20.10.2021

<u>Proof</u> (upward L-S, Malcev thm)

Consider theory $T = Th(M, \underline{a}^M)_{a \in |M|}$ in Language $\mathcal{L}(M)$.

elemental diagram of M

Let $\varkappa$ be any cardinal number.

Let $\{c_\alpha : \alpha < \varkappa\}$ be a set of new constant symbols. Let $T' = T \cup \{c_\alpha \neq c_\beta : \alpha < \beta < \varkappa\}$ in $\mathcal{L}' = \mathcal{L}(M) \cup \{c_\alpha : \alpha < \varkappa\}$. We'll prove that T' is consistent.

Let $X \subseteq T'$ be finite. We expand $(M, a)_{a \in |M|}$ interpreting finitely many c_α's appearing in X as distinct elements of M

$\Downarrow$

We get a model of X, so X is consistent, so T' is (compactness thm)

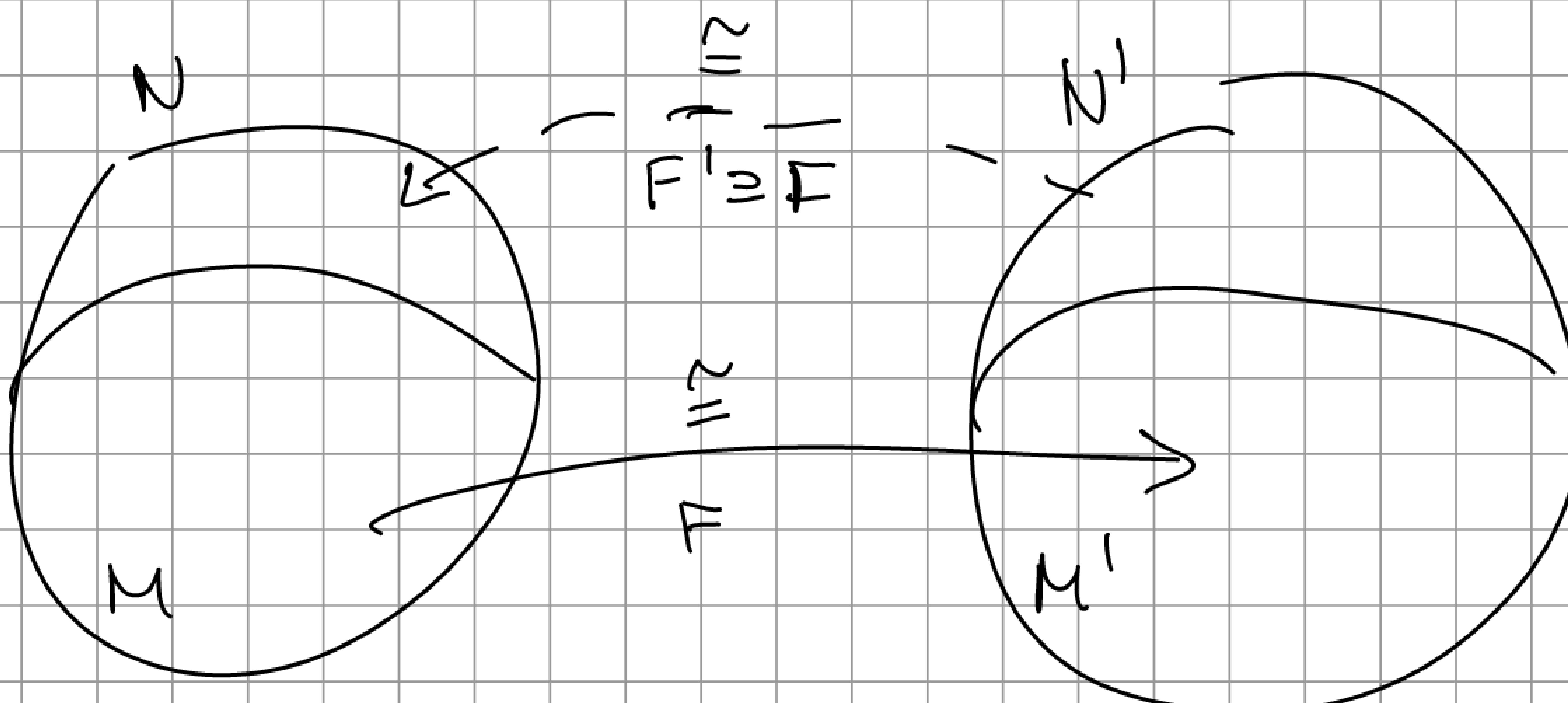
Let $N' \models T' \supseteq T$. Also $N' \models c_\alpha \neq c_\beta$ for all $\alpha < \beta < \kappa$. Hence $\|N'\| \geq \kappa$

$$N' = (N', \underline{a}^N)_{a \in |M|}$$

Exercise: • $\{\underline{a}^N : a \in |M|\} = |M'|$ for some $M' \prec N'$

• $M \cong M'$

$a \mapsto \underline{a}^N$



• We can extend F so that N is isomorphic to N'.

Löwenheim-Skolem paradox

"We cannot express uncountability"

My interpretation of this paradox:

Let $(V, \in) \models ZFC$ be some "big" model that contains $\mathbb{N}$ and is transitive

$\forall A, B \in V$
$A \subseteq B \Leftrightarrow A \in B$

Take subset of ZFC (call it ZFC^{*} s.t. it's finite and $ZFC^{*} \models$ "For every X we have $|\mathcal{P}(X)| \neq |X|$"

(This is Cantor's theorem, it is true in ZFC and we can take some finite number of ZFC axioms that prove this).

Take some $\mathbb{N} \subseteq \underset{\text{countable}}{M} \models ZFC^{*}$ $(M \subseteq V)$ (L-S thm).

We also need M to be transitive (just add some axioms to ZFC^*).

Now, $M \models$ "$\mathcal{P}(\mathbb{N}) \neq \mathbb{N}$" which can be written as

$$M \models \quad \forall X \subseteq \mathbb{N} (X \in A) \longrightarrow |A| \neq |\mathbb{N}|$$

We now that $A \in M (\Longleftrightarrow A \subseteq M)$, M is contable, but proves that A is uncountable! (which is not true in V).

Problem How to determine if $M \equiv N$ or $M \cong N$?

$\Downarrow$

Ehrenfeucht – Fraïssé games

Assume, that $\mathcal{L}$ if a finite relational language. We consider $\mathcal{L}$-structures M, N.

<u>Def. 4.1</u> $\Gamma_n(M,N)$ is a game consisting of n moves (each move is one move for a player).

n-move game

We call the players
↳ I (spoiler) ↳ II (prover)

At each move players pick one element of M and N (one from each). On move i $(1 \leq i \leq n)$ the players have already picked elements $a_1, \dots, a_{i-1} \in M$, $b_1, \dots, b_{i-1} \in N$. Now I picks one of M, N and then one element from it. Player II responds with picking an element from the other model: $a_i \in M$, $b_i \in N$.

After n moves: $\{a_1, \dots, a_n\} \subseteq M$, $\{b_1, \dots, b_n\} \subseteq N$.

$$f : a_i \longmapsto b_i$$

Player II wins if f is an isomorphism of the induced substructures.

<u>Example</u> When $M \cong N$, then II obviously win. Just take $F : M \xrightarrow{\cong} N$ and pick according to it.

<u>Example</u> $(\mathbb{Z}, \leq) \equiv (\mathbb{Z}_1, \leq) \sqcup (\mathbb{Z}_2, \leq)$

$$\not\cong \qquad \mathbb{Z}_1 \cong \mathbb{Z}, \quad \mathbb{Z}_2 \cong \mathbb{Z}$$

$$\underbrace{(\cdots\cdots)}_{\mathbb{Z}_1} < \underbrace{(\cdots\cdots)}_{\mathbb{Z}_2}$$

For instance: $n = 5$

$$\mathrm{I} : 1 \longrightarrow \mathrm{II} : 1_1$$

$$\mathrm{II} : 32 \longleftarrow \mathrm{I} : 1_2$$

$$\vdots$$

Arbitrary number greater than 5 wins!

<u>Proof</u> excersice. ∎

Thm 4.2 (Ehrenfeucht – Fraïssé)

$M \equiv N \iff \forall n$ II has a winning strategy in $\Gamma_n(M,N)$

Def. 4.3 X is a set of axioms of theory T when $Cn(X) = T$.

Examples of theories

Let $L = \{\leq\}$.

1. LO (linear order): axioms "$\leq$ is a linear order".
2. DLO (dense LO): axioms of LO and:
   $$(\forall x,y)(x<y \to \exists z\ x<z<y)$$
3. DLO_0 (DLO without endpoints): axioms of DLO and:
   $$\forall x\, \exists y,z\ \ y<x<z$$

DLO_0 is consistent, because it has a model: $\mathbb{Q}$. Moreover, it is complete and <u>decidable</u>.

Def. 4.3 Theory T is $\varkappa$-categorical if for every $M, N \models T$ s.t. $\|M\| = \|N\| = \varkappa$ we have $M \cong N$.

Remark 4.4 DLO_0 is $\aleph_0$-categorical

Proof (Back-and-forth argument)

Assume $M = \{a_n, n < \omega\} \models DLO_0$

$N = \{b_n, n < \omega\} \models DLO_0$

We construct functions f_n and sets $A_n \subseteq M$, $B_n \subseteq M$ such that

(a) $f_n \subseteq f_{n+1}$ (which implies $A_n \subseteq A_{n+1}$, $B_n \subseteq B_{n+1}$)

(b) $f_n : A_n \xrightarrow{\cong} B_n$ of induced structures

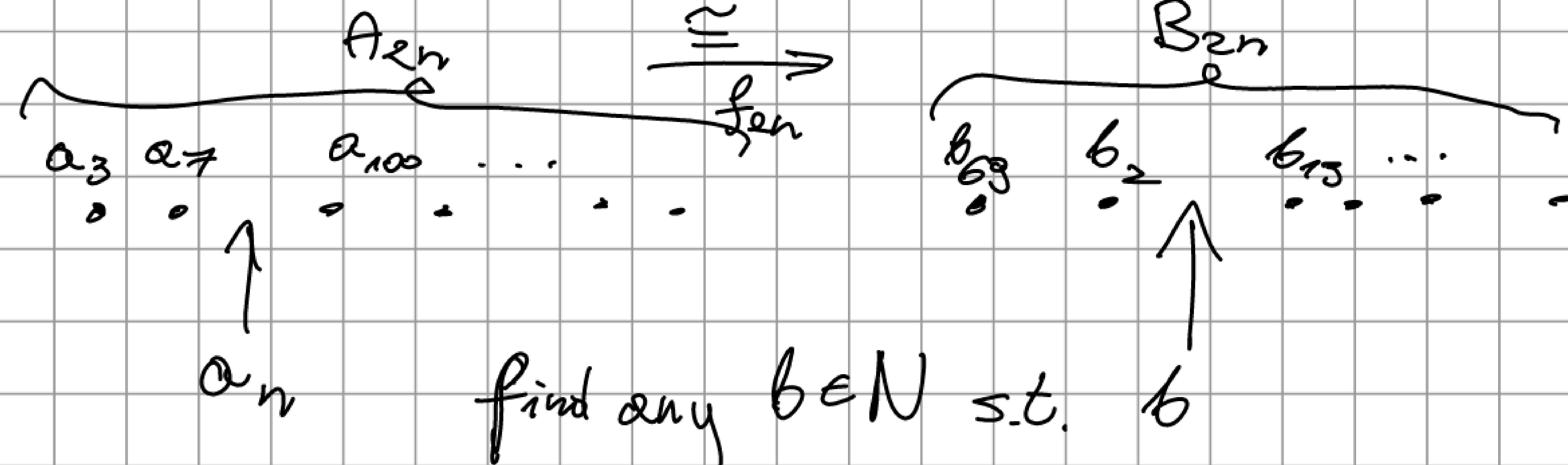
(c) $a_n \in A_{2n+1}$, $b_n \in B_{2n+2}$

Construction We will construct those objects recursively:

- $f_0 = A_0 = B_0 = \emptyset$
- recursive step: $2n \longrightarrow 2n+1 \longrightarrow 2n+2$

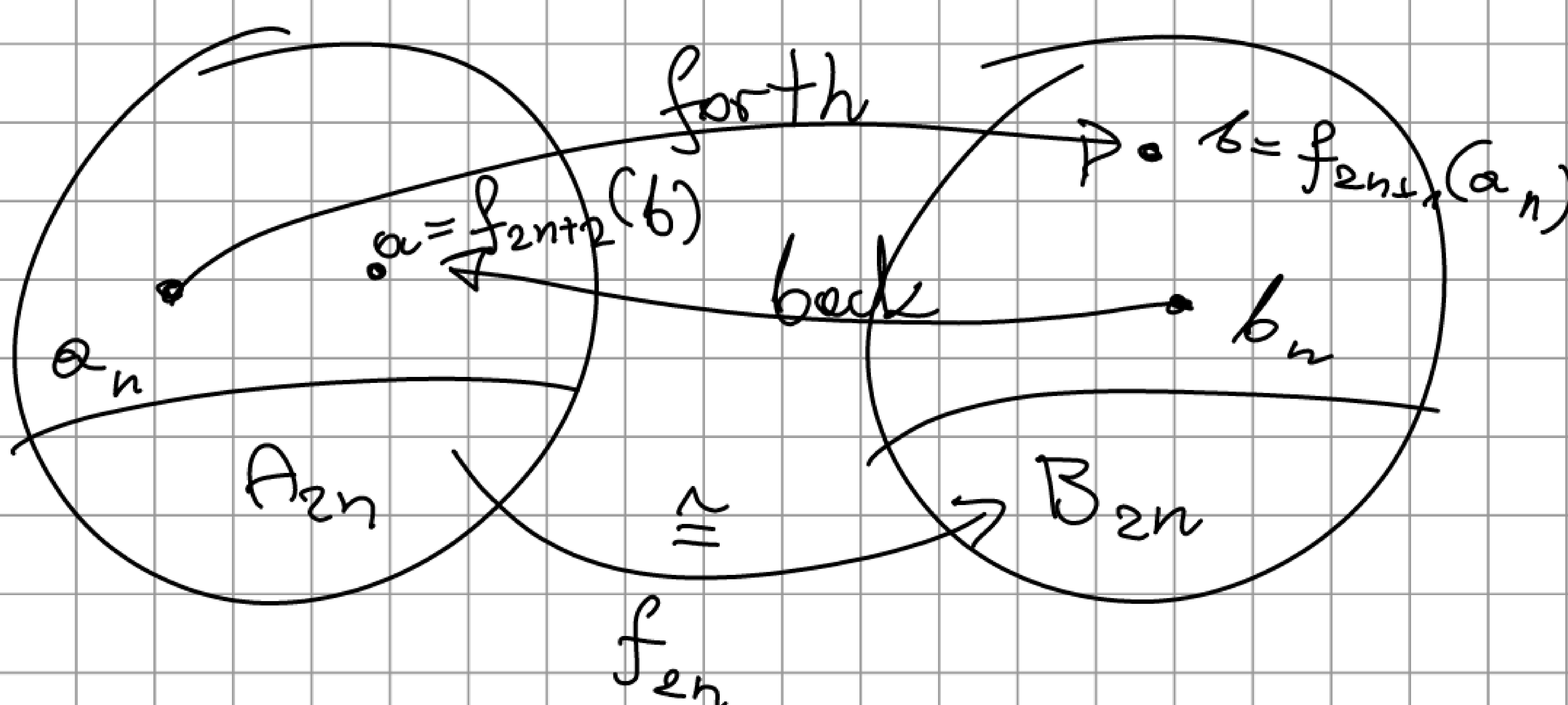
We have $f_{2n} : A_{2n} \longrightarrow B_{2n}$

Look at a_n. If $a_n \in A_{2n}$, then $f_{2n+1} = f_{2n}$.
Else then $A_{2n+1} = A_{2n} \cup \{a_n\}$. Now:



Now $B_{2n+1} = B_{2n} \cup \{b\}$, $f_{2n+1} = f_{2n} \cup \langle a_n, b \rangle$

Similarly we chose a for b_n and construct $f_{2n+2}, A_{2n+2}, B_{2n+2}$.



Now $f = \bigcup_n f_n$, by (c) $\mathrm{Dom} f = M$, $\mathrm{Rng} f = N$ and by (b) f is an isomorphism.

Corollary 4.5 DLO_0 is complete

Proof Let $\varphi \in \mathcal{F}_L$ be a sentence. Suppose (ad absurdum) that $DLO_0 \nvdash \varphi$ and $DLO_0 \nvdash \neg\varphi$.

Hence, by "$\vdash \Leftrightarrow \models$" there is $M, N \models DLO_0$ s.t. $M \models \varphi$, $N \models \neg\varphi$

$$\begin{array}{ccc} M & & N \\ \curlyvee & & \curlyvee \\ M_0 & \cong & N_0 \end{array}$$

, but $M_0 \models \varphi$
$N_0 \models \neg\varphi$ $\lightning$

(downward L-S) countable