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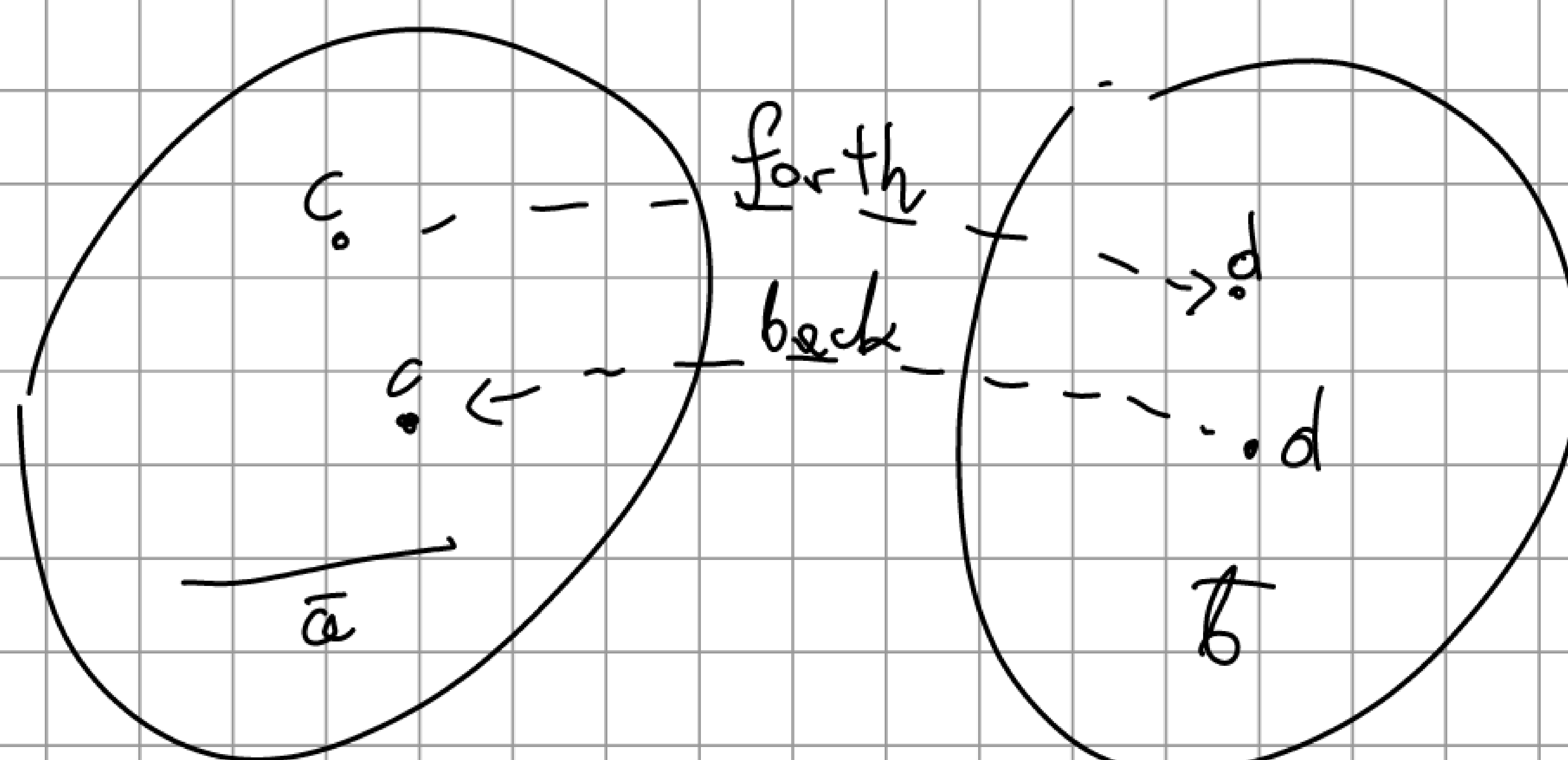
First we define equivalence rel. $\cdot \equiv_\alpha \cdot$ $(\alpha \in Ord)$ on sets M^n, $n < \omega$ (M: fixed)

1) $\bar{a} \equiv_0 \bar{b} \iff tp^M(\bar{a}) = tp^M(\bar{b})$

2) $\delta \in Lim$ $\quad \bar{a} \equiv_\delta \bar{b} \iff \forall \alpha < \delta \quad \bar{a} \equiv_\alpha \bar{b}$

3) $\bar{a} \equiv_{\alpha+1} \bar{b} \iff [\forall c \in M \, \exists d \in M \quad \bar{a}c \equiv_\alpha \bar{b}d] \wedge$
$[\forall d \in M \, \exists c \in M \quad \bar{a}c \equiv_\alpha \bar{b}d]$



Fact $\equiv_\alpha$ is an equivalence relation on M^n

Proof Induction on α

1. $\alpha = 0$ obvious

2. Limit step. $\delta \in Lim$. We assume that $\forall \alpha < \delta$ $\equiv_\alpha$ is an equiv. rel.

$$\Rightarrow \; \cdot \equiv_\delta \cdot \; = \bigcap_{\alpha<\delta} \cdot \equiv_\alpha \cdot$$

3. Successor step. Assume that $\equiv_\alpha$ is an equivalence relation. We prove that $\equiv_{\alpha+1}$ is an eq. rel.. Reflexive, symmetric is trivial. Transitive:

$\bar{a} \equiv_{\alpha+1} \bar{a}' \equiv_{\alpha+1} \bar{a}''$. We want to show that $\bar{a} \equiv_{\alpha+1} \bar{a}''$.

"forth": choose any $c \in M$. Since $\bar{a} \equiv_{\alpha+1} \bar{a}'$ we have $c' \in M$ s.t $\bar{a}c \equiv_\alpha \bar{a}'c'$ and also $c'' \in M$ s.t. $\bar{a}'c' \equiv_\alpha \bar{a}''c'' \overset{\text{Ind. ass.}}{\Rightarrow} \bar{a}c \equiv_\alpha \bar{a}''c''$

"back": similar. $\blacksquare$

<u>Fact</u> $\bar{a} \equiv_\alpha \bar{b},\ \alpha > \beta \Rightarrow \bar{a} \equiv_\beta \bar{b}$ $[\equiv_\alpha \subseteq \equiv_\beta]$

<u>Proof</u> exercise

Lemma Assume M is ctble.

(1) $\forall \bar{a} \subseteq M \; \exists \alpha < \omega_1 \; \forall \beta \in (\alpha, \omega_1) \; [\bar{a}]_{\equiv_\alpha} = [\bar{a}]_{\equiv_\beta}$

(2) $\exists \alpha < \omega_1 \; \forall \bar{a}, \bar{b} \subseteq M \; \left(\bar{a} \equiv_\alpha \bar{b} \Rightarrow \bar{a} \equiv_{\alpha+1} \bar{b} \right)$

Proof

(1) $\langle [\bar{a}]_{\equiv_\alpha} : \alpha < \omega_1 \rangle$ a weakly decreasing sequence of ctble sets.

$$\begin{array}{ccc} [\bar{a}]_{\equiv_\alpha} & \supsetneq & [\bar{a}]_{\equiv_\beta} \\ \psi & & \not\psi \\ b_\alpha & & b_\alpha \end{array}$$

It has to stabilize before ω_1.

(2) From (1) for $\bar{a} \subseteq M$ we pick $\alpha_{\bar{a}} < \omega$ as in (1)

Let $\alpha = \sup_{\bar{a} \subseteq M} \alpha_{\bar{a}} < \omega_1$ α is good in (2)

Def $SH(M) = \min \{ \alpha : \underbrace{\forall \bar{a}, \bar{b} \subseteq M \left(\bar{a} \equiv_\alpha \bar{b} \Rightarrow \bar{a} \equiv_{\alpha+1} \bar{b} \right)}_{\Updownarrow \; \forall \beta > \alpha \; \forall n \; \left(\equiv_\beta = \equiv_\alpha \text{ on } M^n \right)} \}$

Scott height

$SH(M) < \|M\|^+$

Remark Let $\alpha = SH(M)$. Then $\forall \beta > \alpha \ \forall \bar{a}, \bar{b} \subseteq M$

$$(\bar{a} \equiv_\alpha \bar{b} \Rightarrow \bar{a} \equiv_\beta \bar{b})$$

Proof. Induction on β

Comparing structures: $\cdot \equiv_\alpha \cdot$ $(\alpha \in Ord)$: equivalence relation on classes of (ctble) ctbl. L-structures

(a) $M \equiv_0 N \Leftrightarrow M \equiv N$

(b) $(\delta \in Lim)$ $M \equiv_\delta N \Leftrightarrow \forall \alpha < \delta \ M \equiv_\alpha N$

(c) $M \equiv_{\alpha+1} N \Leftrightarrow [\forall \bar{a} \subseteq M \ \exists \bar{b} \subseteq N \ (M, \bar{a}) \equiv_\alpha (N, \bar{b})]$

$\wedge [\forall \bar{b} \subseteq N \ \exists \bar{a} \subseteq M \ (M, \bar{a}) \equiv_\alpha (N, \bar{b})]$

Thm (Dana Scott) Assume M, N ctble L-structures, $\alpha = SH(M) = SH(N)$.

(1) $M \equiv_{\alpha+1} N \Rightarrow M \cong N$

(2) $M \equiv_{\alpha+2} N' \Rightarrow SH(N') = SH(M)$ and $M \cong N'$

N': ctble L-str.

(3) $\exists$ a sentence $\varphi \in \mathcal{L}_{\omega_1, \omega}$ (the Scott sentence) s.t. $M \models \varphi$ and

$$\forall N \ (N \models \varphi \Rightarrow M \cong N)$$

N: ctbl L-str.

??

<u>Proof</u> (3) $\mathcal{L}_{\omega_1\omega}$: the set of generalized formulas

(a) atomic formulas, terms : as before

(b) negation, quantifiers : as before

(c) we admit conjunctions, disjunctions of length $< \omega_1$. If $\{\varphi_i : i < \omega\}$: $\mathcal{L}_{\omega_1\omega}$ formulas

$\bigvee_n \varphi_n$, $\bigwedge_n \varphi_n$: $\mathcal{L}_{\omega_1\omega}$ formulas

Analogously $\mathcal{L}_{\varkappa\lambda}$ — bound on len of conj., bound on len of quant.

Example: Scott sentence $(\mathbb{N}, \leq)$

$\varphi_0(x)$: "x is minimal"

$$\varphi_{n+1}(x) : \exists y \big(\varphi_n(y) \wedge x < y \wedge \wedge \neg \exists z \; y < z < x\big)$$

Scott sentence for $(\mathbb{N}, \leq)$ φ : "$\leq$ is an infinite LO"

$$\wedge \forall x \bigvee_{n<\omega} \varphi_n(x)$$

<u>Lemma 1</u> $\forall \alpha < \omega_1 \; \forall \bar{a} \subseteq M \; \exists \varphi_\alpha(\bar{x})$: a $\mathcal{L}_{\omega_1\omega}$-formula

$$\forall \bar{b} \subseteq M \;\; (\bar{a} \equiv_\alpha \bar{b} \iff M \models \varphi_\alpha(\bar{b}))$$

Proof Induction on α.

(1) $\alpha = 0$: $\varphi_0(\bar{x}) = \bigwedge tp^M(\bar{a})(\bar{x})$

(2) $\delta \in Lim$: $\varphi_\delta(\bar{x}) = \bigwedge_{\alpha<\delta} \varphi_\alpha(\bar{x})$

(3) $\alpha+1$: For any $b \in M$ let $\psi_b(\bar{x}, y)$ be a formula good for $[\bar{a}b]_{\equiv_\alpha}$ class namely: $\forall \bar{c}d \subseteq M \, (\bar{a}b \equiv_\alpha \bar{c}d \iff M \models \psi_b(\bar{c}, d))$ (by ind. ass.)

$$\varphi_{\alpha+1}(\bar{x}): \forall y \bigvee_{b \in M} \psi_b(\bar{x}, y) \quad \left(\text{"} \forall y \exists b \; \bar{a}b \equiv_\alpha \bar{x}y \text{"} \right)$$

$$\wedge \bigwedge_{b \in M} \exists y \, \psi_b(\bar{x}, y) \quad \left(\text{"} \forall b \exists y \; \bar{a}b \equiv_\alpha \bar{x}y \text{"} \right)$$

$\blacksquare$

Lemma 2. $\forall \alpha < \omega_1 \; \forall \underset{\text{ctble}}{M} \left(\exists \varphi : \text{a } L_{\omega_1\omega}\text{-sentence} \right) \forall N$
$\left(M \equiv_\alpha N \iff N \models \varphi \right)$

Proof similar

Using lemmas 1 & 2 (3) follows.

Skolem functions, skolemization. T: a theory in L

Def. T has Skolem functions, if

$\forall \varphi(\bar{x}, y) \in \mathcal{F}_L$ there is a term

$t_\varphi(\bar{x}) \in \mathcal{T}_L$ st. $T \vdash \exists y\, \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, t_\varphi(\bar{x}))$

Def. Assume T has Skolem functions.

Assume $M \models T$ and $A \subseteq |M|$. Then

$\mathcal{H}(A)$ = the smallest subset X of M st.

the Skollem hull of A in M — $A \subseteq X$ and $\forall t_\varphi(\bar{x}) \forall \bar{a} \subseteq X$ $t_\varphi^M(\bar{a}) \in X$

Remark 1. If T has Skolem functions, then T has q.e.

Proof We prove: for every $\varphi \in \mathcal{F}_L$ there is a q.f. $\varphi' \in \mathcal{F}_L$ s.t. $T \vdash \varphi \leftrightarrow \varphi'$. Induction on $|\varphi|$.

- φ is atomic formula: OK
- the connectives: OK
- $\varphi(\bar{x}) = \exists y\, \psi(\bar{x}, y)$

By the ind. ass. $T \vdash \psi \leftrightarrow \psi'$ ← q.f.

wlog. ψ q.f.

$$T \vdash \exists y\, \psi(\bar{x}, y) \leftrightarrow \psi(\bar{x}, t_\psi(\bar{x}))$$

for free

■

<u>Remark 2</u> If T has Skolem functions and $A \subseteq M \models T$, then $\mathcal{H}(A) \prec M$

<u>Proof</u> Apply T–V test. Let $\varphi(\bar{x}, y) \in \mathcal{F}_L$, $\bar{a} \subseteq \mathcal{H}(A)$ s.t. $M \models \exists y\, \varphi(\bar{a}, y)$.

But $T \vdash \exists y\, \varphi(\bar{x}, y) \to \varphi(\bar{x}, t_\varphi(\bar{x}))$

So $M \models \varphi(\bar{a}, \underbrace{t_\varphi(\bar{a})}_{\in \mathcal{H}(A)})$

<u>Remark 3</u> Assume "$\leq$" $\in L$ and M: L-str. s.t. $(M, \leq^M)$ is well ordered. Then in M we have definible Skolem functions, i.e. if for every $\varphi(\bar{x}, y) \in \mathcal{F}_L$ there is a definible

function f in M s.t.

$$M \models \exists y\, \varphi(\bar{x}, y) \longrightarrow \varphi(\bar{x}, f(\bar{x}))$$

<u>Proof.</u> (idea) $f(\bar{x})$ = the $\leq$-minimal y s.t. $\varphi(\bar{x}, y)$