17. 11. 2021

Assume M: a ctbl structure $\Rightarrow$ $\mathrm{Aut}(M)$: the group of authomorphisms of M.

- $\mathrm{Aut}(M) \curvearrowright M$: $\underset{\mathrm{Aut}(M)}{\sigma} \cdot \underset{M}{a} = \sigma(a)$

also on M^n:

$\sigma \cdot \langle a_1, \dots, a_n \rangle = \langle \sigma(a_1), \dots, \sigma(a_n) \rangle$

Corollary (Ryll-Nardzewski) $Th(M)$ is $\aleph_0$-categorical iff $\forall n$ on M^n there are only finetely many orbits of $\mathrm{Aut}(M)$.

Proof "$\Rightarrow$" Fix n. $S_n(\emptyset) = \underbrace{\{p_1, \dots, p_k\}}_{\text{all isolated}}$.

Let $\varphi_i(\bar{x}) \vdash p_i(\bar{x})$, $i = 1, \dots, k$

Clearly: $\bigcup \underbrace{[\varphi_i]}_{\text{clopen}} = S_n(\emptyset) \Rightarrow$ in $\mathcal{L}_n(\emptyset)$

$[\varphi_1 \vee \varphi_2 \vee \dots \vee \varphi_k]$ $\qquad [\varphi_1 \vee \dots \vee \varphi_n]_\sim = \mathbb{1}$

Hence $T \vdash \varphi_1(\bar{x}) \vee \ldots \vee \varphi_k(\bar{x})$

More: $[\varphi_i] \cap [\varphi_j] = \emptyset$ for $i \neq j$, so

$T \vdash \varphi_1(\bar{x}) \dot{\vee} \varphi_2(\bar{x}) \dot{\vee} \ldots \dot{\vee} \varphi_k(\bar{x})$ $\quad \{\bar{a} \in M^n : M \models \varphi_k(\bar{a})\}$

Hence $M^n = \varphi_1(M) \dot{\cup} \ldots \dot{\cup} \varphi_k(M)$

i.e. $T \vdash \forall \bar{x} \left(\bigvee_i \varphi_i(\bar{x}) \wedge \bigwedge_{1 \leq i < j \leq k} \neg(\varphi_i(\bar{x}) \wedge \varphi_j(\bar{x})) \right)$

- $\varphi_1(M), \ldots, \varphi_k(M)$: orbits of the $\mathrm{Aut}(M)$ on M^n:
  - let $\bar{a} \in \varphi_i(M)$, $\bar{b} \in \varphi_j(M)$. If $i \neq j$ then $\neg \exists \sigma \in \mathrm{Aut}(M)\ \sigma(\bar{a}) = \bar{b}$.

    If $i = j$, then $\exists \sigma \ldots$ because

    then $tp(\bar{a}) = tp(\bar{b})$

    $\Downarrow$

    construct σ

    (back-and-forth arg)

"$\Leftarrow$" Fix n. Notice that $\mathcal{L}_n(\emptyset)$ finite.

If not, then there are pair-wise T-contradictory consistent formulas $\varphi_i(\bar{x})$, $i < \omega$

$\mathcal{F}_L(\bar{x})$

This gives infinitely many orbits on M^n.

$\varphi_i(M)$: a union of orbits. $\lightning$

$\mathcal{L}_n(\emptyset)$: finite $\underset{\forall n}{\Longrightarrow}$ $S_n(\emptyset)$ is finite $\Big) \Rightarrow T$ is $\aleph_0$-categorical

$\blacksquare$

<u>Remark</u> Assume $M = \{a_n : n < \omega\}$, $N = \{b_n : n < \omega\}$

: L-structures. s.t. $\forall n$ $tp^M(a_0, \dots, a_{n-1}) = tp^N(b_0, \dots, b_{n-1})$

Then $M \cong N$

$a_i \mapsto b_i$

"Another" proof of Gödels model existence thm:

T: consistent complete $\Rightarrow T$ has a model

We construct sequences: 1) of types $p_n(x)$

2) of new constant symbols c_n

3) of theories T_n s.t.:

a) $T = T_0 \subseteq T_1 \subseteq \dots$

(b) $p_n(x) \in S_1^{T_n}(\emptyset)$

(c) $T_{n+1} = Cn\left(T_n \cup \{\varphi(c_n) : \varphi(x) \in p_n(x)\}\right)$

in language $L_{n+1} = L \cup \{c_i : i \leq n\}$

T_n is complete and consistent: (ind. on n)

- T_0 is from assumptions
- consistency: $p_n(x)$ compl. type in T_n, so

  for $p' \subseteq p_n$ finite

  $T_n \vdash \exists x \bigwedge p'(x) \Rightarrow T_n \cup \{\bigwedge p'(c_{n+1})\}$ consistent

- completeness

  Let $\sigma(c_n)$: a sentence in L_{n+1}

  1° $\sigma(x) \in p_n(x)$ in L_n or $(\neg\sigma(x)) \in p_n(x)$

  $\Downarrow$ $\Downarrow$

  $T_{n+1} \vdash \sigma(c_n)$ $T_{n+1} \vdash \neg\sigma(c_n)$

Let $T' = \bigcup_n T_n$: consistent, complete in $\mathcal{L}' = \mathcal{L} \cup \{c_n : n < \omega\}$. We can ensure in the construction that $\forall \varphi(x) \in \mathcal{F}_{\mathcal{L}'}$ (if $T' \vdash \exists x\, \varphi(x)$, then $\underbrace{T' \vdash \varphi(c_n)}_{\Updownarrow}$ for some n)

$$\varphi(x) \in p_n(x)$$

(how? choose $\varphi(x) \in \mathcal{F}_{\mathcal{L}'}$ s.t. $T' \vdash \exists x\, \varphi(x)$.
Then $\varphi \in \mathcal{F}_{\mathcal{L}_k}$ for some k, $T_k \vdash \exists x\, \varphi(x)$
for some $n \geq k$ we ensure $p_n(x) \ni \varphi(x)$
$\Rightarrow T_{n+1} \vdash \varphi(c_n) \Rightarrow T' \vdash \varphi(c_n)$)

Therefore $\{c_n : n < \omega\}$ form a model of T

Problem of model theory:

- how to: construct models?
- how to: describe models of T?

Stability hierarchy of theories: Assume T complete, consistent with infinite models.

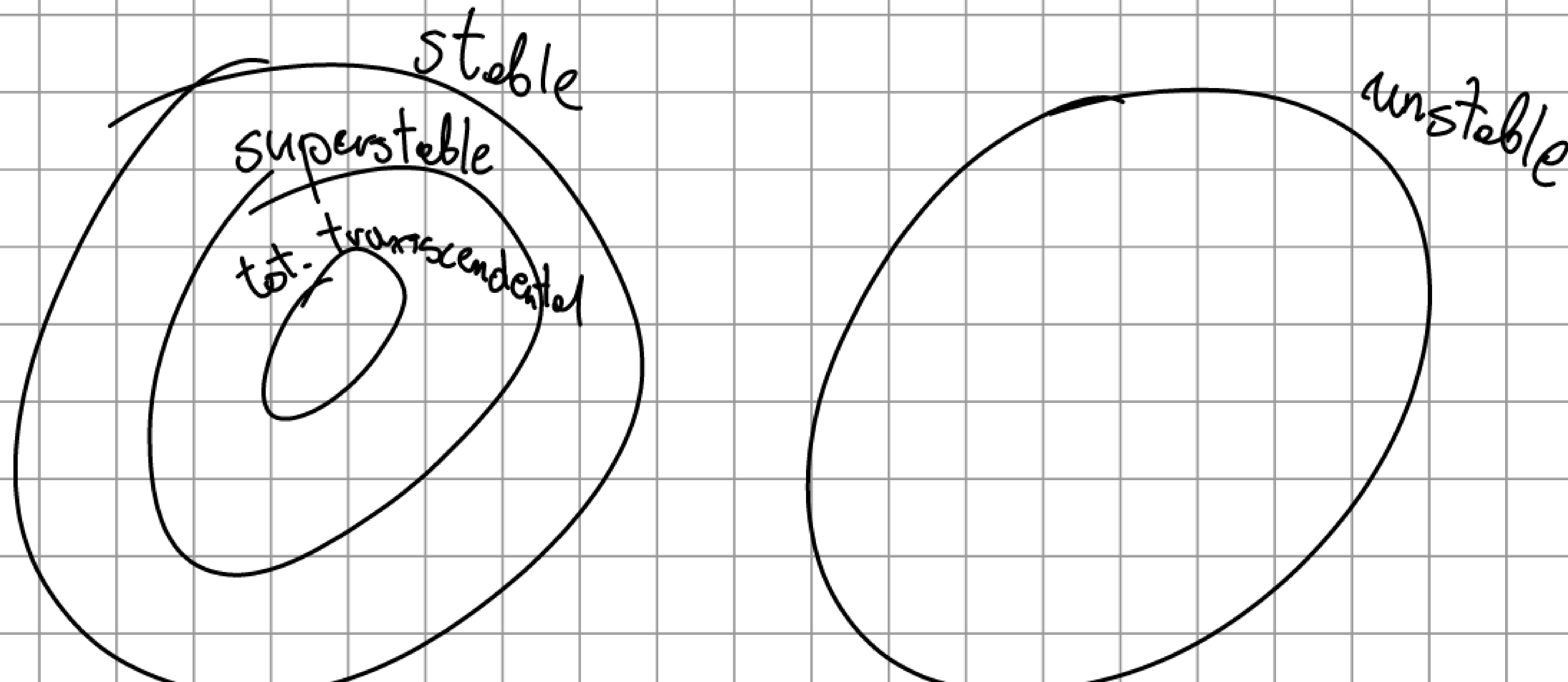
Let $\underset{\mathrm{Cn}}{\varkappa} \geq \aleph_0$.

(1) T is $\varkappa$-stable $\iff \forall M \models T \ \forall \underset{|A| \leq \varkappa}{A \subseteq |M|} \ |S_1(A)| \leq \varkappa$

(2) T is stable $\iff \exists \varkappa (T \ \varkappa\text{-stable})$

(3) T is superstable $\iff \exists \mu \ \forall \varkappa \geq \mu$ T is $\varkappa$-stable

(4) T Totally transcendental $\iff \forall \varkappa$ T $\varkappa$-stable



Remark T is tot. trans. $\iff T$ is $\aleph_0$-stable

Proof "$\Rightarrow$" obv.

"$\Leftarrow$" Suppose T is $\aleph_0$-stable but not $\varkappa$-stable for some $\varkappa$

$M \underset{\underset{A}{\cup}}{}$ FT s.t. $|S(A)| > |A| \geq \aleph_0$, $|A| = \varkappa$

We shall find $A_0 \subseteq A$ with $|S(A_0)| > |A_0| + \aleph_0$
ctbl.

Let $\varphi(x) \in \mathcal{F}_{L(A)}(x)$.

<u>Def.</u> φ is <u>big</u> iff $|S(A) \cap [\varphi]| > |A|$.

$p \in S(A)$ is <u>big</u> iff $\forall \varphi(x) \in p(x)$ φ is big.

<u>Lemma</u> If $\varphi(x)$ is big, then there is $\psi(x) \in \mathcal{F}_{L(A)}(x)$ s.t. both $(\varphi \wedge \psi)(x)$ and $(\varphi \wedge \neg\psi)(x)$ are big.

<u>Proof</u> (a.a.) If not, then there is exactly one big type $p(x) \in S(A) \cap [\varphi]$

$p = \{\psi(x) \in \mathcal{F}_{L(A)}(x) : (\varphi \wedge \psi)(x)$ is big $\}$

big, $\in S(A) \cap [\varphi]$

⋮

From the lemma we construct a tree of big formulas $\varphi_\eta(x) \in \mathcal{F}_{L(A)}(x)$, $\eta \in 2^{<\omega}$

$$\begin{tikzcd}[column sep=small]
 & & & \varphi_\emptyset(x) = (x = x) \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & \varphi_{\langle 0 \rangle} \arrow[dl, dash] \arrow[dr, dash] & & & & \varphi_{\langle 1 \rangle} \arrow[dl, dash] \arrow[dr, dash] & \\
\varphi_{00} & & \varphi_{01} & & \varphi_{10} & & \varphi_{11}
\end{tikzcd}$$

$$T \vdash \varphi_\eta(x) \leftrightarrow \varphi_{\eta 0}(x) \dot{\vee} \varphi_{\eta 1}(x)$$

Let $A_0 \underset{\text{ctbl}}{\subseteq} A$ including parameters of all $\varphi_\eta(x)$

For $\eta \in 2^\omega$ $\underbrace{\{\varphi_{\eta|n}(x) : n < \omega\}}_{\text{a consistent type over } A_0 \text{ in } T.} \subseteq p_\eta(x) \in S(A_0)$

$\eta \neq \eta' \Rightarrow p_\eta \neq p_{\eta'}$ so: $|S(A_0)| \geq 2^{\aleph_0} > |A_0| + \aleph_0$

and T is not $\aleph_0$-stable

Examples

(1) $Th(\mathbb{C},+,\cdot) = ACF_0$: $\aleph_0$-stable

(2) $Th(\mathbb{Z},+)$: superstable, but not tot. trans.

(3) $SCF_{p,l}$: the theory of separably closed fields of char. p and Ersov invariant l is stable, but not superstable

(4) $Th(\mathbb{Q},\leq)$, $Th(\mathbb{Z},+,\cdot)$, BA_0, $Th(\mathbb{R},\cdot,+,\leq)$ are unstable

(5) R: a ring with 1, M: R-module $\Rightarrow Th(M)$ is stable