Lecture 10. 11. 2021

Recall: $\varkappa \in CN$, $\varkappa \geq \aleph_0$, L: ctble, T: complete consistent theory in L

$M \models T$

Def.

(1) M is $\varkappa$-saturated, if $\forall A \subseteq M$ $\forall p \in S_1(A)$ p is realized in M ($|A| < \varkappa$)

(2) M is saturated if M is $\|M\|$-saturated.

Corollary (1) $\forall \varkappa\ \forall M\ \exists N \succ M$ N is $\varkappa$-saturated

(2) If $\varkappa > \aleph_0$ is regular and $2^{<\varkappa} = \varkappa$, then $\forall M$ $\exists N \succ M$ of power $\leq \varkappa$

$\Updownarrow \leftarrow$ exercise

$\varkappa^{<\varkappa} = \varkappa$

($\|N\| = \varkappa$ and N = saturated).

Proof

$S_1(A)$

Remark. For $A \subseteq M$, $|S(A)| \leq 2^{|A| + \aleph_0}$

Proof. $|S(A)| \leq |\mathcal{P}(L_1(A))| \leq 2^{|L_1(A)|}$, $|L_1(A)| = |A| + \aleph_0$.

Idea of the proof of (1).

Let $\mu = 2^\varkappa$. Then $cf(\mu) > \varkappa$

We construct a elementary chain of models N_α, $\alpha < \mu$, of power $\leq \mu$ each such that:

(1) $M = N_0 \prec N_1 \prec \ldots \prec N_\alpha \prec N_\beta \prec \ldots$ for $\alpha < \beta < \mu$

Recursively:

(2) At step $\alpha = \beta + 1$: we have a model N_β.

We choose $N_\alpha \succ N_\beta$ so that:

$(\forall B \subseteq N_\beta,\ |B| < \varkappa)\ \forall p \in S_1(B)$ p is realized in N_α.

and $\|N_\alpha\| \leq \mu$.

- there are $\leq \mu$ types p to consider
- there is a model $N' \succ N_\beta$ realizing all of them
- by downward Löwenheim–Skolem, can assume $\|N'\| \leq \mu$.

(3) Limit step: assume $\alpha \in Lim$ (a limit ordinal) and N_β already are chosen for all $\beta < \alpha$.

Then $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$.

Then let $N = \bigcup_{\alpha<\mu} N_\alpha$

- $M = N_0 \prec N$
- N is $\varkappa$-saturated.

Proof. Assume $A \subseteq N$, $|A| < \varkappa$, $p \in S_1^N(A)$

$cf(\mu) > \varkappa \Rightarrow (\exists \alpha < \mu)\ A \subseteq N_\alpha$

because: Say $A = \{a_\gamma : \gamma < \nu\}$, where $|A| = \nu < \varkappa$.
For $\gamma < \nu$ let $\alpha_\gamma < \mu$ s.t. $a_\gamma \in N_{\alpha_\gamma}$.

~~$\{\alpha_\gamma : \gamma < \nu\} \subseteq$~~

$$\left.\begin{array}{l}\underbrace{\{\alpha_\gamma : \gamma < \nu\}}_{\text{of power } \leq \nu < \varkappa < cf\mu} \subseteq \mu\end{array}\right] \Rightarrow \exists \alpha < \mu \quad \{\alpha_\gamma : \gamma < \nu\} \subseteq \alpha.$$

that is: $(\forall \gamma < \nu)\ a_\gamma \in N_{\alpha_\gamma} \prec N_\alpha$

$a_\gamma \in N_\alpha$

So: $A \subseteq N_\alpha$.

So: $A \subseteq N_\alpha \prec N \implies S_1^N(A) = S_1^{N_\alpha}(A)$

So $p \in S_1^{N_\alpha}(A)$,

By the successor step: p is realized in $N_{\alpha+1}$ by some c.

$N_{\alpha+1} \prec N$, hence c realizes p also in N. $\square$

(2) A similar proof.

Auxiliary lemma on $\prec$:

(1) $\prec$ is transitive [exercise]

(2) If γ is a limit ordinal and $(N_\alpha : \alpha < \gamma)$ is an elementary chain of structures,

[i.e. for $\alpha < \beta < \gamma$ $N_\alpha \prec N_\beta$]

then there is a structure N_γ [called the union of the chain]

s.t.

(a) $|N_\gamma| = \bigcup_{\alpha<\gamma} |N_\alpha|$

(b) $N_\alpha \prec N_\gamma$ for all $\alpha < \gamma$. [Tarski]

Proof

Definition of N_γ: • $|N_\gamma| = \bigcup_{\alpha<\gamma} |N_\alpha|$

• Interpretations of L-symbols in N_γ:

(i) P: relational symbol:

Let $a_1, \dots, a_n \in |N_\gamma|$.

$P^{N_\gamma}(a_1, \dots, a_n) \iff P^{N_\alpha}(a_1, \dots, a_n)$ holds for some [$\equiv$ every] $\alpha < \gamma$ with $a_1, \dots, a_n \in |N_\alpha|$

[does not depend on the choice of α,
because $(N_\alpha)_{\alpha<\gamma}$: elementary].

- f : a function symbol of L :

$a_1, \dots, a_n, a_{n+1} \in |N_\gamma|$

$f^{N_\gamma}(a_1, \dots, a_n) = a_{n+1} \iff$ for $\underset{\text{every}}{\overset{\text{some}}{\equiv}}$ $\alpha < \gamma$ with
$a_1, \dots, a_{n+1} \in |N_\alpha|$
$f^{N_\alpha}(a_1, \dots, a_n) = a_{n+1}$

- c : a constant symbol of L.

$c^{N_\gamma} = c^{N_\alpha}$ for $\underset{\text{every}}{\overset{\text{any}}{\equiv}}$ $\alpha < \gamma$.

(b) : $N_\alpha \prec N_\gamma$ for every $\alpha < \gamma$

Inductive statement:

$$(\forall \underset{\varphi(\bar{x})}{\overset{}{\varphi}} \in \mathcal{F}_L \; \underbrace{\forall \alpha < \gamma \; \forall \bar{a} \subseteq N_\alpha \; [N_\alpha \models \varphi(\bar{a}) \iff N_\gamma \models \varphi(\bar{a})]}_{(*)}$$

Proof by induction on $|\varphi|$:

1. φ quantifier-free.
   Then $(*)$ true, because for every $\alpha < \gamma$, $N_\alpha \subseteq N_\gamma$
   substructure.

2. Induction step:

Assume $\varphi(\bar{x}) = \exists y \underbrace{\psi(\bar{x}, y)}_{\text{shorter, so } (*) \text{ holds for } \psi.}$

$$\overset{\overset{\bar{a}}{\cap}}{N_\alpha} \models \exists y\, \psi(\bar{a}, y) \iff N_\alpha \models \psi(\bar{a}, b) \text{ for some } b \in N_\alpha$$

$$\Downarrow \text{ induction hypothesis}$$

$$N_\gamma \models \psi(\bar{a}, b)$$

$$\Downarrow$$

$$N_\gamma \models \exists y\, \psi(\bar{a}, y)$$

$$N_\gamma \models \exists y\, \psi(\bar{a}, y) \iff N_\gamma \models \psi(\bar{a}, b) \text{ for some } b \in N_\gamma$$

Choose $\alpha < \beta < \gamma$ s.t. $b \in N_\beta$. $\Downarrow$ induction assumption

$$N_\beta \models \psi(\bar{a}, b)$$

$$\Downarrow$$

$$N_\beta \models \exists y\, \psi(\bar{a}, y)$$

$$\Downarrow N_\alpha \prec N_\beta$$

$$N_\alpha \models \exists y\, \psi(\bar{a}, y)$$

Example.

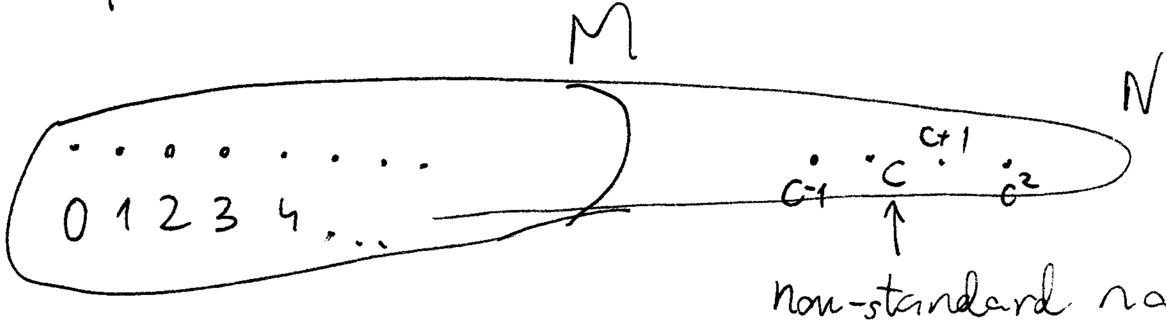
$M = (\mathbb{N}, \overbrace{+, \cdot, 0, 1, <}^{L})$ $TA = Th(M)$ true arithmetic.

For $n \in \mathbb{N}$ let $\underline{n} = \overbrace{\underline{1} + \ldots + \underline{1}}^{n}$ (an L-term) not a number
(numeral n)

Let $p(x) = \{x > \underline{n} : n \in \mathbb{N}\}$. : a consistent type in TA.

let $N \succ M$ countable s.t. p is realized in N, by some $c \in N$.

$N \succ M$



non-standard natural number, infinitely large

$M \models (\forall x \neq 0)\, \exists y \;\; y+1=x$

$\Downarrow$

~~N ⊨ "istnieje poprzednik c"~~ ← tei

$N \models$ "there is a predecessor of c" ← also infinitely large

Similarly: $c < c+1 < c+2 < \ldots < c^2 \ldots$

Example 2. Non-standard analysis.

$\mathcal{R} = (\mathbb{R}, \overbrace{+, \cdot, 0, 1, f, r, R}^{L})$

f : all functions $\mathbb{R}^k \to \mathbb{R}$

R : all relations $\subseteq \mathbb{R}^m$

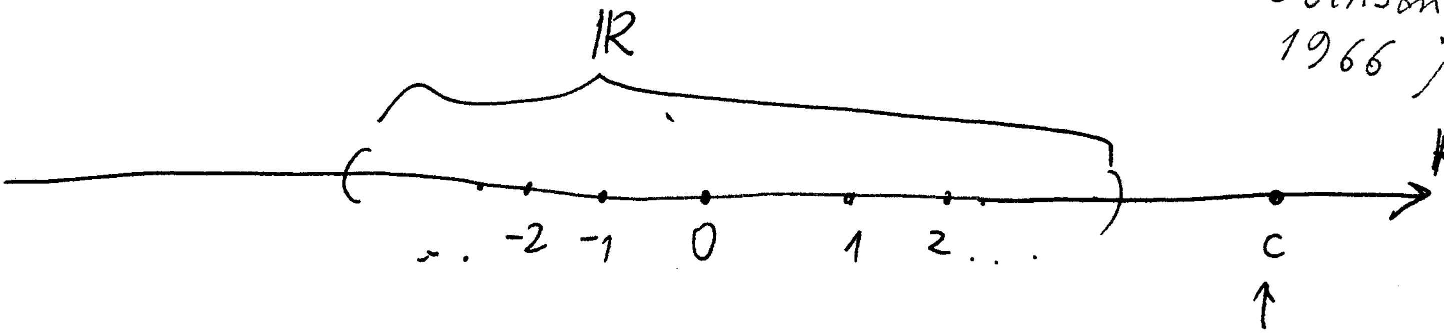
r : all reals

analysis

$A = Th(\mathcal{R})$

Let $\overset{\mathbb{R}^*}{\overset{\|}{M}} \models A$ $\aleph_1$-saturated. Wlog $\mathcal{R} \prec M$

analysis in M: "non-standard analysis" (Abraham Robinson 1966)



infinitely large

- $\frac{1}{c} \in \mathbb{R}^*$: in $\mathbb{R}^*$: • positive, $\neq 0$
  • $< \frac{1}{n}$ for every $n \in \mathbb{N}$] infinitesimal

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Omitting types

Assume T: a complete theory, $M \models T$, $p(x)$ a (consistent) type in T (over $\emptyset$)

<u>Def</u> M omits $p \iff \neg \exists a \in M$ a realizes p

<u>Def</u> $p(x)$ is <u>isolated</u>, if $\exists \varphi(x) \in F_L$

$$T \vdash \exists x \, \varphi(x)$$

and

$$\left[T \vdash \varphi(x) \rightarrow \psi(x) \text{ for every } \psi(x) \in p(x) \right.$$

Notation: $\varphi(x) \vdash p(x)$
[T in the background]

<u>Remark</u> (1). Assume $p(x)$ is a complete type.

Then p is isolated $\iff$ $\{p\}$ is open in $S(\emptyset)$
[i.e. p is isolated in the topological sense].

(2) p is ~~is~~ isolated $\iff$ the ~~set~~ $\{q(x) \in S(\emptyset) : p(x) \subseteq q(x)\}$ has non-empty interior in $S(\emptyset)$
[Stone topology]

<u>Proof</u> (1). $\varphi \vdash p \iff [\varphi] = \{p\}$ in $S(\emptyset)$,
(If $T \vdash \exists x \, \varphi(x)$)

(2): exercise.

<u>Remark</u> If $p(x)$ is isolated, then $\forall M \models T$ p is realized in M [i.e. $p(x)$ can not be omitted].

<u>Pf.</u> $\varphi(x) \vdash p(x)$ $\quad M \models T \quad M \models \exists x \varphi(x)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\in}{c}$ ~~witne~~ witness.

c realizes $p(x)$. $\Leftarrow$

Thm (A. Ehrenfeucht)

<u>If</u> T is countable, complete, consistent, ~~P~~ $p(x)$ is non-isolated, <u>then</u> $\exists M \models T$ M omits p.

<u>Proof</u> (similar to Henkin)

Let $\{c_n : n \in \omega\}$: a set of new constant symbols, $L' = L \cup \underbrace{\{c_n : n \in \omega\}}$.

- $\{\varphi_n(x) : n < \omega\}$: enumeration of $\mathcal{F}_{L'}(x)$.
- $h : \omega \to \omega$ increasing, s.t. $c_{h(n)}$ does not appear in $\varphi_0, \dots, \varphi_n$.
- Let $H_i = \{\exists x \varphi_i \to \varphi_i(c_{h(i)})\}$; Henkin's axiom

We define an increasing sequence of consistent sets $T = T_0 \subseteq T_1 \subseteq \dots \subseteq \mathcal{F}_{L'}$ s.t.

(a) $T_{2i+1} = T_{2i} \cup \{H_i\}$

(b) $T_{2i+2} = T_{2i+1} \cup \{\neg \psi(c_i)\}$ for some $\psi(x) \in p(x)$.

Recursive construction:

- Suppose we have T_{2i} : consistent.

  then $T_{2i+1} = T_{2i} \cup \{H_i\}$ consistent (as in Henkin's proof)

- Assume $T_{2in} = T \cup \{\psi_j(\underbrace{\bar{c}, c_i}) : j < k\}$

  new constants symbols from among $c_n, n<\omega$

  Suppose we can not pick $\psi(x) \in p(x)$ so that

  $T_{2i+1} \cup \{\neg\psi(c_i)\}$ is consitent.

  Hence $T_{2i+1} \vdash \psi(c_i)$ for every $\psi(x) \in p(x)$

Let $\Psi = \bigwedge_{j<k} \psi_j(\bar{c}, c_i)$

by deduction thm: $T \vdash \Psi(\bar{c}, c_i) \to \psi(c_i)$ for every $\psi(x) \in p(x)$

$\bar{c}, c_i$: new constant symbols

$\Downarrow$

$\cancel{T \vdash \psi(\bar{y}, x) \to \psi(x)}$

$$T \vdash \Psi(\bar{y}, x) \to \psi(x)$$ [if needed: $\bar{y}, x$: new variables]

But: $\models [\Psi(\bar{y}, x) \to \psi(x)] \to (\exists \bar{y}\, \Psi(\bar{y}, x) \to \psi(x))$

So: $T \vdash (\exists \bar{y}\, \Psi(\bar{y}, x)) \to \psi(x)$ for every $\psi(x) \in p(x)$

But: $T \vdash \exists x (\exists \bar{y}\, \Psi(\bar{y}, x))$ [as: $T \cup \{\Psi(\bar{c}, c_i)\}$ consistent]

hence: the formula

$\exists \bar{y}\, \Psi(\bar{y}, x)$ isolates $p(x)$ $\lightning$

Therefore can find T_{2i+2} !

Now as in Henkin's proof

let $T_\infty = \bigcup_i T_i$ and $S \supseteq T_\infty$

$\uparrow$ consistent complete theory.

We construct a model M of S from constant symbols c_n, $n<\omega$.

By case $\cancel{A}$ (b) ($T_{2i+1} \rightsquigarrow T_{2i+2}$):

$M \models \neg\psi(c_i)$ for some $\psi \in p$, so

no c_i realizes $p(x)$ and M omits p.

Remark.

It is much harder to omit types in uncountable theories.

Def. ($\varkappa \in CN$, $\varkappa \geq \aleph_0$)

T is $\varkappa$-categorical [categorical in $\varkappa$], if $(\forall M, N \models T)(\|M\| = \|N\| = \varkappa \Rightarrow M \cong N)$

Corollary (Ryll-Nardzewski, Svenonius, Engeler) $\Leftrightarrow$:

(1) T is $\aleph_0$-categorical

(2) $\forall n$ $S_n(\emptyset)$ is finite

Proof (1) $\Rightarrow$ (2) Suppose for some n, $S_n(\emptyset)$ is infinite.

Hence there is a non-isolated $p(\bar{x}) \in S_n(\emptyset)$.

omitting types thm $\Rightarrow$ $\exists M_1 \models T$ omitting $p(\bar{x})$

realizing types thm $\Rightarrow$ $\exists M_2 \models T$ realizing $p(\bar{x})$ $\quad M_1 \not\cong M_2$ $\lightning$

(2) $\Rightarrow$ (1)

Suppose $M = \{a_n : n < \omega\}$, $N = \{b_n : n < \omega\}$ countable models of T.

We will show: $M \cong N$.

We construct a sequence of finite functions

$$\emptyset = f_{-1} \subseteq f_0 \subseteq f_1 \subseteq \dots \subseteq f_n \subseteq \dots \quad n < \omega \text{ s.t.}$$

(a) $\mathrm{Dom}\, f_i \subseteq M$, $\mathrm{Rng}\, f_i \subseteq N$

(b) $a_i \in \mathrm{Dom}\, f_i$, $b_i \in \mathrm{Rng}\, f_i$

(c) f_i is <u>elementary</u>, i.e. $tp^M(d_1, \dots, d_k) = tp^N(f(d_1), \dots, f(d_k))$,

where $\mathrm{Dom}\, f_i = \{d_1, \dots, d_k\}$.

Recursively:

Suppose we have f_i. We will find f_{i+1} $(i \geq -1)$

<u>Step 1</u> (forth) $a_{i+1} \hookrightarrow \mathrm{Dom}\, f_{i+1}$:

If $a_{i+1} \in \mathrm{Dom}\, f_i$, then do nothing

If $a_{i+1} \notin \mathrm{Dom}\, f_i$, then let

$$p = tp^M(\langle d_1, \dots, d_k, a_i \rangle)$$

$p \in S_{k+1}(\emptyset) \leftarrow$ finite $\Rightarrow$ p isolated:

$$\varphi(x_1, \dots, x_k, y) \vdash p(\overset{d_1}{x_1}, \dots, \overset{d_k}{x_k}, \overset{a_i}{y})$$

so $\exists y\, \varphi(x_1, \dots, x_k, y) \in tp^M(d_1, \dots, d_k)$.

$$\parallel$$

$(f_i(d_1) \dots f_i(d_k), y) \Leftarrow tp^N(f_i(d_1), \dots, f_i(d_k))$

$N \models \exists y\, \varphi(\cancel{d_1, \dots, d_k}, y)$

Let b be a witness: ~~$N \models \varphi(d_1, \dots, d_k, b)$.~~

$$N \models \varphi(f_i(d_1), \dots, f_i(d_k), b)$$

φ izolates type in $S_{k+1}(\emptyset)$

so $\underbrace{tp^M(d_1,\dots,d_k,a_i)}_{\varphi} = \underbrace{tp^N(f_i(d_1),\dots,f_i(d_k),b)}_{\varphi}$,

Let $f' = f_i \cup \{\langle a_i, b\rangle\}$. f' elementary.

<u>Step 2</u>. Replace the roles of M, N.
(back) Find $a \in M$ s.t. $f_{i+1} = f' \cup \{\langle a, b_i\rangle\}$
elementary.

Let $f = \bigcup_i f_i$ $\quad f: M \xrightarrow{\equiv} N$ (by (c)), $\quad$ $\mathrm{Dom} f = M$, $\mathrm{Rng} f = N$ $\Rightarrow f: M \xrightarrow{\cong} N$.