'$ we have $c' \in M$ s.t $\bar{a}c \equiv_\alpha \bar{a}'c'$ and also $c'' \in M$ s.t. $\bar{a}'c' \equiv_\alpha \bar{a}''c'' \overset{\text{Ind. ass.}}{\Longrightarrow} \bar{a}c \equiv_\alpha \bar{a}''c''$

"back": similar. $\blacksquare$

<u>Fact</u> $\bar{a} \equiv_\alpha \bar{b}$, $\alpha > \beta \Rightarrow \bar{a} \equiv_\beta \bar{b}$ $\left[\equiv_\alpha \subseteq \equiv_\beta\right]$

<u>Proof</u> exercise

Lemma Assume M is ctble.

(1) $\forall \bar{a} \subseteq M \; \exists \alpha < \omega_1 \; \forall \beta \in (\alpha, \omega_1) \; [\bar{a}]_{\equiv_\alpha} = [\bar{a}]_{\equiv_\beta}$

(2) $\exists \alpha < \omega_1 \; \forall \bar{a}, \bar{b} \subseteq M \; \left(\bar{a} \equiv_\alpha \bar{b} \Rightarrow \bar{a} \equiv_{\alpha+1} \bar{b} \right)$

Proof

(1) $\langle [\bar{a}]_{\equiv_\alpha} : \alpha < \omega_1 \rangle$: a weakly decreasing sequence of ctble sets.

$$\underset{\substack{\in \\ b_\alpha}}{[\bar{a}]_{\equiv_\alpha}} \supsetneq \underset{\substack{\notin \\ b_\alpha}}{[\bar{a}]_{\equiv_\beta}}$$

It has to stabilize before ω_1.

(2) From (1) for $\bar{a} \subseteq M$ we pick $\alpha_{\bar{a}} < \omega$ as in (1)

Let $\alpha = \sup_{\bar{a} \subseteq M} \alpha_{\bar{a}} < \omega_1$ α is good in (2)

Def $SH(M) = \min \{ \alpha : \underbrace{\forall \bar{a}, \bar{b} \subseteq M \left(\bar{a} \equiv_\alpha \bar{b} \Rightarrow \bar{a} \equiv_{\alpha+1} \bar{b} \right)}_{\substack{\Updownarrow \\ \forall \beta > \alpha \; \forall n \left(\equiv_\beta = \equiv_\alpha \text{ on } M^n \right)}} \}$

Scott height

$$SH(M) < \|M\|^+$$

Remark Let $\alpha = SH(M)$. Then $\forall \beta > \alpha \ \forall \bar{a}, \bar{b} \subseteq M$
$(\bar{a} \equiv_\alpha \bar{b} \Rightarrow \bar{a} \equiv_\beta \bar{b})$

Proof. Induction on β

Comparing $\underset{\text{(ctble)}}{\text{structures}}$: $\cdot \equiv_\alpha \cdot$ $(\alpha \in \mathrm{Ord})$: equivalence relation on classes of ctbl. L-structures

(a) $M \equiv_0 N \iff M \equiv N$

(b) $(\delta \in \mathrm{Lim})$ $M \equiv_\delta N \iff \forall \alpha < \delta \ M \equiv_\alpha N$

(c) $M \equiv_{\alpha+1} N \iff [\forall \bar{a} \subseteq M \ \exists \bar{b} \subseteq N \ (M, \bar{a}) \equiv_\alpha (N, \bar{b})]$
$\wedge \ [\forall \bar{b} \subseteq N \ \exists \bar{a} \subseteq M \ (M, \bar{a}) \equiv_\alpha (N, \bar{b})]$

Thm (Dana Scott) Assume M, N ctble L-structures, $\alpha = SH(M) = SH(N)$.

(1) $M \equiv_{\alpha+1} N \Rightarrow M \cong N$

(2) $M \equiv_{\alpha+2} \underset{\substack{\uparrow \\ \text{ctble } L\text{-str.}}}{N'} \Rightarrow SH(N') = SH(M)$ and $M \cong N'$

(3) $\exists$ a $\underset{\text{(the Scott sentence)}}{\text{sentence } \varphi} \in \mathcal{L}_{\omega_1, \omega}$ s.t. $M \models \varphi$ and
$\underset{\text{ctbl } L\text{-str.}}{\forall N} \ (N \models \varphi \Rightarrow M \cong N)$

??

<u>Proof</u> (3) $L_{\omega_1\omega}$: the set of generalized formulas

(a) atomic formulas, terms : as before

(b) negation, quantifiers : as before

(c) we admit conjunctions, disjunctions of length $< \omega_1$. If $\{\varphi_i : i < \omega\}$: $L_{\omega_1\omega}$ formulas

$$\bigvee_n \varphi_n, \bigwedge_n \varphi_n : L_{\omega_1\omega} \text{ formulas}$$

Example: Scott sentence $(\mathbb{N}, \leq)$

$\varphi_0(x)$: " x is minimal"

$\varphi_{n+1}(x) : \exists y \left(\varphi_n(y) \wedge x < y \wedge \wedge \neg \exists z \; y < z < x\right)$

Analogously $L_{\kappa\lambda}$

κ: bound on len of conj.

λ: bound on len of quant.

Scott sentence for $(\mathbb{N}, \leq)$ φ: " $\leq$ is an infinite LO" $\wedge \forall x \bigvee_{n<\omega} \varphi_n(x)$

<u>Lemma 1</u> $\forall \alpha < \omega_1 \; \forall \bar{a} \subseteq M \; \exists \varphi_\alpha(\bar{x})$: a $L_{\omega_1\omega}$-formula

$$\forall \bar{b} \subseteq M \quad (\bar{a} \equiv_\alpha \bar{b} \iff M \models \varphi_\alpha(\bar{b}))$$

<u>Proof</u> Induction on α.

(1) $\alpha = 0$: $\varphi_0(\bar{x}) = \bigwedge tp^M(\bar{a})(\bar{x})$

(2) $\delta \in Lim$: $\varphi_\delta(\bar{x}) = \bigwedge_{\alpha<\delta} \varphi_\alpha(\bar{x})$

(3) $\alpha+1$: For any $b \in M$ let $\psi_b(\bar{x}, y)$ be a formula good for $[\bar{a}b]_{\equiv_\alpha}$ class namely: $\forall \bar{c}d \subseteq M \ (\bar{a}b \equiv_\alpha \bar{c}d \iff M \models \psi_b(\bar{c}, d))$ (by ind. ass.)

$$\varphi_{\alpha+1}(\bar{x}) : \forall y \bigvee_{b\in M} \psi_b(\bar{x}, y) \quad (\text{"}\forall y \exists b \ \bar{a}b \equiv_\alpha \bar{x}y\text{"})$$

$$\wedge \bigwedge_{b\in M} \exists y\, \psi_b(\bar{x}, y) \quad (\text{"}\forall b \exists y \ \bar{a}b \equiv_\alpha \bar{x}y\text{"})$$

$\blacksquare$

Lemma 2. $\forall \alpha < \omega_1 \ \forall \underset{\text{ctble}}{M} (\exists \varphi : \text{a } L_{\omega_1\omega}\text{-sentence}) \forall N$ $(M \equiv_\alpha N \iff N \models \varphi)$

<u>Proof</u> similar

Using lemmas 1 & 2 (3) follows.

Skolem functions, skolemization. T: a theory in L

Def. T has Skolem functions, if

$\forall \varphi(\bar{x}, y) \in \mathcal{F}_L$ there is a term

$t_\varphi(\bar{x}) \in \mathcal{T}_L$ st. $T \vdash \exists y\, \varphi(\bar{x}, y) \to \varphi(\bar{x}, t_\varphi(\bar{x}))$

Def. Assume T has Skolem functions.

Assume $M \models T$ and $A \subseteq |M|$. Then

$\mathcal{H}(A)$ = the smallest subset X of M st.

the Skollem hull of A in M

$A \subseteq X$ and $\forall t_\varphi(\bar{x}) \forall \bar{a} \subseteq X \quad t_\varphi^M(\bar{a}) \in X$

Remark 1. If T has Skolem functions, then T has q.e.

Proof We prove: for every $\varphi \in \mathcal{F}_L$ there is a q.f. $\varphi' \in \mathcal{F}_L$ s.t. $T \vdash \varphi \leftrightarrow \varphi'$. Induction on $|\varphi|$.

- φ is atomic formula: OK
- the connectives: OK
- $\varphi(\bar{x}) = \exists y\, \psi(\bar{x}, y)$

By the ind. ass. $T \vdash \psi \leftrightarrow \underset{\text{q.f.}}{\psi'}$

wlog. ψ q.f.

$$T \vdash \exists y\, \psi(\bar{x}, y) \underset{\substack{\uparrow \\ \text{for free}}}{\longleftrightarrow} \psi(\bar{x}, t_\psi(\bar{x}))$$

▨

<u>Remark 2</u> If T has Skolem functions and $A \subseteq M \models T$, then $\mathcal{H}(A) \prec M$

<u>Proof</u> Apply T-V test. Let $\varphi(\bar{x}, y) \in \mathcal{F}_L$, $\bar{a} \subseteq \mathcal{H}(A)$ s.t. $M \models \exists y\, \varphi(\bar{a}, y)$.

But $T \vdash \exists y\, \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, t_\varphi(\bar{x}))$

So $M \models \varphi(\bar{a}, \underset{\substack{\cap \\ \mathcal{H}(A)}}{t_\varphi(\bar{a})})$

<u>Remark 3</u> Assume "$\leq$" $\in L$ and M: L-str. s.t. $(M, \leq^M)$ is well ordered. Then in M we have definible Skolem functions, i.e. if for every $\varphi(\bar{x}, y) \in \mathcal{F}_L$ there is a definible

function f in M s.t.

$$M \models \exists y\, \varphi(\bar{x}, y) \longrightarrow \varphi(\bar{x}, f(\bar{x}))$$

Proof. (idea) $f(\bar{x})=$ the $\leq$-minimal y s.t. $\varphi(\bar{x}, y)$

Skolemization of:

$\begin{cases} \cdot \text{ language } L \\ \cdot \text{ theory } T \text{ (in } L) \\ \cdot \text{ structure } M \end{cases}$

① Single step: $L \rightsquigarrow L'$, $T \rightsquigarrow T'$, $M \rightsquigarrow M'$:
[here: $M \models T$, T: a theory in L).

For each formula $\varphi(\bar{x}, y) \in \mathcal{F}_L$ let $t_\varphi(\bar{x})$: a new function symbol.

$L' = L \cup \{t_\varphi : \varphi \in L\}$

$T' = Cn(T \cup \{\exists y\, \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, t_\varphi(\bar{x})) : \varphi \in \mathcal{F}_L\}$

$M' = M$ expanded to an L'-structure:
t_φ^M such that $M' \models \exists y\, \varphi(\bar{x}, y) \rightarrow \varphi(t_\varphi(\bar{x}))$.

So: $M' \models T'$, hence T': consistent.

② Iteration:

$L^{(0)} = L$, $L^{(n+1)} = (L^{(n)})'$
$T^{(0)} = T$, $T^{(n+1)} = (T^{(n)})'$, $n = 0, 1, 2, \ldots$
$M^{(0)} = M$, $M^{(n+1)} = (M^{(n)})'$

③ Result: skolemization of:

$L : L^S = \bigcup_n L^{(n)}$, $T : T^S = \bigcup_n T^{(n)}$, $M : M^S =$
$= M$ expanded to an L^S-structure... $M^{(n)}$, $n < \omega$.

Remark $M^s \models T^s$, T^s has Skolem functions.

Proof Let $\varphi(\bar{x}, y) \in \mathcal{F}_{L^s}$. Then $\varphi(\bar{x}, y) \in \mathcal{F}_{L^{(n)}}$ for some n,

hence $t_\varphi(\bar{x}) \in L^{(n+1)} \subseteq L^s$

and
$$\begin{array}{l} T^{(n+1)} \vdash \exists y\, \varphi(\bar{x}, y) \to \varphi(\bar{x}, t_\varphi(\bar{x})) \\ \ \cap\!| \\ T^s \quad\ \vdash \exists y\, \varphi(\bar{x}, y) \to \varphi(\bar{x}, t_\varphi(\bar{x})). \end{array}$$

Fact. T^s is a conservative extension of T, that is:

If $\varphi \in \mathcal{F}_L$, then $T \vdash \varphi \iff T^s \vdash \varphi$.

Proof Wlog $\varphi = \bar{\varphi}$: a sentence. $\boxed{\Rightarrow}$ clear.

$\Leftarrow$: Suppose $T \nvdash \varphi$. So there is $M \models T \cup \{\neg\varphi\}$.

$$\begin{array}{ccc} & & \Downarrow \\ T^s \nvdash \varphi & \Leftarrow & M^s \models T^s \cup \{\neg\varphi\} \end{array}$$

Ramsey thm Assume $f : \underbrace{[\mathbb{N}]^k}_{\substack{\| \\ \{X \in \mathcal{P}(\mathbb{N}) : |X| = k\}}} \to \{0, 1\}$.
$k \geq 1$

Then $\exists \underset{\text{infinite}}{X \subseteq \mathbb{N}}\ \exists t \in \{0, 1\}\ f\restriction_{[X]^k} \equiv t$

(X is called "homogeneous for f").

Proof. Induction on k. $k = 1$: OK.

and $\mathbb{N}$ replaced by any infinite set.

Induction step $k \mapsto k+1$:

Assume $f: [\mathbb{N}]^{k+1} \to \{0,1\}$.

We define $a_n \in \mathbb{N}$ and $\underset{\text{infinite}}{X_n} \subseteq \mathbb{N}$ (for $n \in \mathbb{N}$) s.t.

(1) $a_i \in X_{i-1} \setminus X_i$ $\qquad$ [and $X_{-1} = \mathbb{N}$]

(2) $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$

(3) $\forall i \; \exists t_i \in \{0,1\} \; \forall Y \in [X_i]^k \; f(\{a_i\} \cup Y) = t_i$

Construction:

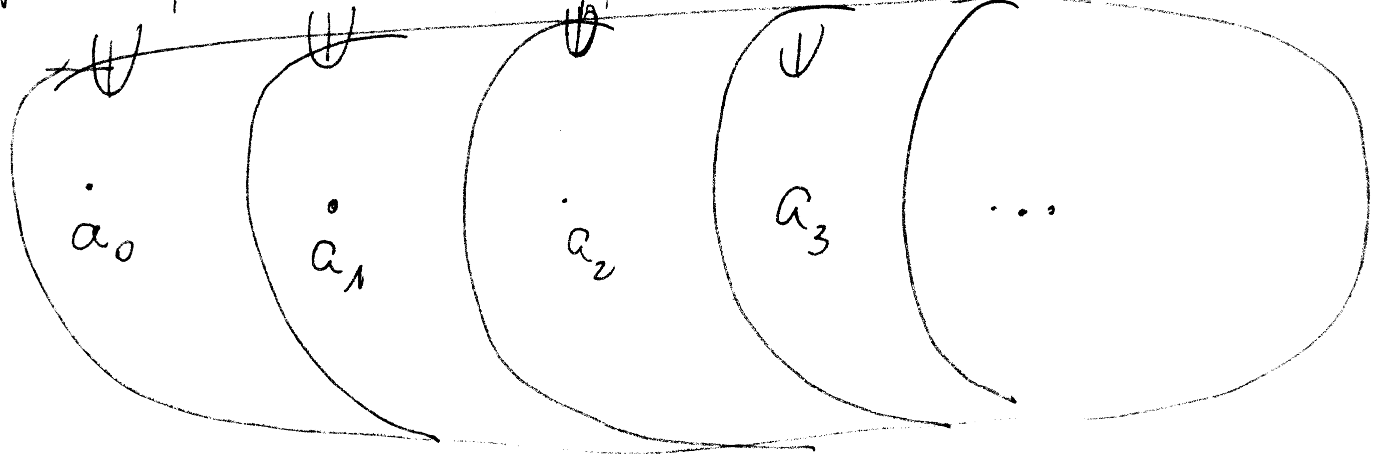
- Let $\underline{\text{any}}$ $a_i \in X_{i-1}$. Define $f': [X_{i-1} \setminus \{a_i\}]^k \to \{0,1\}$ by $f'(Y) = f(Y \cup \{a_i\})$

by inductive assumption there is $\underset{\text{infinite}}{X_i} \subseteq X_{i-1} \setminus \{a_i\}$, homogeneous for f'.

... $f' \restriction_{[X_i]^k} \equiv t_i$

~~Picture~~ Drawing:

$\mathbb{N} = X_{-1} \supseteq X_0 \supseteq X_1 \supseteq X_2 \ldots$



Let $t \in \{0,1\}$ s.t. $\{i : t = t_i\}$ is infinite.

Then $X := \{a_i : t = t_i\}$ good for f

homogeneous

Why? Let $Y \in [X]^{k+1}$ then $f(Y) = t$, because:

$$Y = \{a_{i_1}, a_{i_2}, \dots, a_{i_{k+1}}\} = \{a_{i_1}\} \cup Y', \quad Y' \in [X_{i_1}]^k$$

$$\text{so: } f(\{a_{i_1}\} \cup Y') = \underset{\substack{\uparrow \\ \text{step } i_1}}{f'}(Y') = t_{i_1} = t.$$

Corollary (Ramsey thm)

$$f : [\mathbb{N}]^k \longrightarrow \{0,1,\dots,l\} \Rightarrow \exists \underset{\text{infinite}}{X \subseteq \mathbb{N}} \; \exists t \in \{0,1,\dots,l\} \quad f\restriction_{[X]^k} \equiv t.$$

<u>many variants</u>, like:

Corollary (finite version of Ramsey theorem)

~~$\forall k, l \in \mathbb{N}^+ \exists n \in \mathbb{N}^+ \forall$~~

$$\forall k,l,m \in \mathbb{N}^+ \; \exists n \in \mathbb{N}^+ \; \underset{|X| \geqslant n}{\forall X} \; \forall f : [X]^k \longrightarrow \{0,\dots,l\} \quad \exists Y \in [X]^m \; f\restriction_{[Y]^k} \text{ constant}$$

Order indiscernible sets

Def Assume $(I, \leq)$: a linear ordering (e.g. $I = \mathbb{N}$)

and $A = \{a_i : i \in I\} \subseteq M$ (an I-indexed set)

We say that:

A is <u>order indiscernible</u> if $\overset{\forall n}{\forall}\ \varphi(x_1,\dots,x_n) \in \mathcal{F}_L$

$$\forall i_1 < \dots < i_n,\ j_1 < \dots < j_n \in I$$

~~$M \models \varphi(a_{i_1},\dots,a_{i_n}) \Leftrightarrow M$~~

$$M \models \varphi(a_{i_1},\dots,a_{i_n}) \Longleftrightarrow \varphi(a_{j_1},\dots,a_{j_n}). \qquad (*)$$

<u>Thm</u> If T is a complete theory with infinite models and $(I, \leq)$ is a linear order, then

$\exists M \models T\ \exists A = \{a_i : i \in I\} \subseteq M$ A infinite, order indiscernible.

<u>Proof</u> Let $L' = L \cup \{c_i : i \in I\}$ new constant symbols.

$$T' = T \cup \bigcup_{\varphi} S_\varphi, \qquad \varphi = \varphi(x_1,\dots,x_n) \in \mathcal{F}_L$$

where $S_\varphi = \{\varphi(c_{i_1},\dots,c_{i_n}) \Longleftrightarrow \varphi(c_{j_1},\dots,c_{j_n}) : \begin{array}{l} i_1 < \dots < i_n \in I \\ j_1 < \dots < j_n \in I \end{array}\}$

<u>It is enough to show that T' is consistent.</u>

Let $\varphi_1,\dots,\varphi_k$: formulas, $\underset{\text{finite}}{S'_{\varphi_t}} \subseteq S_{\varphi_t}$ for $t = 1,\dots,k$

We will show that $T \cup \bigcup_t S'_{\varphi_t}$ is consistent.

Let $N \models T$ infinite

<u>Lemma</u>. If $A = \{a_n, n \in \mathbb{N}\} \subseteq N$ and $\varphi(x_1, \dots, x_k) \in \mathcal{F}_L$, then $\exists X \subseteq \mathbb{N}$ infinite $A' = \{a_n : n \in X\}$ is order φ-indiscernible [i.e. (*) holds for φ]

<u>Proof</u> For $Y = \{n_1 < \dots < n_k\} \in [\mathbb{N}]^k$

let
$$f(Y) = \begin{cases} 0, & \text{when } \mathbb{N} \models \neg\varphi(a_{n_1}, \dots, a_{n_k}) \\ 1, & \text{when } \mathbb{N} \models \varphi(a_{n_1}, \dots, a_{n_k}) \end{cases}$$

$$f : [\mathbb{N}]^k \longrightarrow \{0, 1\}.$$

Let $X \subseteq \mathbb{N}$: homogeneous for f infinite $\quad X$: good. $\square$ Lemma.

Applying Lemma a few times we find

$A = \{a_n : n \in \mathbb{N}\} \subseteq N$ s.t. $\forall 1 \leq t \leq k$ (*) holds for φ_t

Assume S'_{φ_t} $(1 \leq t \leq k)$ use only constants
$$\begin{array}{ccc} & c_{i_1}, \dots, & c_{i_N}, \quad i_1 < \dots < i_N \\ \text{interpretation} & \downarrow & \downarrow \\ & a_1, \dots, & a_N \in N \end{array}$$

So : $N \models T \cup \bigcup_t S'_{\varphi_t}$.

Corollary (Ehrenfeucht – Mostowski)

Assume T is countable, consistent, with infinite models.

Then $\exists M \models T$ ctble $\quad \mathrm{Aut}(\mathbb{Q}, \leq) \hookrightarrow \mathrm{Aut}(M)$.

Proof ~~Proof~~

$T \rightsquigarrow T^s$. Let $M^s \models T^s$ ctble, containing

$$A = \{a_q : q \in \mathbb{Q}\} \text{ order indiscernible}$$

Let $N^s = \mathcal{H}(A) \prec M^s$

(†) For $f \in Aut(Q, \leq)$ let $\hat{f} : N^s \longrightarrow N^s$ given by:

(a) $\hat{f}(a_q) = a_{f(q)}$

(b) If $t(\bar{x})$: a term of L^s, then

$$\hat{f}(t^{N^s}(a_{q_1}, \dots, a_{q_n})) = t^{N^s}(\hat{f}(a_{q_1}), \dots, \hat{f}(a_{q_n})).$$

$\hat{f} \in Aut(N^s) \implies \hat{f} \in Aut(N)$, where $N = N^s\restriction_L$.

$$f \longmapsto \hat{f}$$

$Aut(Q, \leq) \hookrightarrow Aut(N)$. N ctble, $N \models T$

N^s is called an Ehrenfeucht-Mostowski model.

Comments

- the definition of $\hat{f}$ is correct:
  - each $a \in N^s$ is of the form $a = t^{N^s}(\bar{a}_{\bar{q}})$, $\bar{a}_{\bar{q}} \subseteq A$
  - if $a = t^{N^s}(\bar{a}_{\bar{q}}) = t'^{N^s}(\bar{a}_{\bar{q}'})$, then

$$N^s \models t(\bar{a}_{\bar{q}}) = t'(\bar{a}_{\bar{q}'}) \implies N^s \models t(\bar{a}_{f(\bar{q})}) = t'(\bar{a}_{f(\bar{q}')})$$

[A: order indiscernible, f preserves $\leq$]

- ~~$\hat{f} \in \mathrm{Aut}(N^s)$~~ : $f \in \mathrm{Aut}(N^s)$

$$N^s \models \varphi(a_{q_1},\dots,a_{q_n}) \Leftrightarrow \varphi(\hat{f}(a_{q_1}),\dots,\hat{f}(a_{q_n}))$$ [by order indiscernibility].

Ultraproducts

(a) **Products.** M_i, $i \in I$: L-structures.

$\rightsquigarrow M = \prod_I M_i$ product of structures M_i, $i \in I$.

$$|M| = \prod_{i \in I} |M_i| = \{f : I \longrightarrow \bigcup_{i \in I} |M_i| : \forall i \ f(i) \in |M_i|\}.$$

L-structure on M:

- P : relational symbol of L.

$$P^M(f_1,\dots,f_n) \Leftrightarrow \forall i \ P^{M_i}(f_1(i),\dots,f_n(i))$$

- F : function symbol of L

$$F^M(f_1,\dots,f_n) = \langle F^{M_i}(f_1(i),\dots,f_n(i)) : i \in I \rangle \in |M|.$$

- constant symbol c of L

$$c^M = \langle c^{M_i} : i \in I \rangle.$$

Examples products of groups, linear spaces, fields, rings...

Troubles 1. Product $K_1 \times K_2$ of fields is not a field.

2. $\mathrm{Th}(\prod_I M_i)$ unrelated to $\mathrm{Th}(M_i)$, $i \in I$.

Solution ultraproducts (families of "large" sets) LR2/9

Def.: $\mathcal{U}$ is an ultrafilter on I $\Leftrightarrow$

$\mathcal{U}$ is an ultrafilter in algebra $\mathcal{P}(I)$

- An ultrafilter $\mathcal{U}$ on I is principal if $\exists a \in I$

$$\mathcal{U} = \mathcal{U}_a = \{X \subseteq I : a \in X\}.$$

Otherwise: $\mathcal{U}$: non-principal.

Properties of ultrafilter $\mathcal{U}$:

(1) $\forall X \subseteq I \ (X \in \mathcal{U} \ \underline{\vee} \ X^c \in \mathcal{U})$

(2) $\forall X \in \mathcal{U} \ \forall \underset{X \subseteq}{Y} \subseteq I \ \ Y \in \mathcal{U}$

(3) Every proper filter on I extends to an ultrafilter on I.

8.12.2021

$\mathcal{U}$: ultrafilter on I

$\{M_i\}_{i\in I}$: a family of L-structures

<u>Def</u> <u>Ultraproduct</u> $M = \prod_I M_i / \mathcal{U}$

(a) $\sim$: an equivelance relation on $\prod_{i\in I} |M_i|$

$$f \sim g \iff \{i \in I : f(i) = g(i)\} \in \mathcal{U}$$

(b) Let $|M| = \prod_I |M_i| / \sim$. We define an L-structure on M:

1) P - predicate symbol, then

$$P^M([f_1]_\sim, \dots, [f_n]_\sim)$$
$$\Updownarrow$$
$$\{i \in I : M_i \models P^{M_i}(f_1(i), \dots, f_n(i)\}$$

2) F: func. symbol of L:

$$F^M([f_1]_\sim, \dots, [f_n]_\sim) = [g]_\sim$$

where $g(i) = F^{M_i}(f_1(i), \dots, f_n(i))$

3) c: constant symbol of L:

$$c^M = [\langle c^{M_i} \rangle_{i\in I}]_\sim$$

This definition is correct.

Let's see for 1):

Suppose $f_1 \sim f_1', \dots, f_n \sim f_n' \Rightarrow$

$A := \{i \in I : f_1(i) = f_1'(i) \wedge \dots \wedge f_n(i) = f_n'(i)\} \in \mathcal{U}$

Let $B = \{i \in I : P^{M_i}(f_1(i), \dots, f_n(i))\}$

$C = \{i \in I : P^{M_i}(f_1'(i), \dots, f_n'(i))\}$

$B \in \mathcal{U} \Leftrightarrow B \cap A \in \mathcal{U}$

$C \in \mathcal{U} \Leftrightarrow C \cap A \in \mathcal{U}$, but $B \cap A = C \cap A$!

Let's see for 2):

$$\underset{\substack{\cap \\ \mathcal{U}}}{A} \subseteq \underset{\substack{\cap \\ \mathcal{U}}}{\{i \in I : F^{M_i}(f_1(i), \dots, f_n(i)) = F^{M_i}(f_1'(i), \dots, f_n'(i))\}}$$

$$\Longrightarrow \quad \Longrightarrow \langle F^{M_i}(f_1(i), \dots, f_n(i))\rangle_{i \in I} \overset{?}{=} \langle F^{M_i}(f_1'(i), \dots, f_n'(i))\rangle_{i \in I}$$

Remark

If $\underset{\substack{\shortparallel \\ \mathcal{U}_a,\ a \in I}}{\mathcal{U}}$ is principal then $M \cong M_a$

Proof $$\begin{array}{ccc} M & \xrightarrow{\cong} & M_a \\ [f]_\sim & \longmapsto & f(a) \end{array} \quad \blacksquare$$

Theorem (Łoś)

For $\varphi(x_1, \dots, x_n) \in \mathcal{F}_L$ and $[f_1]_\sim, \dots, [f_n]_\sim \in |M|$, where $M = \prod_{i \in I} M_i / \mathcal{U}$:

$$M \models \varphi([f_1]_\sim, \dots, [f_n]_\sim) \iff \underbrace{\{i \in I : M_i \models \varphi(f_1(i), \dots, f_n(i))\}}_{\overset{\cap}{\mathcal{U}}}$$

An auxiliary fact

If $t(x_1, \dots, x_n) \in \mathcal{T}_L$ then

$$t^M([f_1]_\sim, \dots, [f_n]_\sim) = [\langle t^{M_i}(f_1(i), \dots, f_n(i))\rangle_{i \in I}]_\sim$$

Proof: exc.

Proof (Łoś)

Induction on $|\varphi|$

(a) φ atomic, $\bar{x} = (x_1, \dots, x_n)$

(i) $\varphi : t(\bar{x}) = t'(\bar{x})$

$$M \models \varphi(\overline{[f]_\sim}) \iff t^M(\overline{[f]_\sim}) = t'^M(\overline{[f]_\sim})$$

$$\iff \langle t^{M_i}(\overline{f_i(i)}) : i \in I \rangle \sim \langle t'^{M_i}(\overline{f_i(i)}) : i \in I \rangle$$

$$\overset{\text{aux. fact}}{\Longleftrightarrow} \{ i \in I : M_i \models t(\bar{f}(i)) = t'(\bar{f}(i)) \} \in \mathcal{U}$$

$$(ii)\ \varphi :\ P(\underbrace{t_1(\bar{x}), \dots, t_n(\bar{x})}_{\bar{t}(\bar{x})})$$

$$M \models \varphi([\bar{f}]_\sim) \Leftrightarrow M \models P^M([\bar{t}^M([\bar{f}]_\sim)]_\sim)$$

$$\Leftrightarrow \{ i \in I : P^{M_i}(\bar{t}^{M_i}(\bar{f}(i))) \} \in \mathcal{U}$$

$$\Leftrightarrow \{ i \in I : M_i \models \underbrace{P(\bar{t}(\bar{f}(i)))}_{\varphi(\bar{f}(i))} \}$$

(b) $\neg$, $\wedge$

$$(i)\ \varphi : \neg\psi .\quad M \models \varphi([\bar{f}]_\sim) \Leftrightarrow M \not\models \psi(\bar{f})$$

$$\overset{\text{ind.}}{\Longleftrightarrow} \{ i \in I : M_i \models \psi(\bar{f}(i)) \} \notin \mathcal{U}$$

$$\overset{\mathcal{U}:\text{uf}}{\Longleftrightarrow} \{ i \in I : M_i \not\models \psi(\bar{f}(i)) \} \in \mathcal{U}$$

$$\Longleftrightarrow \{ i \in I : M_i \models \varphi(\bar{f}(i)) \} \in \mathcal{U}$$

$(ii)\ \varphi : \varphi_1 \wedge \varphi_2$ easy

$(iii)\ \varphi : \exists y\, \psi(\bar{x}, y)$

"$\Rightarrow$" $M \models \varphi([\bar{f}]_\sim) \Rightarrow \exists g \in |M|\ \ M \models \psi([\bar{f}]_\sim, [g]_\sim)$

$$\overset{\text{ind.}}{\Longleftrightarrow} \{ i \in I : M_i \models \psi(\bar{f}(i), g(i)) \} \in \mathcal{U}$$

$$\cap|$$

$$A := \{ i \in I : \underbrace{M_i \models \exists y\, \psi(f(i), y)}_{\varphi(\overline{f}(i))} \} \in \mathcal{U}$$

„$\Leftarrow$" Assume $A \in \mathcal{U}$

We define a witness $[g]_\sim \in M$ for y in $\varphi([f]_\sim) = \exists y\, \psi([f]_\sim, y)$

- when $i \in A$ $g(i) \in |M_i|$ s.t.
  $$M_i \models \psi(f(i), g(i))$$
- when $i \notin A$ then $g(i) \in |M_i|$ arbitrary

So:

$$\begin{array}{ccl} A & \subseteq & \{ i : M_i \models \psi(f(i), \overline{g}(i)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathcal{U} & \Longrightarrow & \mathcal{U} \end{array}$$

by the ind. hypothesis for ψ.

$$M \models \psi([f]_\sim, [g]_\sim) \Rightarrow M \models \varphi([f]_\sim).$$

∎

A special case

$M_i = N$ for all $i \in I$

$M = \prod_I M_i/\mathcal{U} = \underbrace{N^I/\mathcal{U}}_{\text{Ultrapower}}$

Let $f : N \longrightarrow N^I/\mathcal{U}$

$f(a) = [f_a]_\sim$, $f_a : I \longrightarrow |N|$

$f_a \equiv a$

(a diagonal embedding)

Remark

(a) f is 1-1

(b) $f[N] \prec N^I/\mathcal{U}$

(c) $f : N \xrightarrow{\equiv} N^I/\mathcal{U}$

Proof (c) Let $\varphi(\bar{x}) \in \mathcal{F}_L$, $\bar{a} \subseteq N$.

$N \models \varphi(\bar{a}) \iff \{i \in I : N \models \varphi(\underbrace{f_{\bar{a}}(i)}_{\overset{\parallel}{\bar{a}}})\} \in \mathcal{U}$

$\overset{\text{Łoś thm}}{\iff} N^I/\mathcal{U} \models \varphi(f(\bar{a}))$

Compactness thm (new proof)

Assume T is a set of sentencess s.t. every finite $T_0 \subseteq T$ has a model.

Then T has a model.

Proof Let $I = [T]^{<\aleph_0} = \{T_0 \subseteq T : T_0 \text{ is finite}\}$

For $i \in I$ let $M_i \models i$

Let $\varphi \in T$. $\{i \in I : M_i \models \varphi\}$

$$\supseteq$$

$$X_\varphi := \{i \in I : \varphi \in i\}$$

$\{X_\varphi : \varphi \in T\} \subseteq \mathcal{P}(I)$

has f. i. p. $\longrightarrow \{\varphi_1, \dots, \varphi_n\} \in X_{\varphi_1} \cap \dots \cap X_{\varphi_n} \neq \emptyset$

(finite intersection property)

so $\exists\, \mathcal{U}$: an uf on I st. $\forall \overset{T}{\varphi}\ X_\varphi \in \mathcal{U}$

Let $M = \prod_I M_i/\mathcal{U}$

- $M \models T$

Let $\varphi \in T$. $M \models \varphi \iff \{i \in I : M_i \models \varphi\} \in \mathcal{U}$

$\supseteq$

$X_\varphi \in \mathcal{U}$ ✓

Example Let $\mathcal{U}$: a non-principal ultrafilter on $\mathbb{N} = \omega$

(a Frechet ultrafilter)

If $\{M_i : i < \omega\}$ s.t. $\|M_i\| = k < \omega$

then $\|\prod_{\omega} M_i / \mathcal{U}\| = k$

Proof Look at φ_k: "there're exactly k-many elements"

Special case Let $\mathcal{R} = (R, +, \cdot, r, f, P)_{r, f, P : \text{all}}$

Let $\mathcal{U}$: a Frechet ultrafilter on ω.

$$\begin{array}{l} R^* = N^{\omega}/\mathcal{U} \\ \curlyvee \\ R \end{array}$$

Properties of R^*:

(a) $\|R^*\| = 2^{\aleph_0}$

(b) Every ctble consistent type $p(x)$ over R^* is realised in R^*

Set theory as "meta theory" for mathematics.

- What exists?
  - for sure: $\emptyset$ (in style of Descartes)

Set theory:

(1) describes procedures of creating sets

(2) describes properties of sets.

For simplicity: „every individual is a set'

- Language of set theory: $L = \{ \in \}$

relational binary symbol

The first try: "META-META SET THEORY"

Informal

If $\varphi(x) \in F_L$ then
there is $y = \{ x : \varphi(x) \}$

Formally: $\exists y \forall x \, (x \in y \leftrightarrow \varphi(x))$

Russel antynomy:

Let $\varphi(x): x \notin x$

$\exists y \forall x (x \in y \leftrightarrow x \notin x)$ is contradictory, it has no model.

$$\text{If } (M, \underset{\underset{\in^M}{\|}}{E}) \models \underset{\underset{a \in M}{\uparrow}}{\exists y}\, \underset{\underset{a \in M}{\uparrow}}{\forall x} (x \in y \leftrightarrow x \notin x) \qquad a \in a \leftrightarrow a \notin a$$

Therefore: need care when constructing sets.

- General rule: allow as many sets as possible (by Gödel, freedom principal)

Axioms of ZFC (Zermelo-Frenkel set theory with choice axiom)

A1 (Existence) "$\exists \phi$"

$$\exists x \forall y \;\; y \notin x$$

A2 (Extensionality) If x, y have the same elements, then $x = y$

A3 (Pair axiom)

$$(\forall x, y) \exists z \; \forall t \; (t \in z \leftrightarrow t = x \vee t = y)$$

A4 (Comprehension)

If $\varphi(x, \bar{y})$ is a formula, X : a set then

$$\forall \bar{y} \, \exists Y = \{x \in X : \varphi(x, \bar{y})\}$$

A5 (Union) $\forall X$ $\bigcup X$ exists

A6 (Power set) $\forall X$ $\mathcal{P}(X)$ exists

A7 (Replacement)

~~LR~~ LR/3 ①

~~Aksjomaty~~ ZFC Axioms

A1. (~~istnienia~~ existence) : "~~istnieje~~ exists $\emptyset$" : $\exists x \forall y \; y \notin x$

A2. (~~ekst~~ extensionality)

If x, y have the same elements, then $x = y$

A3. (Pair) $\forall x, y \, \exists z \, \forall t \, (t \in z \Leftrightarrow t = x \vee t = y)$

A4. Comprehension (wyróżniania)

Assume ~~$\varphi(\bar{z})$~~ $\varphi(x, \bar{z}) \in \mathcal{F}_{L_{ZF}}$, X : a set $\rightsquigarrow$

$\exists Y = \{x \in X : \varphi(x, \bar{y})\}$

Formally: axiom scheme for a formula $\varphi(x, \bar{z})$.

$\forall x \, \exists y \, \forall t \, (t \in y \Leftrightarrow t \in x \wedge \varphi(t, \bar{z}))$

A5. Union : ~~$\exists$~~ $\forall x$ $\bigcup x$ exists

A6. Power set : $\forall x \, \exists \mathcal{P}(x)$

A7. Replacement (zastępowania)

If $\varphi(x, y, \bar{z}) \in \mathcal{F}_{L_{ZF}}$ is a function $[\text{i.e. } \forall x \, \exists! y \; \varphi(x, y, \bar{z})]$

then $\forall X \, \exists Y = \{y : (\exists x \in X) \, \varphi(x, y, \bar{z})\}$

A8. Infinity ("There is an infinite set")

How to write it down?

For example: $\underset{\text{(inductive)}}{\operatorname{ind}(x)} = (\forall y \in x)(y \cup \{y\} \in x) \wedge \emptyset \in x$

Axiom: $\exists x \operatorname{ind}(x)$

A9. Regularity $\forall \underset{\neq \emptyset}{x} \, \exists y \in x \; y \cap x = \emptyset$.

LR/3 (2

A10. Choice Axiom:

If $\forall a \in A\ a \neq \emptyset$, then $\exists f : A \longrightarrow \bigcup A\ \forall a \in A\ f(a) \in a$

Remark 1. (ZFC) $\neg \exists x \forall y\ y \in x$ (There is no set of all sets)

Proof Suppose not and x is the set (capo di tutti capi) of all sets. Then x violates several axioms:

- comprehension: let $x' = \{y \in x : y \notin y\}$ ⇝ Russell antinomy

(• regularity + pair) $x \in x \Rightarrow \underline{\underline{\{x\}}}$

Therefore classes

Let $\varphi(x, \bar{y}) \in F_{L_{ZF}}$

then for every $\bar{y}$ we think of $\varphi(x, \bar{y})$ as $C_{\varphi, \bar{y}} = \{x : \varphi(x, \bar{y})\}$ class

A class is proper if ~~there~~ it is not a set.

$C_{\varphi, \bar{y}}$

Example: $C_= = \{x : x = x\}$ is a proper class.

Natural numbers: $\mathbb{N} = \bigcap \{x : ind(x)\}$: a set because: let x_0: inductive (by ∞-axiom).

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5. Wymagania techniczne i edytorskie

5.1 Zaleca się przygotowywanie prac dyplomowych przy użyciu programu TeX, z uwagi na przystosowanie tego programu do profesjonalnego składu tekstów matematycznych. Dopuszczalne jest przygotowywanie prac dyplomowych przy użyciu programu Microsoft Word lub podobnych edytorów tekstu, pod warunkiem zachowania zasad składu tekstów matematycznych.

5.2 Strona tytułowa pracy dyplomowej powinna być zgodna ze wzorem umieszczonym na stronie Instytutu Matematycznego (zakładka Praca dyplomowa).

5.3 Strony pracy dyplomowej powinny być numerowane zaczynając od strony tytułowej.

Then $\mathbb{N} = \{ t \in x_0 : \forall x \ (\mathrm{ind}(x) \longrightarrow t \in x) \}$

exists by comprehension axiom.

$\mathbb{N}$: the smallest inductive set.

$$\underset{\underset{0}{\|}}{\emptyset} \in \mathbb{N}, \quad \underset{\underset{1=\{0\}}{\|}}{\emptyset \cup \{\emptyset\}} \in \mathbb{N}, \quad \underset{\underset{1\cup\{1\}=\{0,1\}}{\|}}{2} \in \mathbb{N}, \dots, \overset{\overset{\mathbb{N}}{\Psi}}{n} = \{0, 1, \dots, n-1\}$$

Def. $\mathrm{Ord}(x) = \begin{cases} \text{"}(x, \in) \text{ is a well-ordered set"} \\ \text{and} \\ \underline{\forall y \in x \ \ y \subseteq x} \end{cases}$

that is: "x is transitive".

<u>Fact</u>. (ZF) $\mathrm{Ord}(x) \wedge \mathrm{Ord}(y) \longrightarrow \begin{cases} x \in y \ \underline{\vee} \ y \in x \ \underline{\vee} \ x = y \\ \text{and} \\ \mathrm{Ord}(x \cup \{x\}) \end{cases}$

(2) $\mathrm{Ord}(x) \wedge y \in x \Rightarrow \mathrm{Ord}(y) \wedge$ "y : an initial segment of x".

x transitive $\Rightarrow \Downarrow$ ~~x transit~~

$y \subseteq x$

(3) $\neg \exists x \quad x = \{ y : \mathrm{Ord}(y) \}$ (But ~~Ale~~ Ord ~~∈~~ $= \{ y : \mathrm{Ord}(y) \}$ is a (proper) class)

<u>Proof</u> (3) Suppose $x = \{ y : \mathrm{Ord}(y) \}$.

Then $\mathrm{ord}(x)$, hence $x \in x$.

Let $x' = \{x\}$. x' violates regularity axiom

since: $y \in x' \Rightarrow y \cap x' \neq \emptyset$

<u>Remark</u> Ord is a well-ordered (by $\in$) class.

$$\mathrm{Ord}(x) \Rightarrow \underset{x+1}{\overset{\text{transitive}}{x \cup \{x\}}} \in \mathrm{Ord} \ : \text{a successor of } x$$

LR/3 ④

Examples of ordinal numbers: (~~{~~ Ord)

$$\underbrace{0, 1, 2, \ldots}_{\substack{\| \\ \mathbb{N}}} ; \underset{\substack{\| \\ \mathbb{N}}}{\omega}, \omega+1, \omega+2, \ldots, \underset{\substack{\| \\ \bigcup_{n<\omega} \omega+n}}{\omega+\omega}, \ldots \leftarrow \text{replacement axiom}$$

$$\underset{\substack{\| \\ \{\alpha \in Ord : \alpha \text{ countable}\}}}{\omega_1} \nwarrow \text{uncountable}, \quad \omega_1+1, \ldots$$

Von Neumann hierarchy of sets V_α, $\alpha \in Ord$:

- $V_0 = \emptyset$
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
- $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$, when $\overset{\lambda}{\cancel{\alpha}}$: limit ordinal

] transfinite recursion.

How to ~~honest~~ define V_α honestly, by an L_{ZF}-formula?

$$x = V_\alpha \iff (\exists f: \text{function})\left(\text{Dom } f = \alpha+1 \wedge \begin{cases} f(0) = \emptyset \\ (\forall \beta \cancel{<} \alpha)\ f(\beta+1) = \mathcal{P}(f(\beta)) \\ (\forall \beta < \alpha)\ (Lim(\beta) \to f(\beta) = \bigcup_{\gamma<\beta} f(\gamma)) \end{cases} \text{and } f(\alpha) = x\right)$$

f: a constructor function/sequence

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5. Wymagania techniczne i edytorskie

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Fact. (1) $V_\alpha \subseteq V_\beta$ for $\alpha < \beta$

(2) $\alpha \in V_{\alpha+1} \setminus V_\alpha$

(3) $\forall x\, \exists \alpha\ x \in V_\alpha$ hence $\forall x\ x \in V$

[here: $V = \bigcup_{\alpha \in Ord} V_\alpha$: a class proper class

$x \in V \iff \exists \alpha\, (Ord(\alpha) \wedge x \in V_\alpha)$]

Proof. (1), (2): exercises.

(3) (a.a.) Suppose $x \notin V$.

Claim $\exists x' \notin V\ \forall y \in x'\ y \in V$

Proof Case 1 $\forall y \in x\ y \in V$, Then let $x' = x$.

Case 2.: $\boxed{\neg \text{Case } 1}$

Let $x_1 = \underbrace{x \cup \bigcup x \cup \bigcup\bigcup x \cup \bigcup\bigcup\bigcup x \cup \dots}_{\overset{\shortparallel}{TC(x)}}$,

$TC(x)$, transitive closure of x
(the smallest transitive set y s.t. $x \subseteq y$) ~~and~~

[Exercise:
$a \in TC(x) \Rightarrow a \subseteq x$.

Now: if $x_1 \in V$, then $x \subseteq x_1$, hence $x \in V$ ↯.

So $x_1 \notin V$.

Let $y = \{t \in x_1 : t \notin V\}$. ~~B~~

If $y = \emptyset$, then $x_1 \subseteq V$ and we ~~are~~ are done in the Claim.

Otherwise $y \neq \emptyset$. By regularity choose $(*)\begin{cases} x' \in y \text{ with} \\ x' \cap y = \emptyset. \end{cases}$

So $x' \in x_1 \Rightarrow x' \subseteq x_1 \cancel{\Rightarrow}$

and by $(*)$ $\underbrace{\forall v \in x' \; v \in V}_{x' \subseteq V.}$ claim $\dashv$

Now we define

$$f : x' \longrightarrow \mathrm{Ord} \quad f(\underset{\substack{\cap \\ x'}}{v}) = \min\{\alpha : \cancel{x}\, v \in V_\alpha\}$$

By replacement axiom:

Rng f is a set. Hence $\bigcup \mathrm{Rng} f = \beta \in \mathrm{Ord}$.

$x' \subseteq V_\beta \Rightarrow x' \in V_{\beta+1}$. ↯.

Thm (on transfinite induction).

Let $\varphi(\alpha, \bar{y}) \in \mathcal{F}_{L_{ZF}}$.

If $\varphi(0, \bar{y})$ and $(\forall \alpha \in \mathrm{Ord})(\forall \beta < \alpha \; \varphi(\beta, \bar{y}) \rightarrow \varphi(\alpha, \bar{y}))$

then $\forall \alpha \in \mathrm{Ord} \; \varphi(\alpha, \bar{y})$.

V is very ample (obszerny): interprets (practically) all mathematics

Troubles (1) V is too ample (allows existence of pathological objects, like Banach-Tarski paradox)

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5. Wymagania techniczne i edytorskie

5.1 Zaleca się przygotowywanie prac dyplomowych przy użyciu programu TeX, z uwagi na przystosowanie tego programu do profesjonalnego składu tekstów matematycznych. Dopuszczalne jest przygotowywanie prac dyplomowych przy użyciu programu Microsoft Word lub podobnych edytorów tekstu, pod warunkiem zachowania zasad składu tekstów matematycznych.

5.2 Strona tytułowa pracy dyplomowej powinna być zgodna ze wzorem umieszczonym na stronie Instytutu Matematycznego (zakładka Praca dyplomowa).

5.3 Strony pracy dyplomowej powinny być numerowane zaczynając od strony tytułowej.

(2) There are simple L_{ZF}-sentences undecidable in ZFC, e.g. CH: $(\forall X\subseteq\mathbb{R})$ (X countable or $X\sim\mathbb{R}$)

Positive aspects of V:

V ~~is over~~ interprets all mathematics

$\mathbb{N}=\{0,1,\dots\}=\omega\in V_{\omega+1}$. $\mathbb{R}=\mathcal{P}(\mathbb{N})\in V_{\omega+2}$.

$\mathbb{R}^{\mathbb{R}}\in V_{\omega+5}$ (mathematical analysis)

CH:

① Gödel (~1930) $\vdash \mathrm{Con}\,ZFC\longrightarrow \mathrm{Con}(ZFC+CH)$

② Cohen (~1960) $\vdash \mathrm{Con}\,ZFC\longrightarrow \mathrm{Con}(ZFC+\neg CH)$

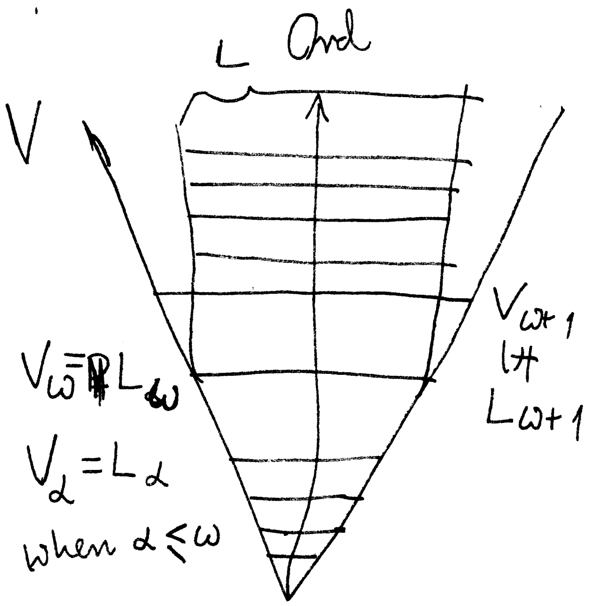
Ad ① : constructible universe L :
(Gödel universe)

$$L_0=\emptyset,\quad L_{\alpha+1}=\mathrm{Def}(L_\alpha,\in)=$$

$$=\{X\subseteq L_\alpha : \underbrace{\overline{\exists\varphi(x,\bar{y})\in F_{L_{ZF}}\ \exists\bar{a}\subseteq L_\alpha\ X=\{b\in L_\alpha:(L_\alpha,\in)\models\varphi(b,\bar{a})\}}}_{\text{how to write it down in } L_{ZF}?}\}$$

$$L_{\underset{\underset{\mathrm{Lim}}{\cap}}{\delta}}=\bigcup_{\alpha<\delta}L_\alpha$$

$L=\bigcup_{\alpha\in\mathrm{Ord}}L_\alpha$: a proper class



Fact

(1) $\mathrm{Ord} \subseteq L \subseteq V$

(2) $\alpha \subseteq L_\alpha \subseteq V_\alpha$, $\alpha \in L_{\alpha+1}$

(3) "$L \models ZFC + CH$"
(But: L is not a set)

(4) there is a total well-ordering on L

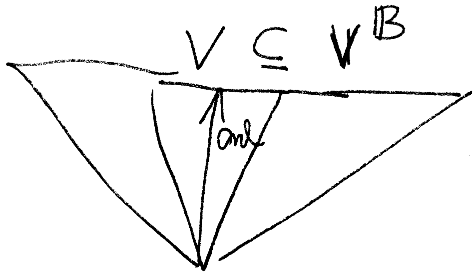
(5) $L \models (ZFC + V = L)$

Ad ② Cohen forcing, boolean models :

Idea 1. ~~boolean~~ logical values of sentences $\in \mathbb{B}$: a boolean algebra.

$$\varphi \Longrightarrow [\![\varphi]\!]_{\mathbb{B}} \in \mathbb{B}.$$

2. Construct a boolean universe $V^{\mathbb{B}} \supseteq V$ consisting of "boolean terms"



$$V_0^{\mathbb{B}} = \emptyset$$

$$V_{\alpha+1}^{\mathbb{B}} = \{ f : f \text{ a function}, \ \mathrm{Dom}\, f \subseteq V_\alpha^{\mathbb{B}}, \ \mathrm{Rng}\, f \subseteq \mathbb{B} \}$$

f = a set of pairs $\langle x, b \rangle$, where $x \in V_\alpha^{\mathbb{B}}$, $b \in \mathbb{B}$

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Idea $[\![t \in f]\!]_{\mathbb{B}} = b$, where $\langle t, b \rangle \in f$.

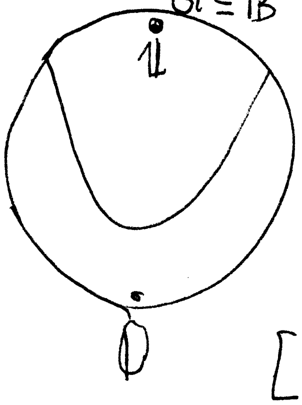
$\mathbb{B}$ chosen so that:

- $[\![\varphi]\!]_{\mathbb{B}} = \mathbb{1}_{\mathbb{B}}$ for every $\varphi \in ZFC$
- $[\![CH]\!]_{\mathbb{B}} \neq \mathbb{1}_{\mathbb{B}}$

hence: If ZFC is consistent, then $ZFC \nvdash CH$

3. Additional trick. (Solovay, 1970)

$G \subseteq \mathbb{B}$: a filter, even ultrafilter, so $\mathbb{B}/G \cong \{0, \mathbb{1}\}$



"$V^{\mathbb{B}}/G$" $= V[G]$: a "real model" of $ZFC + \neg CH$

G: a generic ultrafilter in $\mathbb{B}$

[this trick is possible under some additional asscumptions, like that V: countable]

But:

Gödel $\vdash \underbrace{Con(ZFC)}_{} \longrightarrow ZFC \nvdash Con(ZFC)$

i.e: "ZFC is consistent".

So if ZFC is consistent, then $ZFC \nvdash \exists M \models ZFC$.

(nothing to say of a countable $M \models ZFC$)

But. Def $\overset{\text{CN}}{\overset{\downarrow}{\varkappa}}$ is (strongly) inaccessible

if $\varkappa > \aleph_0$, $\varkappa$ regular (i.e. $cf(\varkappa) = \varkappa$) and $\forall \alpha < \varkappa \ 2^{\alpha} < \varkappa$.

Fact $\varkappa$ inaccessible $\Longrightarrow (V_{\varkappa}, \in) \models ZFC$.

Corollary:

$$\underbrace{ZFC + \text{"}\exists \kappa : \text{inaccessible"}} \vdash \mathrm{Con}(ZFC)$$

$\parallel$

call it: ZFC'

But: (Gödel thm). $\mathrm{Con}\, ZFC' \Rightarrow ZFC' \nvdash \mathrm{Con}(ZFC')$.

How to write down a sentence: "Con ZFC"?

$L_{ZF} \subseteq V_\omega$ (a natural interpretation or a Gödel numbering)

<u>Remark</u> The relation $\mathrm{Sat}(M, \varphi, \bar{a}) \iff (M, \in) \models \varphi(\bar{a})$

[where $\varphi \in L_{ZF}$, $\bar{a} \subseteq M$]

is definable $\overset{\text{"}\varphi(\bar{x})\text{"}}{}$ by a formula of L_{ZF} [exercise].

But (thinking of L_{ZF} as a subset of V_ω):
The relation $\mathrm{True}(\varphi) \iff V \models \varphi$ is not definable in V

<u>Warning</u>! (Tarski thm on non-definability of truth, a special case)

Hence: research on "large cardinal numbers".

For example: $\kappa \in Cn$ is measurable if ~~$\exists$~~ $\kappa > \aleph_0$ and

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and $\exists\, \mathcal{U} \subseteq P(\varkappa)$ an ultrafilter (non-principal), closed under intersections of length $< \varkappa$.

[like a $\varkappa$-additive $(0-1)$-measure on $P(\varkappa)$]

Def. An ultrafilter $\mathcal{U}$ on I is called $\varkappa$-complete if $\mathcal{U}$ is closed under intersections of length $< \varkappa$.

Remark $ZFC \vdash$ Peano arithmetic PA is consistent

(as $ZFC \vdash \mathbb{N} \models PA$).

Interpretations in ZFC:

- an ordered pair: $\underbrace{\langle a, b\rangle}_{\text{remembers of its 1. and 2. coordinate}} = \{\{a\}, \{a, b\}\}$ (Kuratowski) may be defined in many ways.
- more generally:

  $\langle a_1, \ldots, a_n\rangle = \langle a_1, \langle a_2, \ldots, \langle a_{n-1}, a_n\rangle \ldots\rangle\rangle$ nested sharp parentheses.
- relation R in a set A:

  $R \subseteq A^k = \{\langle a_1, \ldots, a_k\rangle : a_i \in A\}$
- function $f : A \longrightarrow B$ "=" 1-valued relation $f \subseteq A \times B$.
- similarly formulas of L_{ZF} may be regarded as some elements of V_ω (so that $L_{ZF} \subseteq V_\omega$)
- logical connectives, quantifiers as some operators

on $L_{ZF} \subseteq V_\omega$

$\wedge(\varphi, \psi) = \varphi \wedge \psi$, $\exists x_i \varphi$ ...

Assume $M = (A, R)$: an L-structure, where $L = \{\underline{R}\}$

A ↑ finite $\neq \emptyset$ / from the real world.

$\underline{R}$ ↑ binary relational symbol

Then $\exists M' \in V_\omega$ $M' \cong M$.

Proof ① We find $A' \in V_\omega$ $|A'| = |A|$.

(2) We find $R' \subseteq A' \times A'$, $R' \in V_\omega$ s.t.
$\langle A', R' \rangle \cong \langle A, R \rangle$

③ $M' = \langle A', R' \rangle \in V_\omega$. $M' \cong M$, but ~~M count~~ M matters up to $\cong$!

Alternative set theories:

Example: New Foundations: NF, Quine ~~≈~~ ~ 1940.

$L = \{ L_{ZF} = \{ \in \}$.

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Idea: avoid Russell antinomy

$$\{x : \underbrace{x \notin x}_{\substack{\neg x \in x \\ \text{strange formula}}} \}$$

Avoid strange formulas.

<u>Def</u> $\varphi \in L_{ZF}$ is <u>stratifiable</u> if

$\exists \underset{\text{all variables in } \varphi}{f : Var(\varphi)} \longrightarrow \mathbb{N} \quad \forall x, y \in Var(\varphi)$

$$\begin{cases} \text{(If } x = y \text{ appears in } \varphi \text{, then } f(x) = f(y) \\ \text{If } x \in y \text{ appears in } \varphi \text{, then } f(y) = f(x) + 1. \end{cases}$$

Axioms of NF:

1. Extensionality: ~~x~~ $\forall t \, (t \in x \Leftrightarrow t \in y) \longrightarrow x = y$
2. "Stratified" full comprehension:
   Assume $\varphi(x, \bar{y})$ is stratifiable. Then
   $\exists z \, \forall x \, (x \in z \Longleftrightarrow \varphi(x, \bar{y}))$ (there is $z = \{x : \varphi(x, \bar{y})\}$)

[·NF is finitely axiomatizable

· In NF: there is a set of all sets.

Logic R: Notes 4.

Troubles with set theory (ZFC) as a "metatheory" for mathematics.

- too many sets ⤳ byproduct: pathological objects.
- $\left.\begin{array}{l}\text{independence of ZFC} \\ \text{undecidability in ZFC}\end{array}\right]$ of fundamental conjectures.

Reaction:

- restrict to objects, whose existence is not problematic:
  - computable objects.

Computability:

Objects: for example natural numbers represented as:

$n \Longleftrightarrow \underbrace{|||||\ldots|}_{n}$ (n-many sticks, matches)

or ~~$n = (011001)$~~ [redacted] $n = (1011001)_2$: binary representation.

Generally:

$\emptyset \neq \Sigma$: [redacted] a finite set of "concrete" objects.. e.g. $\Sigma = \{0, 1\}$.
("alphabet)

$\Sigma^* = \{ \underbrace{\text{finite tuples of elements of } \Sigma}_{} \}$ (
words over Σ ← still concrete objects.
(~~computable~~)

For example
$\mathbb{N} \approx \Sigma^*$ for $\Sigma = \{1\}$ or $\mathbb{N} \approx \Sigma^*$ for $\Sigma = \{0, 1\}$.

Other ~~computable~~ "concrete" objects:

- subsets of Σ^*, but: not all! (like in ZFC)

LR-N4/2

Intention: identify $\underbrace{\text{concrete}}_{\substack{\parallel \\ \text{Computable.}}}$ subsets of Σ^*

Similarly: $f: \Sigma^* \longrightarrow \Sigma^*$ we want to focus on "concrete" = computable functions.

$\boxed{\text{Computable} = ?}$

Turing Machine TM : an abstract ~~comp~~ computer.

Alan Turing... died ~1956 ?

[there are many other equivalent formalizations of computability]

Let Σ : a finite alphabet.

Turing machine M over Σ consists of:

(1) working $\overset{\text{scanners/writers}}{\text{heads}}$ $G_0, \dots, G_n$ (głowice robocze)

(2) working $\underset{\text{taśmy}}{\underline{\text{tapes}}}$ $T_1, \dots, T_n$; input tape T_0.

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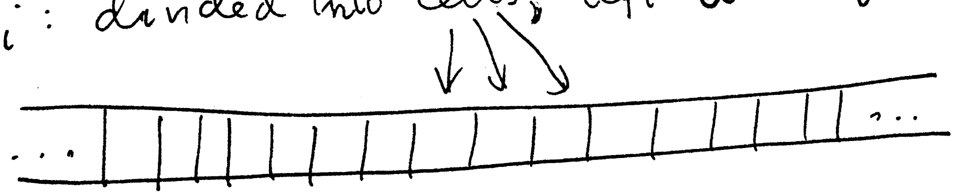
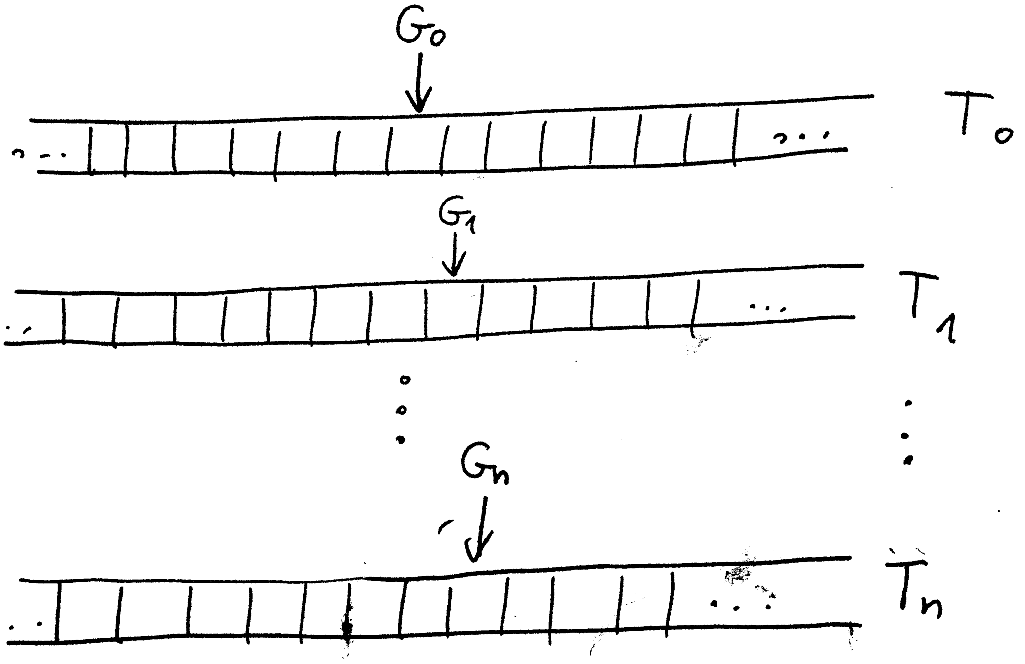
Tape T_i : divided into cells, left- and right-infinite.



M:



Each head G_i sees a single cell of T_i, [in any given moment] each cell contains a letter $\in \Sigma$ or is empty blanc B

(3). A finite set S of states of M

- A transition function $f : S \times (\Sigma \cup \{B\})^{n+1} \longrightarrow S \times (\Sigma \cup \{B\})^{n+1} \times \overbrace{\{L, R, 0\}}^{n+1}$
- distinguished states $\in S$:
  - s_0 : initial state
  - end : final state
  - yes, no $\in S$.

Operation of M : in time, divided into moments: 0, 1, 2, 3, ...

in steps $t = 1, 2, 3, \ldots$

Step t : ~~the~~ operation of M between moment $t-1$ and moment t.

LR-N4/4

(1) configuration of M in moment t:

(a) each cell of each T_i contains a letter $\in \Sigma$ or blanc (B)

(b) each head G_i sees a single cell of T_i with content $c_i \in \Sigma \cup \{B\}$.

(c) M is in a state $s \in S$.

(2) step t+1 of M (from moment t to moment t+1)

(a) calculates $f(s, c_0, \dots, c_n) = (s', c_0', \dots, c_n', v_0, \dots, v_n)$, $v_i \in \{L, R, 0\}$

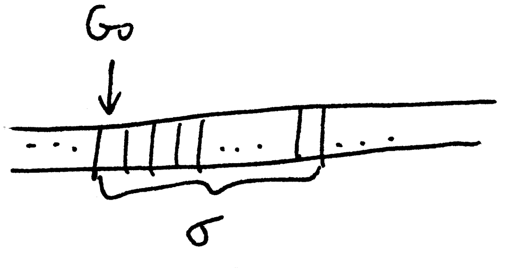
(b) replaces content c_i of the cell of T_i scanned by G_i, by c_i'.

(c) if $v_i = L$, moves G_i one cell left
if $v_i = R$, moves G_i one cell right
if $v_i = 0$, ~~des~~ does not move G_i.

(d) changes the state of M from s to s'.

(3) configuration of M in moment t = 0:

- state $s = s_0$
- on T_0: an initial word $\sigma \in \Sigma^*$
  G_0 sees the cell with the first letter of σ



for $i \geq 0$ G_i sees a cell of T_i, all cells of T_i are empty (blanc).

(4) In moment t:

- if s = end, yes or no, then M ends operation.
  terminates

(a) if s = end, then the word currently written on T_0 is called the outcome of M on inputed $\overset{\uparrow}{\sigma}$.
output   initial word.

(b) If s = yes, we say that M accepts σ

(c) If s = no, we say that M rejects σ.

Def Let $\underset{\substack{\uparrow\\ \text{"language"}}}{L} \subseteq \Sigma^*$. M recognizes L $\Leftrightarrow$

$$(\forall \sigma \in \Sigma^*) \begin{cases} \sigma \in L \Rightarrow M \text{ accepts } \sigma \\ \sigma \notin L \Rightarrow M \text{ rejects } \sigma \end{cases}$$

Def. Let $f : \Sigma^* \underset{\substack{\uparrow\\ \text{partial function}}}{-\!\!\ominus\!\!\rightarrow} \Sigma^*$. M computes $f \Leftrightarrow \forall \sigma \in \Sigma^*$
(i.e. Dom $f \subseteq \Sigma^*$)

$$\begin{cases} f(\sigma)\downarrow \Rightarrow \text{on inputed } \sigma \text{ M terminates with } \text{~~oup~~ output } f(\sigma) \\ f(\sigma)\uparrow \Rightarrow \text{on input } \sigma \text{ M does not terminate its operation.} \end{cases}$$

where:
$f(\sigma)\downarrow$ = "$\sigma \in \text{Dom} f$", $f(\sigma)\uparrow$ = "$\sigma \notin \text{Dom} f$".

Def.

(1) $L \subseteq \Sigma^*$ is TM-computable $\Leftrightarrow \exists M{:}TM$ M recognizes L

(2) $f : \Sigma^* -\!\!\ominus\!\!\rightarrow \Sigma^*$ is TM-computable $\Leftrightarrow \exists M{:}TM$ M computes f.

Example Let $\Sigma = \{1\}$, $L = \{\underbrace{1\ldots1}_{n} : n \text{ even}\}$ $\overset{\mathbb{N}}{\cup\!|}$ ($\approx$ {even numbers})

L is TM-Computable:

M with 1 tape only: T_0

$$\overset{\text{Go} \quad \text{state } s = s_0}{\ldots|B|B|\overset{\downarrow}{1}|\underbrace{1|1|1|1|1|1}_{n}|B|B|\ldots}$$

transition function: (a) $f(s_0, B) = (\text{yes}, B, 0)$

(b) $f(s_0, 1) = (s_1, 1, R)$

LR-N4/6

(c) $f(s_1, B) = (no, B, 0)$

(d) $f(s_1, 1) = (s_0, 1, R)$

Representation of $\mathbb{N}$:

(a) $\Sigma = \{1\}$, $\mathbb{N} \approx \Sigma^*$ (b) $\Sigma = \{0, 1\}$, $\mathbb{N} \approx \Sigma^*$

so natural numbers $\approx$ words over Σ. binary representation.

Def. $L \subseteq \mathbb{N}$ is TM-computable $\Leftrightarrow \exists M$: TM M recognises L

$f : \mathbb{N} \xrightarrow{\circ} \mathbb{N}$ is TM-computable $\Leftrightarrow \exists M$: TM M computes f

Different approach to COMPUTABILITY.

Recursive functions $f : \mathbb{N}^n \xrightarrow{\circ} \mathbb{N}$:

(a) basic functions: $S : \mathbb{N} \longrightarrow \mathbb{N}$, $S(x) = x + 1$ successor function

$O : \mathbb{N}^n \longrightarrow \mathbb{N}$, $O(\bar{x}) = 0$

$I : \mathbb{N} \longrightarrow \mathbb{N}$, $I(x) = x$, $I^n_j : \mathbb{N}^n \longrightarrow \mathbb{N}$

$I^n_j(x_1, \ldots, x_n) = x_j$

(b) defining schemes:

(a) composition: Given $f(x_1, \ldots, x_n)$, $g_1(\bar{y}_1), \ldots, g_n(\bar{y}_n)$

obtain $h(\bar{y}_1, \ldots, \bar{y}_n) = f(g_1(\bar{y}_1), \ldots, g_n(\bar{y}_n))$

(b) simple recursion:

Given $f(\bar{x})$, $g(\bar{x}, y, z)$ obtain $h(\bar{x}, y)$ such that

$$\begin{cases} h(\bar{x}, 0) = f(\bar{x}) \\ h(\bar{x}, n+1) = g(\bar{x}, n, h(n, \bar{x})) \end{cases}$$

(3) operation minimum:

given $f(\bar{x}, y)$ obtain $h(\bar{x})$ such that

$$h(\bar{x}) = \min\{y : f(\bar{x}, y) = 0\}.$$

Warning to defining schemes (1)–(3):

functions f, g may be partial, then h also may be partial:

Ad (1): $h(\bar{y}_1,\dots,\bar{y}_n)\downarrow \iff g(\bar{y}_1)\downarrow,\dots,g(\bar{y}_n)\downarrow$ and

$$f(g(\bar{y}_1),\dots,g(\bar{y}_n))\downarrow$$

Ad (2): $h(\bar{x}, 0)\downarrow \iff f(\bar{x})\downarrow$

~~h(n+1)~~ $h(\bar{x}, n+1)\downarrow \iff h(\bar{x}, n)\downarrow$ and $g(\bar{x}, n, h(\bar{x}, n))\downarrow$

Ad (3):

$h(\bar{x})\downarrow \iff$ there is y s.t. $f(\bar{x}, y) = 0$ and

$$\forall y' < y \, (f(\bar{x}, y')\downarrow \text{ and } f(\bar{x}, y') \neq 0)$$

Def. Rec = the smallest family of functions $f : \mathbb{N}^n \rightharpoonup \mathbb{N}$, $n \geq 0$, containing base functions and closed under defining schemes.

- f is recursive $\iff f \in$ Rec

Def. $A \subseteq \mathbb{N}^n$ is recursive $\iff \chi_A \in$ Rec.

Examples:

$$+ : \begin{cases} x + 0 = 0 = O(x) \\ x + (n+1) = (x+n) + 1 = S(x+n) \end{cases}$$

$$\cdot : \begin{cases} x \cdot 0 = 0 \\ x \cdot (n+1) = x \cdot n + x. \end{cases}$$

LR-N4 /8

- The set $\mathbb{P}$ of prime numbers is recursive.
- the function ($n \longmapsto p_n$ = n-th prime number) is recursive

Proof

① $P(x)$: $\begin{cases} P(0)=0 \\ P(n+1)=n \end{cases}$ predecessor function $P \in Rec.$

② $x \dot{-} y = \begin{cases} x-y, \text{ when } x \geq y \\ 0, \text{ when } x<y \end{cases}$ $\begin{cases} x \dot{-} 0 = x \\ x \dot{-} (n+1) = P(x \dot{-} n) \end{cases}$

natural subtraction

(3) Let $H(x,y) = (x \dot{-} y) + (y \dot{-} x)$: $x=y \Leftrightarrow H(x,y)=0$

(4) $f(y) \in Rec \Rightarrow f'(x) = \prod_{y<x} f(y)$ recursive.

$$\begin{cases} \prod_{y<0} f(y) = 1 = S(0) \\ \prod_{y<n+1} f(y) = f(n) \cdot \prod_{y<n} f(y) \end{cases}$$

(5) $x \in \mathbb{P} \Leftrightarrow \forall y<x \, \forall z<x \;\; y \cdot z \neq x$

$\Leftrightarrow \forall y<x \, \forall z<x \;\; H(x, y \cdot z) \neq 0$

$\Leftrightarrow g(x) = \prod_{y<x} \prod_{z<x} H(x, y \cdot z) \neq 0$

(6) Let $h(x) = \min(g(x), 1) = 1 \dot{-} (1 \dot{-} g(x))$, $h: N \to \{0,1\}$

[similarly $F: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$]

$F(x,y) = \begin{cases} 1, & x=y \\ 0, & x \neq y \end{cases}$

$x \in$ ~~Prime~~ $\mathbb{P} \Leftrightarrow h(x)=1$

$Rec \ni h = \chi_{\mathbb{P}}$ so $\mathbb{P} \in Rec.$

(7) $p_0 = 2$

$p_{n+1} = \min\{x : x > p_n \text{ and } x \in \mathbb{P}\}$

$= \min\{x : (p_n+1) \dot{-} x = 0 \text{ and } 1 \dot{-} h(x) = 0\}$

$= \min\{x : ((p_n+1) \dot{-} x) + (1 \dot{-} h(x)) = 0\}.$

Thm (1) $A \subseteq \mathbb{N}^n$ is recursive $\Leftrightarrow$ A is TM-computable

(2) $f : \mathbb{N}^n \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{N}$ is recursive $\Leftrightarrow$ f is TM-computable.

Proof. $\Rightarrow$ obvious.

[· basic functions are TM-computable,
· TM-computable functions are closed under defining schemes]

$\Leftarrow$ (2), $n=1$, sketch.

Assume M: TM computing $f : \mathbb{N} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{N}$
(under some representation of natural numbers as words).

Assume M has k tapes, the set of states S, transition function F.

· conf(M) = [content of tapes, state(M), positions of working heads of M]
configuration

≈ coded as a natural number n.

for example $n = p_0^{\varepsilon_0} p_1^{\varepsilon_1} \dots p_k^{\varepsilon_k}$ codes $\langle \varepsilon_0, \varepsilon_1, \dots, \varepsilon_k \rangle$.

We define $g : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

· $g(t, n)$ = configuration of M on input n, in moment t.

· $g \in$ Rec

Let $h(n) = \min\{t :$ M stops in moment t, on input $n\}$

$\Updownarrow$

[state s recovered from $g(t, n)$ is end]

~~Therefore~~

Let $f(n)$ = content of the input tape in mom~~ent~~ $h(n)$ LR-N4/10

(may be recursively recovered from $g(n, h(n))$)].

Therefore $f \in Rec$.

Def $A \subseteq \mathbb{N}$ is recursively enumerable $\iff$

$A = \emptyset$ or $\exists \underset{\text{recursive}}{f : \mathbb{N} \longrightarrow \mathbb{N}}$ $A = Rng(f)$

Church thesis, Assume $A \subseteq \mathbb{N}$.

Then A is recursive $\iff$ A is computable (i.e. there is an algorithm determining, for $n \in \mathbb{N}$, if $n \in A$)

Fact Assume $A \subseteq \mathbb{N}$. If both A and $\mathbb{N} \setminus A$ are recursively enumerable.

Proof Wlog $A \neq \emptyset \neq \mathbb{N} \setminus A$. Choose total recursive f, g with $A = Rng f$, ~~B~~ $\mathbb{N} \setminus A = Rng g$.

An algorithm determining for $n \in \mathbb{N}$: if $n \in A$

1. ~~Calcul~~ Compute $f(0), g(0), f(1), g(1), \dots$
2. When in sequence $f(0), g(0), f(1), g(1), \dots$, n appears then answer if $n = f(i)$, then $n \in A$
   if $n = g(i)$, then $n \notin A$

5. Wymagania techniczne i edytorskie

5.1 Zaleca się przygotowywanie prac dyplomowych przy użyciu programu TeX, z uwagi na przystosowanie tego programu do profesjonalnego składu tekstów matematycznych. Dopuszczalne jest przygotowywanie prac dyplomowych przy użyciu programu Microsoft Word lub podobnych edytorów tekstu, pod warunkiem zachowania zasad składu tekstów matematycznych.

5.2 Strona tytułowa pracy dyplomowej powinna być zgodna ze wzorem umieszczonym na stronie Instytutu Matematycznego (zakładka Praca dyplomowa).

5.3 Strony pracy dyplomowej powinny być numerowane zaczynając od strony tytułowej.

Formally:

Let $r(n) = \min \{ i :$ ~~f(i)n~~ $f(i) = n \text{ or } g(i) = n \} =$

$= \min \{ i : H(f(i), n) \cdot H(g(i), n) = 0 \}$

$r \in Rec$

$$n \in A \iff f(r(n)) = n \iff H(f(r(n)), n) = 0$$

$$\iff \underbrace{1 \dot{-} H(f(r(n)), n)}_{\chi_A \in Rec.} = 1$$

Thm. · Rec is countable.

· There are countably many recursively enumerable sets.

Thm. There is a recursively enumerable, non-recursive set $A \subseteq \mathbb{N}$.

Proof. 1. $\exists \underset{\text{recursive}}{f} : \mathbb{N} \times \mathbb{N} \mathrel{-\!\circ\!\!\rightarrow} \mathbb{N} \ \forall \underset{\text{recursive}}{g} : \mathbb{N} \mathrel{-\!\circ\!\!\rightarrow} \mathbb{N} \ \exists n \ f(n, \cdot) = g(\cdot)$

[f is called a universal recursive function]

proof of 1:

- we enumerate effectively "recipes" for recursive functions. $\alpha_0(\cdot), \alpha_1(\cdot), \alpha_2(\cdot), \ldots$
- $f(n, m) = (\text{recipe } \alpha_n \text{ applied to } m)$

  $f \in Rec.$

2\. Let $A = \{ x : f(x, x) = 0 \}$: recursively enumerable.

proof: an algorithm generating A:

- i-th step: perform i-many steps of computation of $f(x, x)$ for all $x \leq i$.

~~the~~ We list those $x \leq i$ such that in this stage $f(x,x)$ is computed and equals 0.

In this way we create an (infinite) recursive list of natural numbers, enumerating A,
[in each stage we add finitely many members to the list]

(3) A is not recursive.

proof (a.a.) Suppose $\chi_A \in Rec$. Then $\chi_A(\cdot) = f(n,\cdot)$ ~~f~~ for some $n \in \mathbb{N}$

and $f(n,\cdot)$ is total

Then $f(n,n) = 0 \Leftrightarrow n \in A \Leftrightarrow \chi_A(n) \neq 0 \Leftrightarrow f(n,n) \neq 0$ $\lightning$.

Thm: There is a set $A \subseteq \mathbb{N}$, recursively enumerable, but not recursive

Proof: (1) $\exists f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ recursive $(\underset{\text{recursive}}{\forall g})\ \mathbb{N} \to \mathbb{N}\ (\exists n)\ f(n, \cdot) = g(\cdot)$

f - a universal recursive function - for every partial recursive function there is a section equal to g

(effectively)

— We enumerate the recipes for recursive functions: $\alpha_0, \alpha_1, \dots$ (by algorithm)

$f(n, m) = [\alpha_n \text{ applied to } m]$

(2) ~~Let~~ $A = \{x \in \mathbb{N} : f(x,x) = 0\}$: rec. enumerable

recursive

Proof: An algorithm, acts in steps, generates a list of nat numbers

[step i] we render i steps in calc. of $f(x,x)$ for all $x \leq$

· whenever $f(x,x)$ is calculated, we add it to the list

We have enumerated A.

(3) A is not recursive:

Suppose $\chi_A \in Rec$, then $\chi_A(\cdot) = f(n, \cdot)$ for some n

$f(n, \cdot)$ is total

$f(n,n) = 0 \iff n \in A \iff \chi_A(n) \neq 0 \iff f(n,n) \neq 0$, a contradiction

Languages, coding, decidability

Let $L = \{P_i, f_j, c_t\}_{i,j,t}$ - a finite language.

Formulas of L are computable objects - having a string of symbols we can decide whether it is indeed a formula

$$\mathcal{F}_L \subseteq \Big(L \cup \{(,),x,|,=,\wedge,\neg,\exists\}\Big)^*$$

$\Uparrow$

$x_2 = x_{11}$

We will think of formulas as of numbers (Gödel coding)

$\longleftarrow p_2 = p_{11}$

$(,), x, |, =, \wedge, \neg, \exists, \forall, p, f, c$
$1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11\ \ 12$

Ex: $\forall x_3\ P_1(x_3) \longleftarrow \varphi$

$\forall x_{111}\ P_1(x_{111}) \longleftarrow \varphi$ in our language

$\Downarrow$

$\langle 9,3,4,4,4,10,4,1,3,4,4,4,2 \rangle \rightsquigarrow$ the Gödel code

$$2^9 \cdot 3^3 \cdot 5^4 \cdot 7^4 \cdot 11^4 \cdot 13^{10} \cdot 17^4 \cdot 23^1 \cdot 29^3 \cdot 31^4 \cdot 37^4 \cdot 41^4 \cdot 43^2 = \ulcorner \varphi \urcorner$$

Def. (1) $A \subseteq \mathcal{F}_L$ is $\overset{\text{(computable)}}{\text{recursive}}$ $\Leftrightarrow$ $\{\ulcorner\varphi\urcorner : \varphi \in A\}$ recursive $\Leftrightarrow$ A is TM-computable

(2) $A \subseteq \mathcal{F}_L$ is recursively enumerable $\Leftrightarrow$ $\{\ulcorner\varphi\urcorner : \varphi \in A\}$ is r.e $\Leftrightarrow$ A is TM-computable enu

$\Leftrightarrow$ $A = \emptyset$ or there is a total TM-computable $f : \mathbb{N} \to \mathcal{F}_L$ with $A = f[\mathbb{N}]$

Example: $\{\varphi \in \mathcal{F}_L : \vdash \varphi\}$ is r.e.

We have a semi-algorithm

Given φ:

we write down all formal proofs (in KRL, L)

we look at their conclusions

if φ is a conclusion, we stop and answer yes

Def. $T \subseteq \mathcal{F}_L$ is decidable if T is recursive

Thm. If T is r.e. and complete, then T is decidable.

Peano arithmetic

Language: $L_{PA} = \{+, \cdot, 0, S, <\}$ $\quad PA \subseteq \overbrace{Th(\mathbb{N}, +, \cdot, 0, S, <)}^{TA\text{ - true arithmetic}}$

Classically: primitive notions: $\underset{\text{constant}}{0}$ $\quad \underset{\text{successor}}{S}$

Axioms: 1) $0 \neq Sx$

2) $Sx = Sy \Rightarrow x = y$

induction scheme 3) $\varphi(x, \dots)$ a formula in our lang. $[\varphi(0, \dots) \wedge \forall x(\varphi(x, \dots) \rightarrow \varphi(Sx, \dots)) \rightarrow$

$\rightarrow \forall_x \varphi(x, \dots)$

+ a rule for introducing new function~~s~~ symbols.

Suppose f, g - function symbols of suitable arities

Then we introduce a function symbol h and new axioms for it

$$\begin{cases} h(0, \bar{x}) = f(\bar{x}) \\ h(Sn, \bar{x}) = g(\bar{x}, n, h(n, \bar{x})) \end{cases} \qquad \text{(simple) recursion}$$

Example: $+ : \begin{cases} 0+y=x \\ S_n+x = S(n+x) \end{cases}$ $\cdot : \begin{cases} 0\cdot x = 0 \\ S_n \cdot x = n\cdot x + x \end{cases}$

PA4
PA5

PA6
PA7

PA1: $0 \neq S_x$
PA2: $S_x = S_y \to x = y$
PA3: the induction scheme
PA4 – PA7

- PA is undecidable (Rosser)
- PA is incomplete (Gödel)

Example: $PA \vdash$ $x+0=x$ $\varphi(x)$

- $\varphi(0)$ (by PA(4) for $x=0$)
- Suppose $\varphi(x)$. We will show $\varphi(S_x)$
  $x+0=x$ $S_x+0=S_x$

From PA5 we have $S_n+0 = S(x+0) = S_x$
Now we apply PA3(φ).

Similarly: $PA \vdash +, \cdot$ are associative, commutative, distributive
and „$\leq$" $\underbrace{x \leq y \overset{df}{\iff} (\exists z)\, x+z=y}$ is correct

For $n \in \mathbb{N}$ let $\underline{n} = \underbrace{S \dots S}_{n} 0 \in \mathcal{T}_{L_{PA}}$. It is called a numeral

Lemma (representability of recursive sets and functions in PA)

(1) Assume $A \subseteq \mathbb{N}$ is recursive. Then there is a formula $\varphi_A(x) \in \mathcal{F}_{L_{PA}}$ s.t.
$(\forall n \in \mathbb{N})$ $n \in A \Rightarrow PA \vdash \varphi_A(\underline{n})$ and $n \notin A \Rightarrow PA \vdash \neg\varphi_A(\underline{n})$)
(2) Assume $f : \mathbb{N}^k \xrightarrow{rec.} \mathbb{N}$. Then there is a formula $\varphi_f(\overline{x}, y) \in \mathcal{F}_{L_{PA}}$
s.t. $PA \vdash (\forall \overline{x})(\exists^{\leq 1} y)\ \varphi_f(\overline{x}, y)$ and $\forall \overline{n} \in \mathbb{N}^k$ (if $f(\overline{n})\downarrow$ then $PA \vdash \varphi_f(\overline{\underline{n}}, \underline{f(\overline{n})})$))

consider χ_A

Proof: Enough to prove (2).

Induction on the length of def of f

1° the basic functions: obvious

2° composition scheme: obvious

minimum operator: $f(\bar{n}) = \min\{y : g(\bar{n}, y) = 0\}$

} ind. hyp.

$\varphi_f(\bar{x}, y)$ represented by $\varphi_g(\bar{x}, y, z)$

$$\varphi_g(\bar{x}, y, 0) \wedge (\forall y' < y)((\exists z)\, \varphi_g(\bar{x}, y', z) \wedge \neg\varphi_g(\bar{x}, y', 0))$$

$\varphi_f(\bar{x}, y)$ represents f.

(a) Assume $f(\bar{n})\downarrow, = k$ then $g(\bar{n}, k) = 0$ and $(\forall k' < k)\, g(\bar{n}, k')\downarrow \neq 0$

$$PA \vdash \varphi_g(\underline{\bar{n}}, \underline{k}, \underline{0}) \wedge (\forall y' < \underline{k})((\exists z)\, \varphi_g(\underline{\bar{n}}, y', z) \wedge \neg\varphi_g(\underline{\bar{n}}, y', 0))$$

Fact: $P \vdash x < k \Leftrightarrow (x = \underline{0} \vee \dots \vee x = k-1)$

(b) $PA \vdash (\exists^{\leq 1} y)\, \varphi_f(\bar{x}, y) \leftarrow$ ex.

(d) recursion scheme:

$$\begin{cases} h(0, \bar{x}) = f(\bar{x}) \\ h(S_n, \bar{x}) = g(\bar{x}, n, h(n, \bar{x})) \end{cases}$$

We have φ_f, φ_g rep. f, g

We want φ_h rep. h.

Trick: coding sequences

Idea: $h(n, \bar{x}) = m \Longleftrightarrow (\exists \langle a_0, \dots, a_n \rangle) \begin{cases} a_0 = f(\bar{x}) \\ (\forall i < n)\,[a_{S_i} = g(\bar{x}, i, a_i)] \\ a_n = m \end{cases}$

We use the Chinese remainder thm to make it quantify over 1 thing

~~ER~~LR.N5/1

Notes 5.

Thm (Church) PA is undecidable.

Proof (a.a.) Suppose PA: decidable, i.e.

the set $\{\ulcorner\varphi\urcorner : PA \vdash \varphi\}$ recursive.

Let $\{\varphi_0(x), \varphi_1(x), \dots\}$: a recursive enumeration of all formulas of L_{PA} with free variable x.

Let $A = \{n \in \mathbb{N} : PA \vdash \neg\varphi_n(\underline{n})\}$.

PA: decidable $\Rightarrow$ A recursive.

By representability lemma there is a formula $\varphi_A(x)$ representing A.

$\shortparallel$

$\varphi_n(x)$ for some n.

$$\text{So: } PA \vdash \varphi_A(\underline{n}) \underset{\text{Lemma}}{\iff} n \in A \underset{\text{def. } A}{\iff} PA \vdash \neg\varphi_n(\underline{n}) \quad ↯$$

because PA: consistent (in ZFC)

Corollary (1) (Rosser) If $T \subseteq$ ~~$\mathcal{F}_{PA}$~~ $\mathcal{F}_{L_{PA}}$ consistent theory, $PA \subseteq T$, then T is not decidable.

(2) (Gödel 1st incompleteness thm).

If $T \subseteq \mathcal{F}_{L_{PA}}$ theory, T recursively enumerable consistent, $PA \subseteq T$, then T is incomplete.

Proof (1) the same ~~dead~~ proof as Church thm.

(2) follows from (1).

Corollary (Turing, 1936)

There is no algorithm deciding if $\models \varphi$, for $\varphi \in \mathcal{F}_L$, $L \geq \{+, \cdot, S\}$

Proof Representability lemma holds also for a finite

$PA_0 \subseteq PA$ in place of PA.

(PA_0 needed to prove the Chinese remainder theorem + ...)

Therefore: PA_0 undecidable.

Suppose (a.a). that the set $\{\varphi \in \mathcal{F}_L : \models \varphi\}$ is recursive.

Then for $\varphi \in \mathcal{F}_{L_{PA}}$

$$PA_0 \vdash \varphi \underset{\substack{\uparrow \\ \text{deduction} \\ \text{thm}}}{\Longleftrightarrow} \vdash \bigwedge PA_0 \to \varphi$$

$$\Updownarrow$$

$$\underbrace{\models \bigwedge PA_0 \to \varphi}_{\text{decidable}},$$

so PA_0: decidable, a contradiction.

Corollary. If ZFC is consistent, then ZFC is undecidable and incomplete.

Proof PA is interpretable in ZFC.

~~Corollary Assume $PA \subseteq T \subseteq TA$. Then there~~ / later.
~~rec. enumerable theory~~

Diagonal lemma.

For every formula $G(x) \in \mathcal{F}_{L_{PA}}(x)$ there is a sentence $F \in \mathcal{F}_{L_{PA}}$ such that $PA \vdash F \leftrightarrow G(\underline{\ulcorner F \urcorner})$.

Proof

1. There is a recursive function $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

s.t.

$$(\forall \varphi(x) \in \mathcal{F}_{L_{PA}}(x))(\forall n \in \omega)\ f(\ulcorner \varphi(x) \urcorner, n) = \ulcorner \varphi(\underline{n}) \urcorner$$

(via TM-computability).

Idea:

Let $H(x) =$ "$G(f(x,x))$" (diagonal argument.)

Let $F = H(\underline{\ulcorner H(x) \urcorner})$.

Notice $f(\ulcorner H(x) \urcorner, \ulcorner H(x) \urcorner) = \ulcorner F \urcorner$

$$\text{so } G(\underline{\ulcorner F \urcorner}) \overset{''}{\Leftrightarrow}{}'' \; G(\ulcorner F \urcorner) = G(f(\ulcorner H(x) \urcorner, \ulcorner H(x) \urcorner)) =$$

$$= H(\ulcorner H(x) \urcorner) \Leftarrow\!\!\Rightarrow H(\underline{\ulcorner H(x) \urcorner}) = F$$

"$\boxed{\text{provably" in PA}}$

Formally.

represents ~~φ~~ $f(x,x)$

By Representability Lemma: $\varphi_f(x,y) \in \mathcal{F}_{L_{PA}}$ s.t.

$PA \vdash \forall x \exists^{\leq 1}_{y} \varphi_f(x,y)$ and

for all $n, m \in \mathbb{N}$

$f(n,n) = m \Rightarrow PA \vdash \varphi_f(\underline{n}, \underline{m})$

$$H(x) = \exists y\ (\varphi_f(x,y) \wedge G(y))$$

$$\text{Let } F = H(\underline{\ulcorner H(x) \urcorner}) = \exists y\ (\varphi_f(\underline{\ulcorner H(x) \urcorner}, y) \wedge G(y))$$

$$f(\ulcorner H(x)\urcorner, \ulcorner H(x)\urcorner) = \ulcorner H(\underline{\ulcorner H(x)\urcorner})\urcorner$$

$$\Downarrow$$

$$PA \vdash \varphi_f(\underline{\ulcorner H(x)\urcorner}, \underline{\ulcorner H(\underline{\ulcorner H(x)\urcorner})\urcorner})$$

$$\| \leftsquigarrow F = H(\underline{\ulcorner H(x)\urcorner})$$

$$\underline{\ulcorner F\urcorner}$$

$$\boxed{\text{unique } y} \text{ ~~s.t.~~} \leftarrow PA \vdash \exists^{\leq y} \varphi_f(\underline{\ulcorner H(x)\urcorner}, y).$$

$PA \vdash F \Leftrightarrow G(\underline{\ulcorner F\urcorner})$:

In a model of PA (semantically):

$\boxed{\rightarrow}$ Assume F, i.e.: $\underbrace{H(\underline{\ulcorner H(x)\urcorner})}_{\|} = \cancel{\exists y\, \varphi}$

$$\exists y\, (\varphi_f(\underline{\ulcorner H(x)\urcorner}, y) \wedge G(y))$$

$$\Uparrow \quad \Downarrow \quad \begin{array}{l} PA \vdash \varphi_f(\underline{\ulcorner H(x)\urcorner}, \underline{\ulcorner F\urcorner}) \\ PA \vdash \exists^{\leq 1}_{y}\, \varphi_f(\underline{\ulcorner H(x)\urcorner}, y) \end{array}$$

the only witness $\underline{\ulcorner F\urcorner}$, so $G(\underline{\ulcorner F\urcorner})$.

$\leftarrow$: Assume $G(\underline{\ulcorner F\urcorner})$ holds.

But also ~~PA~~ $\varphi_f(\underline{\ulcorner H(x)\urcorner}, \underline{\ulcorner F\urcorner})$ holds.

so: $\exists y\, (\varphi_f(\underline{\ulcorner H(x)\urcorner}, y) \wedge G(y))$ holds.

so $PA \vdash F \Leftrightarrow G(\underline{\ulcorner F\urcorner})$.

Def A formula $G(x) \in \mathcal{F}_{L_{PA}}(x)$ is a truth definition in ~~$(\mathbb{N}, +, \cdot, S, 0)$~~ $(\mathbb{N}, \cdot, +, 0, S)$

if $(\forall \underset{\text{sentence}}{\varphi} \in \mathcal{F}_{L_{PA}})$ $\mathbb{N} \models G(\ulcorner \underline{\varphi} \urcorner) \iff \mathbb{N} \models \varphi$.

Corollary (Tarski, undefinability of truth).

There is no truth definition in $(\mathbb{N}, +, \cdot, 0, S)$

Proof Suppose (a.a.) $G(x)$ is a truth definition in $\mathbb{N}$. Apply the diagonal lemma for $\neg G(x)$.

Get $F \in \mathcal{F}_{L_{PA}}$ sentence s.t. $PA \vdash F \iff \neg G(\ulcorner \underline{F} \urcorner)$

hence $\mathbb{N} \models F \longleftrightarrow \neg G(\ulcorner \underline{F} \urcorner)$

F says: "I am false" (Liar's paradox, self reference)

Hence $\mathbb{N} \models F \iff \mathbb{N} \not\models G(\ulcorner \underline{F} \urcorner) \underset{\substack{\uparrow \\ G\text{: a truth definition.}}}{\iff} \mathbb{N} \not\models F$ $\square$

Similarly in ZFC:

If ZFC is consistent, then there is no formula $G(x) \in \mathcal{F}_{L_{ZF}}$ s.t. for every sentence $F \in \mathcal{F}_{L_{ZF}}$

$$ZFC \vdash F \iff G(\ulcorner \underline{F} \urcorner).$$

Assume $PA \subseteq \underset{\uparrow}{T} \subseteq \mathcal{F}_{L_{PA}}$

consistent, recursively enumerable set of sentences
~~theory~~

- there is a formula $Prov_T(x)$ s.t for every sentence $\varphi \in \mathcal{F}_{L_{PA}}$

$$T \vdash \varphi \iff \mathbb{N} \models Prov_T(\underline{\ulcorner \varphi \urcorner})$$

~~More~~ : $T \vdash \varphi \iff PA \vdash Prov_T(\underline{\ulcorner \varphi \urcorner})$.
Stronger: ↗

Idea. unary relation on $\mathbb{N}$

$$Prov_T(\ulcorner \varphi \urcorner) \iff \exists y\ \exists k\ (y \text{ is a proof of } \varphi \text{ in } T,$$

of length k, using only the first k elements of T, the first k axioms of KRL...)

$$\iff \exists y \exists k\ (\underbrace{\overset{y}{\overset{\shortparallel}{\langle (y)_0, \ldots, (y)_k \rangle}}, \ulcorner \varphi \urcorner}_{\text{recursive relacion}})$$

recursive relacion

$R(y, k, \ulcorner \varphi \urcorner)$

$\Downarrow$

represented in PA by a formule

$\varphi_R(y, k, z)$

$$Prov_T(z) = \cancel{\exists z\, \varphi}\ \exists y\ \exists k\ \ \varphi_R(y, k, z).$$

Now let F be the sentence from the diagonal lemma for $G(x) = \neg \mathrm{Prov}_T(x)$.

so $PA \vdash F \longleftrightarrow \neg \mathrm{Prov}_T(\ulcorner \underline{F} \urcorner)$

F says (according to PA) : ~~# Accordin~~.

"I can not be proved in T".

$\mathrm{Con}(T) = \neg \mathrm{Prov}_T(\ulcorner \underline{0 = 1} \urcorner)$: "T is consistent".

Fact (1) $T \nvdash F$

(2) If $\mathbb{N} \models T$ (i.e. $T \subseteq TA$), then $T \nvdash \neg F$

(3) $PA \vdash \mathrm{Con}(T) \Longleftrightarrow F$

Proof (1). Suppose $T \vdash F$. Then

$$\begin{tikzcd} & PA \vdash \mathrm{Prov}_T(\ulcorner \underline{F} \urcorner) \arrow[d, Rightarrow] \\ T \vdash \neg F & PA \vdash \neg F \arrow[l, Rightarrow, "PA \subseteq T"'] \end{tikzcd}$$

but T consistent ↯.

(2) Suppose $T \vdash \neg F$. $\Rightarrow$ ~~$\mathbb{N} \models \neg F$~~

$$\begin{tikzcd} T \vdash \neg F \arrow[d, Rightarrow, "T \supseteq PA"] & \\ T \vdash \mathrm{Prov}_T(\ulcorner \underline{F} \urcorner) \arrow[r, Rightarrow] & \mathbb{N} \models \mathrm{Prov}_T(\ulcorner \underline{F} \urcorner) \arrow[d, Rightarrow] \\ & T \vdash F \end{tikzcd}$$

↯.

(3) $PA \vdash F \leftrightarrow \neg Prov_T(\underline{\ulcorner F \urcorner})$

$$\Downarrow$$

$$PA \vdash Prov_T(\underline{\ulcorner F \leftrightarrow \neg Prov_T(\underline{\ulcorner F \urcorner}) \urcorner})$$

$$\Downarrow \text{ properties of } Prov_T$$

$$PA \vdash Prov_T(\underline{\ulcorner F \urcorner}) \leftrightarrow Prov_T(\underline{\ulcorner \neg Prov_T(\underline{\ulcorner F \urcorner}) \urcorner})$$

<u>But</u>: $PA \vdash Prov_T(\underline{\ulcorner F \urcorner}) \longrightarrow Prov_T(\underline{\ulcorner Prov_T(\underline{\ulcorner F \urcorner}) \urcorner})$
(*)

<u>Hence</u>: $PA \vdash Prov_T(\underline{\ulcorner F \urcorner}) \longrightarrow \neg Con(T)$

and $PA \vdash Con(T) \longrightarrow \neg Prov_T(\underline{\ulcorner F \urcorner})$

$PA \vdash Con(T) \longrightarrow F.$

$\leftarrow$:

$PA \vdash \neg Con(T) \longrightarrow Prov_T(\underline{\ulcorner F \urcorner})$

$PA \vdash \neg Con(T) \longrightarrow \neg F$

$PA \vdash F \longrightarrow Con(T).$

<u>Corollary</u> (Gödel's 2nd incompleteness thm)

(1) If T is consistent, then $T \cup \{\neg Con(T)\}$ is consistent.

(2). If $\mathbb{N} \models T$, then $T \cup \{Con(T)\}$ is consistent.
(i.e. $T \subseteq TA$)

Proof (1)

$T \vdash F \leftrightarrow Con(T)$ and $T \nvdash F$, so $T \nvdash Con(T)$

$$\Downarrow$$

$T \cup \{\neg Con(T)\}$ consistent

(2) Since T is consistent, $\mathbb{N} \models Con(T)$

so if $\mathbb{N} \models T$ then $\mathbb{N} \models \underbrace{T \cup \{Con(T)\}}_{\text{consistent}}$.

Corollary Assume $\mathbb{N} \models T$.

Let $A(x)$ = "x is a proof of $\overset{0=1}{\cancel{0 \neq 1}}$ in T".

so $\underline{PA \vdash \neg A(\underline{n})}$ for every $n \in \mathbb{N}$

but $PA, \underline{T \nvdash \forall x \neg A(x)}$; because

$T \cup \{\neg Con(T)\}$ consistent,

If $M \models T \cup \{\neg Con(T)\}$ then $M \models \exists \underset{\uparrow}{x} \neg A(x)$

non-standard "proof" of $0 = 1$ in T.

Corollary

Similarly if ZFC is consistent,

then $ZFC \cup \{\neg Con(ZFC)\}$ is consistent.

On (*):

Σ_1-formulas in L_{PA}:

of the form $\exists \bar{x}\, \psi$, where in ψ only bounded quantifiers:

$\exists x \leq y$, $\forall x \leq y$.

Lemma Assume $D(x_1, .., x_n) \in F_{L_{PA}}$ is Σ_1-formula.

Then $PA \vdash D(x_1, .., x_n) \longrightarrow Prov_{PA}(\underline{\ulcorner D(\underline{x}_1, .., \underline{x}_n)\urcorner})$.

Explanation:

For $x_1, ..., x_n \in \mathbb{N}$

$$(x_1, ..., x_n) \xrightarrow[\text{recursive}]{f_D} \ulcorner D(\underline{x}_1, .., \underline{x}_n)\urcorner \in \mathbb{N}$$

$$\Downarrow$$

represented by $\cancel{\Phi}\, \varphi_{f_D}(\bar{x}, y)$

$$Prov_{PA}(\ulcorner D(\underline{x}_1, .., \underline{x}_n)\urcorner = \exists y\, (\varphi_{f_D}(\bar{x}, y) \wedge Prov_{PA}(y))$$

Corollary. Let $n \in \mathbb{N}$

$$PA \vdash D(\underline{n}) \longrightarrow Prov_{PA}(\underline{\ulcorner D(\underline{n})\urcorner}).$$

Apply this to $D(x) = Prov_T(x)$: a Σ_1-formula.