3.11.2021

Corollary BA = the theory of algebras of sets

$= \{\varphi \in \mathcal{F}_{L_{BA}} : \varphi$ holds in every algebra of sets$\}$

Proof: ex.

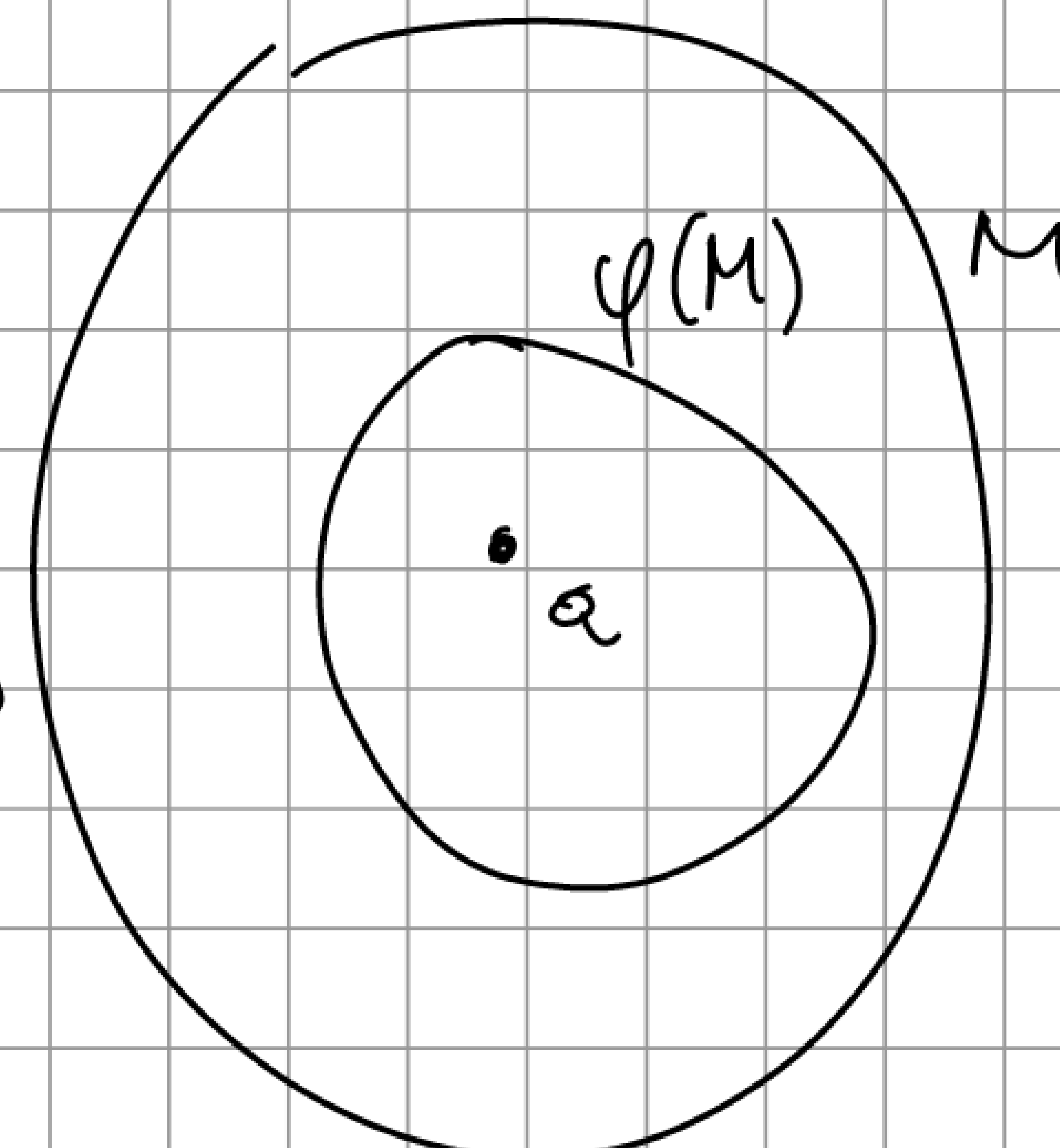
Types, definable sets. L: fixed (cntbl), T: complete consistent theory in L.

$M \models T \quad a \in M$

$\varphi(x) \in \mathcal{F}_L$

$\varphi(M) = \{b \in M : M \models \varphi(b)\}$

definable (sub)set of M



Def. $tp^M(a) = \{\varphi(x) \in \mathcal{F}_L : M \models \varphi(a)\}$

type of a in M

Def. $\mathcal{F}_L(x) = \{$formulas of $\mathcal{F}_L$ of the form $\varphi(x)\}$

Likewise: $\mathcal{F}_L(\bar{x})$

$x_1, \ldots, x_n$

Def. $\varphi(x) \in \mathcal{F}_L(x)$ is T-consistent

$$\Updownarrow$$

$$T \vdash \exists x\, \varphi(x)$$

$$\Updownarrow$$

$$M \models \exists x\, \varphi(x)$$

$$\Updownarrow$$

$$\varphi(M) \neq \emptyset$$

Def. $p(x) \subseteq \mathcal{F}_L(x)$ is T-consistent

$p(x)$ is a type in variable x — a 1-type

$\Leftrightarrow \forall p_0(x) \subseteq p(x) \left(\bigwedge_i p_0(x)_i \text{ is } T\text{-consistent} \right)$
finite $\Leftrightarrow T \vdash \exists x \bigwedge_i p_0(x)_i$

Likewise: $p(\bar{x}) \subseteq \mathcal{F}_L(\bar{x})$: a k-type in variables $\bar{x}$

Def. A type $p(x)$ is <u>complete</u> $\Leftrightarrow$ p is consistent and $\forall \varphi(x) \in \mathcal{F}_L(x) \left(\varphi \in p(x) \vee \neg\varphi \in p(x) \right)$

<u>Example</u> $tp^M(a)$ is a complete type.

<u>Remark</u> $a \in M \prec N \Rightarrow tp^M(a) = tp^N(a)$

A generalization: $A \subseteq |M|$ a set of parameters, $L(A) = L \cup \{\underline{a} : a \in A\}$ new constant symbols

$T(A) := Th(M, \underset{\overset{\shortparallel}{a}}{\underline{a}^M})_{a \in A}$ - a structure in $L(A)$

A type over A (in T) = a type in $T(A)$

<u>Def.</u> A type $p(x)$ is realized in M by a, if
$\forall \varphi(x) \in p(x)\ M \models \varphi(a) \iff p(x) \in tp^M(a)$

<u>Remark</u> Assume $p(x)$ is a consistent type over A.
Then there is an elementary extension $N \succcurlyeq M$ s.t. $p(x)$ is realised N.

<u>Proof</u> Let $T' = Th(M, m)_{m \in M} \cup \{\varphi(c) : \varphi \in p(x)\}$

a new constant symbol

T' is consistent (compactness theorem)

⋮

Corollary Every consistent type extends to a complete type.

Proof $p(x)$ over A in M — $p(x)$ is realised in $N \succ M$ by c^N, c^N

$$\Longrightarrow p(x) \subseteq \underbrace{tp^N(c^N/A)}_{\text{a complete type over } A \text{ in } T} := tp^N(c^N) \text{ in } (N,a)_{a\in N}$$

$$= \{\varphi(x) \in \mathcal{F}_{L(A)}(x) : M \models \varphi(a)\}$$

Def. $Def(M) = \{$definible subsets of $M\}$

Remark

(1) $Def(M)$ is BA over subsets of M

(2) For every $a \in M$ $\{\varphi(M) : \varphi(x) \in tp^M(a)\}$ is an ultrafilter in $Def(M)$

$\sim$ on $\mathcal{F}_L(x)$: $\varphi \sim \psi \Leftrightarrow T \vdash \varphi(x) \leftrightarrow \psi(x)$

$\Leftrightarrow \varphi(M) = \psi(M)$

Proof „$\Rightarrow$" is clear. „$\Leftarrow$" (ad absurdum)

suppose $T \nvdash \varphi(x) \leftrightarrow \psi(x) \Leftrightarrow T \nvdash \forall x(\varphi(x) \leftrightarrow \psi(x))$

$\Leftrightarrow T \vdash \exists x(\varphi(x) \Delta \psi(x)) \Leftrightarrow M \models \exists x(\varphi(x) \Delta \psi(x))$ ↯

T is complete

$\underbrace{\mathcal{L}_1(\emptyset)} = \left(\mathcal{F}_L(x)/_\sim, \wedge, \vee, ', \mathbb{0}, \mathbb{1} \right)$

Lindenbaum algebra

- $[\varphi]_\sim \wedge [\psi]_\sim = [\varphi \wedge \psi]_\sim$
- $[\varphi]_\sim \vee [\psi]_\sim = [\varphi \vee \psi]_\sim$
- $[\varphi]_\sim' = [\neg \varphi]_\sim$
- $\mathbb{0} = [x \neq x]_\sim$
- $\mathbb{1} = [x = x]_\sim$

Remark

(1) $[\varphi(x)]_\sim \xmapsto{F} \varphi(M)$

gives $F : \mathcal{L}_1(\emptyset) \xrightarrow{\cong} Def(M)$

(2) $tp^M(a)/_\sim$: an ultrafilter in $\mathcal{L}_1(\emptyset)$

(3) $p(x)/_\sim$ is an ultrafilter in $\mathcal{L}_1(\emptyset)$ for every complete type $p(x)$ in T.

(4) $\{$complete types in T, in $x\} \underset{\text{onto}}{\overset{1-1}{\longleftrightarrow}} S(\mathcal{L}_1(\emptyset))$

$\{$consistent types in T, in $x\}$ "$\longleftrightarrow$" filters in $\mathcal{L}_1(\emptyset)$

Ad 4 Assume that $p(x)$ is a complete type in T. Then $p(x)$ is closed under $\sim$

$\Longrightarrow p(x)/_{\sim} \in S(L_1(\emptyset))$

Vice versa, if $\mathcal{U} \in S(L_1(\emptyset))$, then

$p_{\mathcal{U}} = \{\varphi(x) \in \mathcal{F}_L(x) : [\varphi]_{\sim} \in \mathcal{U}\}$ is a complete consistent theory in T, in X.

For $A \subseteq M$ $\rightsquigarrow$ $Def_A(M) = \{\varphi(M) : \varphi(x) \in \mathcal{F}_{L(A)}\}$

$$\varphi(\bar{x}, \underset{\underset{M}{\cap}}{\bar{a}}) \in \mathcal{F}_{L(A)}(\bar{x}) \rightsquigarrow \underbrace{\varphi(M)}_{} = \{\bar{b} \in M^k : M \models \varphi(\bar{b}, \bar{a})\}$$

where $\varphi(\bar{x}, \bar{y}) \in \mathcal{F}_L$

[set definible in M over parameters A]

[A-definible in M]

$Def_A(M)$ – the algebra of A-definable subsets of M.

$$\begin{array}{ccc}
L & \rightsquigarrow & L(A) \\
\downarrow & & \downarrow \\
L_1(\emptyset) & & L_1(A) \\
\downarrow & & \downarrow \\
S(L_1(\emptyset)) & & S(L_1(A))
\end{array}$$

the corresponding Lindenbaum algebra of formulas $\varphi(x) \in \mathcal{F}_L(x)$

$S_1(A)$ = the space of all complete 1-types over A in T.

$$S_1(A) \longleftrightarrow \underbrace{S(\mathcal{L}_1(A))}_{\text{Stone space topology}}$$

<u>Recall</u> assume $\mathbb{A}$: BA

$$\mathbb{A} \rightsquigarrow S(\mathbb{A})$$

$\mathbb{A} \ni a \rightsquigarrow [a] = \{\mathcal{U} \in S(\mathbb{A}) : a \in \mathcal{U}\}$

$\{[a] : a \in \mathbb{A}\}$ is a basis of Stone topology in $S(\mathbb{A})$. It is compact, Hausdorff, 0-dimensional, $[a]$ - clopen

Topology on $S_1(A)$:

Basic open sets: $\underset{\substack{\cap \\ \mathcal{F}_{L(A)}(x)}}{[\varphi]} = \{p(x) \in S_1(A) : \varphi(x, \bar{a}) \in p(x)\}$

Likewise: $L_k(A)$, $S_k(A)$ k-types over A in T in variables $\overline{x}$.

<u>Def</u> Let κ be any cardinal number. We say that M is <u>κ-saturated</u> if $(\forall A \subseteq |M|, |A| < \kappa)$ $(\forall p \in S_1(A))$ p is realised in M.

M is saturated if it is $\|M\|$-saturated

<u>Corollary</u> (1) $\forall M \forall \kappa \exists N \succ M$ N is κ-saturated

(2) If $\underbrace{\kappa \text{ is regular and } 2^{<\kappa} = \kappa}$

$\Updownarrow$

$\kappa^{<\kappa} = \kappa$

$2^{<\kappa} = \sup_{\mu < \kappa} 2^{\mu}$

$(I, \leq)$ a linear ordering.

$cf(I) = \min\{|J| : J \subseteq I\}$

cofinality

cofinal = unbounded

$\kappa \in \mathrm{Ord}$ $\kappa = \{\alpha \in \mathrm{Ord} : \alpha < \kappa\}$

$\cap$ CN

$(\kappa, <)$ well-ordered