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Henkin's proof of Gödel's model existance thm.

Why Henkin's axiom? Recall:

S - a consistent set of formulas

$\cap$

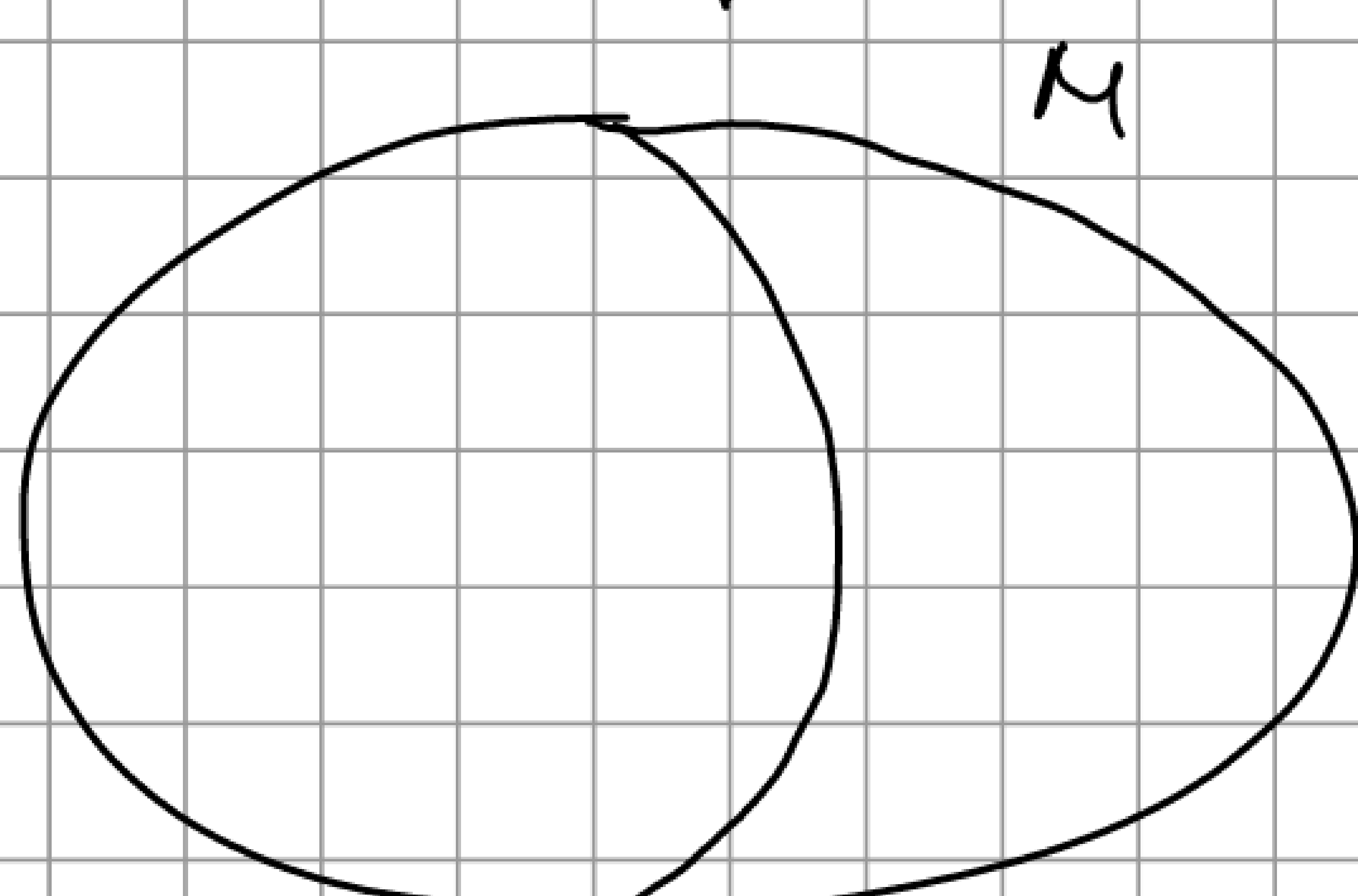
S' in $\mathcal{L}' = \mathcal{L} \cup \{c_n : n \in \mathbb{N}\}$

consistent
complete

Contains H_n: $\exists x\, \varphi_n(x) \longrightarrow \varphi_n(c_{f(n)})$

Enumeration of all formulas in $\mathcal{F}_{\mathcal{L}'}$

Suppose $M \models S'$



$N = \{c_n^M : n < \omega\}$

Henkin's axioms + completness of S' ensures

that N satisfies T-V test

$M \models \exists x\, \underbrace{\varphi(x, \bar{c})}_{\varphi_n(x) \text{ for some } n}$

S' is complete, so

$S' \vdash \exists x\, \varphi_n(x)$

Also $S' \vdash H_n \overset{MP}{\Rightarrow} S' \vdash \varphi_n(c_{f(n)}) \Rightarrow M \models \varphi_n(c_{f(n)})$

the witness from N

So $N \preccurlyeq M$, $N \models S'$

S' contains the "atomic diagram" of N

- $P_i(\bar{c})$ is decided is S'
- $f_j(\bar{c}) = c'$ — " —
- $c_n = c_m$ — " —

: this is enough to determine structure of N

DLO_0 : $\aleph_0$-categorical $\Rightarrow$ complete

Corollary 4.6. DLO_0 is decidable

Proof(a) Let $\varphi \in F_L$: a sentence

Algorithm: generate proofs in DLO_0 : $\bar{\alpha}_0, \bar{\alpha}_1, \ldots$ (in steps $0, 1, \ldots$)

cntbly many

It exists because set of axioms is finite

In step i, $\bar{\alpha}_i$ is generated.

We verify if the conclusion of $\bar{\alpha}_i$ happens to be φ or $\neg\varphi$.

If YES, then we get either $DLO_0 \vdash \varphi$ or $DLO_0 \nvdash \varphi$, STOP.

If NOT, proceed to step $i+1$.

There is some proof of φ or $\neg\varphi$, because DLO_0 is complete.

<u>Remark 4.7</u> If T is a complete theory with **recursively enumerable** set of axioms, then T is decidable.

There is an effective way of generating these axioms in a list

<u>Proof (b)</u> (more practical)

<u>Fact 4.8</u> DLO_0 is **quantifier eliminable** (q.e.)

that is: $\forall \varphi \in \mathcal{F}_L \; \exists \psi \in \mathcal{F}_L \;\; DLO_0 \vdash \varphi \leftrightarrow \psi$

with no quantifiers (quantifiers free)

<u>Proof (Fact 4.8)</u> Let $\varphi(\bar{x}) \in \mathcal{F}_L$.

There are finitely many configurations $C_1(\bar{x}), \dots, C_n(\bar{x})$

A **configuration** $C(\bar{x})$: a conjunction of atomic, negated atomic formulas about $\bar{x}$ completely describing ordering $<$ on $\bar{x}$

$DLO_0 \vdash \dot{\bigvee}_{i=1}^{n} c_i(\bar{x})$

exclusive alternative

$\Updownarrow$

$(\mathbb{Q}, \leq) \models \dot{\bigvee}_{i=1}^{n} c_i(\bar{x})$

<u>Exersice</u> T is complete consistent theory, $M \models T$, $\varphi \in F_L$, then
$T \vdash \varphi \iff M \models \varphi$

Also $\forall i \left[DLO_0 \vdash c_i(\bar{x}) \to \varphi(\bar{x}) \text{ or } DLO_0 \vdash c_i(\bar{x}) \to \neg\varphi(\bar{x}) \right]$

because: if $\bar{a}, \bar{b} \in \mathbb{Q}$, both satisfing $c(\bar{x})$, then

$\exists f \in Aut(\mathbb{Q}, \leq)\ f(\bar{a}) = \bar{b}$ fact ∎

Let $I = \{ i \in \{1, \dots, n\} : DLO_0 \vdash c_i(\bar{x}) \to \varphi(\bar{x}) \}$

<u>excersice</u>: $DLO_0 \vdash \varphi(\bar{x}) \leftrightarrow \underbrace{\bigvee_{i \in I} c_i(\bar{x})}$

ψ: quantifiers free

$\Downarrow$

DLO_0 is decidable

(We trasform φ to ψ and look if it holds in $(\mathbb{Q}, \leq)$) ∎

Examples of theories with q.e.

(1) ACF_p = the theory of algebraically closed fields of char. $p \geq 0$

(2) RCF = the theory of real closed fields
$$Th(\mathbb{R}, +, \cdot, 0, 1, \leq)$$

(3) Many easy theories:

e.g. the theory T of independent unary predicates $P_n(x)$ $n<\omega$

axioms: $A_{I,J}: \exists x \left(\bigwedge_{n \in I} P_n(x) \wedge \bigwedge_{n \in J} \neg P_n(x) \right)$

$I, J \subseteq \omega$ finite, disjoint

A model M of T:
$$M = (2^{\omega}, P_n)_{n<\omega}, \quad P_n(f) \Leftrightarrow f(n) = 0$$

(4) Abelian groups, modules over a fixed ring

They have only reduction of quantifiers to the level of $\underbrace{pp}_{\text{positive primitive}}$-formulas

$$\exists \bar{y}\, \underset{\substack{\uparrow \\ \text{matrix}}}{A} \bar{y} = \bar{x}$$

(5) $(\mathbb{R}, +, \cdot, <, \exp, \text{analytic functions with compact supports})$

$\Downarrow$

"almost" q.e. $\quad$ <u>o-minimal</u>

<u>Corollary 4.9</u> (weak Hilbert nullstellensatz)

If $\underset{\substack{\shortparallel \\ ACF_p}}{K} \subseteq \underset{\shortparallel}{L}$ and $F(\bar{x})$: a system of equations and non-equations with parameters in K, solvable in L, then $F(\bar{x})$ has a solution in K.

<u>Proof</u> In ACF_p: $F(\bar{x}) = F(\bar{x}, \underset{\substack{\cap \\ K}}{\bar{a}}) \rightsquigarrow F(\bar{x}, \bar{y})$

by q.e. $ACF_p \vdash \exists \bar{x}\, F(\bar{x}, \bar{y}) \longleftrightarrow \underset{\text{q.f.}}{\psi(\bar{y})}$

So $L \models \exists \bar{x}\, F(\bar{x}, \bar{a}) \iff L \models \psi(\bar{a})$

$$\begin{matrix} L \models \psi(\bar{a}) \\ \cup \quad \Downarrow \\ K \models \psi(\bar{a}) \end{matrix}$$

because ψ has no quant.

So $ACF_0 \dashv K \models \exists \bar{x}\, F(\bar{x}, \bar{a})$ $\blacksquare$

Def 5.1 An algebra of sets: $\mathcal{C} \subseteq \mathcal{P}(X)$ st.

(1) $\emptyset, X \in \mathcal{C}$

(2) $\mathcal{C}$ closed under Boolean operations

$^c, \cup, \cap, \triangle, \ldots$

$(\mathcal{C}, \cup, \cap, {}^c, \underset{\emptyset}{\underset{\shortparallel}{\mathbb{0}^{\mathcal{C}}}}, \underset{X}{\underset{\shortparallel}{\mathbb{1}^{\mathcal{C}}}})$: a Boolean algebra

Generally $L_{BA} = \{\vee, \wedge, {}', \mathbb{0}, \mathbb{1}\}$ BA: the theory of Boolean algebras

Axioms of BA

(1) $\vee, \wedge$: associative, commutative, distributive,

(2) $\mathbb{0}' = \mathbb{1}$, $\mathbb{1}' = \mathbb{0}$, $x'' = x$, $(x \vee y)' = x' \wedge y'$

$(x \wedge y)' = x' \vee y'$

(3) $x \vee (x \wedge y) = x$, $x \wedge (x \vee y) = x$

Thm (Stone) Every B. Algebras is $\cong$ an algebra of sets

Let $\mathbb{A} = (A, \wedge, \vee, ', \mathbb{0}, \mathbb{1})$: a BA.

- For $x, y \in \mathbb{A}$ $x \leq y \iff x \vee y = y$
  $\iff x \wedge y = x$

  a partial ordering

- $a \in A$, $a \neq \mathbb{0}$ is an atom if $\forall x \in A$ $(x \leq a \to x = \mathbb{0} \vee x = a)$
- $\mathbb{A}$ is atomless if it has no atoms.

<u>Example</u> the algebra of sets $\mathcal{C} \subseteq \mathcal{P}(\mathbb{Q})$ generated by open intervals with rational endpoints is atomless BA.

<u>Def. 5.2</u> $\emptyset \neq \mathcal{U} \subseteq \mathbb{A}$ is a **filter** if

(1) $x \leq y$ and $x \in \mathcal{U} \Rightarrow y \in \mathcal{U}$

(2) $x, y \in \mathcal{U} \Rightarrow x \wedge y \in \mathcal{U}$

<u>Def. 5.3</u> $\mathcal{U}$ is a **proper filter** if $\mathbb{0} \notin \mathcal{U}$

<u>Def. 5.4</u> $\mathcal{U}$ is an **ultrafilter** in $\mathbb{A}$ if it is a maximal proper filter in $\mathbb{A}$

Remark 5.5 Every proper filter extends to an ultrafilter.

Def. 5.6 $S(\mathbb{A}) = \{\text{ultrafilters in } \mathbb{A}\}$: a topological space

Stone space of $\mathbb{A}$

the basis of topology (for $a \in \mathbb{A}$): $[a] = \{\mathcal{U} \in S(\mathbb{A}) : a \in \mathcal{U}\}$

(it is clopen)

Proof of Stone representation thm

$\mathcal{C} = \{[a] : a \in \mathbb{A}\} \subseteq P(S(\mathbb{A}))$

$\mathbb{A} \ni a \xrightarrow{\cong} [a]$

∎