Notes 5.

Thm (Church) PA is undecidable.

Proof (a.a.) Suppose PA: decidable, i.e.

the set $\{\ulcorner \varphi \urcorner : PA \vdash \varphi\}$ recursive.

Let $\{\varphi_0(x), \varphi_1(x), \dots\}$: a recursive enumeration of all formulas of L_{PA} with free variable x.

Let $A = \{n \in \mathbb{N} : PA \vdash \neg\varphi_n(\underline{n})\}$.

PA: decidable $\Rightarrow$ A recursive.

By representability lemma there is a formula $\varphi_A(x)$ representing A.

$\varphi_A(x) = \varphi_n(x)$ for some n.

So: $PA \vdash \varphi_A(\underline{n}) \underset{\text{Lemma}}{\iff} n \in A \underset{\text{def. } A}{\iff} PA \vdash \neg\varphi_n(\underline{n})$ ↯

because PA: consistent (in ZFC)

Corollary (1) (Rosser) If $T \subseteq \mathcal{F}_{L_{PA}}$ consistent theory, $PA \subseteq T$, then T is not decidable.

(2) (Gödel 1st incompleteness thm).

If $T \subseteq \mathcal{F}_{L_{PA}}$, T recursively enumerable, $PA \subseteq T$, then theory consistent T is incomplete.

Proof (1) the same ~~dec~~ proof as Church thm.

(2) follows from (1).

Corollary (Turing, 1936)
There is no algorithm deciding if $\models \varphi$, for $\varphi \in \mathcal{F}_L$, $L \supseteq \{+, \cdot, S\}$

<u>Proof</u> Representability lemma holds also for a finite

$$PA_0 \subseteq PA \text{ in place of } PA.$$

(PA_0 needed to prove the Chinese remainder theorem + ...)

Therefore : PA_0 undecidable.

Suppose (a.a). that the set $\{\varphi \in \mathcal{F}_L : \models \varphi\}$ is recursive.

Then for $\varphi \in \mathcal{F}_{L_{PA}}$ $\quad PA_0 \vdash \varphi \Leftrightarrow \vdash \bigwedge PA_0 \to \varphi$

deduction thm

$$\Updownarrow$$

$$\underbrace{\models \bigwedge PA_0 \to \varphi}_{\text{decidable}},$$

so PA_0 : decidable, a contradiction.

Corollary. If ZFC is consistent, then ZFC is undecidable and incomplete.

<u>Proof</u> PA is interpretable in ZFC.

~~<u>Corollary</u> Assume $PA \subseteq T \subseteq TA$. Then there~~ / later.
~~rec. enumerable theory~~

<u>Diagonal lemma.</u>

For every formula $G(x) \in \mathcal{F}_{L_{PA}}(x)$ there is a sentence $F \in \mathcal{F}_{L_{PA}}$ such that $PA \vdash F \leftrightarrow G(\underline{\ulcorner F \urcorner})$.

Proof

1. There is a recursive function $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

s.t.

$(\forall \varphi(x) \in \mathcal{F}_{L_{PA}}(x))(\forall n \in \omega)\ f(\ulcorner \varphi(x) \urcorner, n) = \ulcorner \varphi(\underline{n}) \urcorner$

(via TM-computability).

Idea:

Let $H(x) =$ "$G(f(x,x))$" (diagonal argument.)

Let $F = H(\underline{\ulcorner H(x) \urcorner})$.

Notice $f(\ulcorner H(x) \urcorner, \ulcorner H(x) \urcorner) = \ulcorner F \urcorner$

so $G(\underline{\ulcorner F \urcorner})$ "$\Longleftrightarrow$" $G(\ulcorner F \urcorner) = G(f(\ulcorner H(x) \urcorner, \ulcorner H(x) \urcorner)) =$

"provably" in PA

$= H(\ulcorner H(x) \urcorner) \Longleftrightarrow H(\underline{\ulcorner H(x) \urcorner}) = F$

Formally.

By Representability Lemma: $\varphi_f(x,y) \in \mathcal{F}_{L_{PA}}$ s.t. represents $f(x,x)$

$PA \vdash \forall x \exists^{\leq 1} y\ \varphi_f(x,y)$ and

for all $n, m \in \mathbb{N}$

$f(n,n) = m \Rightarrow PA \vdash \varphi_f(\underline{n}, \underline{m})$

$H(x) = \exists y\ (\varphi_f(x,y) \wedge G(y))$

Let $F = H(\underline{\ulcorner H(x) \urcorner}) = \exists y\ (\varphi_f(\underline{\ulcorner H(x) \urcorner}, y) \wedge G(y))$

$$f(\ulcorner H(x)\urcorner, \ulcorner H(x)\urcorner) = \ulcorner H(\underline{\ulcorner H(x)\urcorner})\urcorner$$

$$\Downarrow$$

$$PA \vdash \varphi_f(\underline{\ulcorner H(x)\urcorner}, \underline{\ulcorner H(\underline{\ulcorner H(x)\urcorner})\urcorner})$$

$$\underline{\ulcorner F\urcorner} \;\; \Leftarrow \;\; F = H(\underline{\ulcorner H(x)\urcorner})$$

unique y ~~s.t.~~ $\leftarrow PA \vdash \exists^{\leq y} \varphi_f(\underline{\ulcorner H(x)\urcorner}, y)$.

$PA \vdash F \Leftrightarrow G(\underline{\ulcorner F\urcorner})$:

In a model of PA (semantically):

$(\rightarrow)$ Assume F, i.e.: $\underbrace{H(\underline{\ulcorner H(x)\urcorner})}_{=\;\exists y(\varphi_f(\underline{\ulcorner H(x)\urcorner}, y) \wedge G(y))} = \cancel{\exists y\, \varphi}$

$$\exists y(\varphi_f(\underline{\ulcorner H(x)\urcorner}, y) \wedge G(y))$$

$\Downarrow$ $PA \vdash \varphi_f(\underline{\ulcorner H(x)\urcorner}, \underline{\ulcorner F\urcorner})$

$PA \vdash \exists^{\leq 1}_y \varphi_f(\underline{\ulcorner H(x)\urcorner}, y)$

$\underline{\ulcorner F\urcorner}$ the only witness, so $G(\underline{\ulcorner F\urcorner})$.

$\leftarrow$: Assume $G(\underline{\ulcorner F\urcorner})$ holds.

But also ~~PA~~ $\varphi_f(\underline{\ulcorner H(x)\urcorner}, \underline{\ulcorner F\urcorner})$ holds.

so: $\exists y(\varphi_f(\underline{\ulcorner H(x)\urcorner}, y) \wedge G(y))$ holds.

so $PA \vdash F \Leftrightarrow G(\underline{\ulcorner F\urcorner})$.

<u>Def</u> A formula $G(x) \in \mathcal{F}_{L_{PA}}(x)$ is a truth definition in ~~$(\mathbb{N}, +, \cdot, S, 0)$~~ $(\mathbb{N}, \cdot, +, 0, S)$ if $(\forall \underset{\text{sentence}}{\varphi} \in \mathcal{F}_{L_{PA}})\ \mathbb{N} \models G(\ulcorner \underline{\varphi} \urcorner) \Leftrightarrow \mathbb{N} \models \varphi$.

Corollary (Tarski, undefinability of ~~[illegible]~~ truth).
There is no truth definition in $(\mathbb{N}, +, \cdot, 0, S)$

<u>Proof</u> Suppose (a.a.) $G(x)$ is a truth definition in $\mathbb{N}$. Apply the diagonal lemma for $\neg G(x)$.

Get $\underset{\text{sentence}}{F} \in \mathcal{F}_{L_{PA}}$ s.t. $PA \vdash F \Leftrightarrow \neg G(\ulcorner \underline{F} \urcorner)$

hence $\mathbb{N} \models F \longleftrightarrow \neg G(\ulcorner \underline{F} \urcorner)$

F says: "I am false" (Liar's paradox, self reference)

Hence $\mathbb{N} \models F \Longleftrightarrow \mathbb{N} \not\models G(\ulcorner \underline{F} \urcorner) \underset{\substack{\uparrow \\ G\text{: a truth definition.}}}{\Longleftrightarrow} \mathbb{N} \not\models F$ $\square$

Similarly in ZFC:

If ZFC is consistent, then there is no formula $G(x) \in \mathcal{F}_{L_{ZF}}$ s.t. for every sentence $F \in \mathcal{F}_{L_{ZF}}$

$$ZFC \vdash F \Longleftrightarrow G(\ulcorner \underline{F} \urcorner).$$

Assume $PA \subseteq T \subseteq \mathcal{F}_{L_{PA}}$

consistent, recursively enumerable set of sentences ~~theory~~

- there is a formula $Prov_T(x)$ s.t. for every sentence $\varphi \in \mathcal{F}_{L_{PA}}$

$$T \vdash \varphi \iff \mathbb{N} \models Prov_T(\underline{\ulcorner \varphi \urcorner})$$

~~More~~ Stronger: $T \vdash \varphi \iff PA \vdash Prov_T(\underline{\ulcorner \varphi \urcorner})$.

<u>Idea</u>. unary relation on $\mathbb{N}$

$Prov_T(\ulcorner \varphi \urcorner) \iff \exists y \, \exists k$ (y is a proof of φ in T, of length k, using only the first k elements of T, the first k axioms of KRL...)

$$\iff \exists y \exists k \; (\underbrace{\langle (y)_0, \ldots, (y)_k \rangle}_{=y}, \ulcorner \varphi \urcorner \;)$$

recursive relation

$R(y, k, \ulcorner \varphi \urcorner)$

$\Downarrow$

represented in PA by a formula

$\varphi_R(y, k, z)$

$$Prov_T(z) = \cancel{\exists z \, \varphi} \, \exists y \, \exists k \; \varphi_R(y, k, z).$$

Now let F be the sentence from the diagonal lemma for $G(x) = \neg \mathrm{Prov}_T(x)$.

so $PA \vdash F \longleftrightarrow \neg \mathrm{Prov}_T(\underline{\ulcorner F \urcorner})$

F says (according to PA) : ~~" Accordin.~~

" I can not be proved in T ".

$\mathrm{Con}(T) = \neg \mathrm{Prov}_T(\underline{\ulcorner 0 = 1 \urcorner})$: " T is consistent".

Fact (1) $T \nvdash F$

(2) If $\mathbb{N} \models T$ (i.e. $T \subseteq TA$), then $T \nvdash \neg F$

(3) $PA \vdash \mathrm{Con}(T) \Leftrightarrow F$

Proof (1). Suppose $T \vdash F$. Then $PA \vdash \mathrm{Prov}_T(\underline{\ulcorner F \urcorner})$

$$\Downarrow$$

$$T \vdash \neg F \xLeftarrow[PA \subseteq T]{} PA \vdash \neg F$$

but T consistent ↯.

(2) Suppose $T \vdash \neg F$. $\Rightarrow$ ~~$\mathbb{N} \models \neg F$~~

$$\Downarrow T \supseteq PA$$

$$T \vdash \mathrm{Prov}_T(\underline{\ulcorner F \urcorner}) \Rightarrow \mathbb{N} \models \mathrm{Prov}_T(\underline{\ulcorner F \urcorner})$$

$$\Downarrow$$

$$T \vdash F \text{ ↯.}$$

(3) $PA \vdash F \leftrightarrow \neg Prov_T(\ulcorner \underline{F} \urcorner)$

$$\Downarrow$$

$PA \vdash Prov_T(\underline{\ulcorner F \leftrightarrow \neg Prov_T(\ulcorner \underline{F} \urcorner) \urcorner})$

$$\Downarrow \text{ properties of } Prov_T$$

$PA \vdash Prov_T(\ulcorner \underline{F} \urcorner) \leftrightarrow Prov_T(\underline{\ulcorner \neg Prov_T(\ulcorner \underline{F} \urcorner) \urcorner})$

<u>But</u>: $PA \vdash Prov_T(\ulcorner \underline{F} \urcorner) \rightarrow Prov_T(\underline{\ulcorner Prov_T(\ulcorner \underline{F} \urcorner) \urcorner})$
(*)

<u>Hence</u>: $PA \vdash Prov_T(\ulcorner \underline{F} \urcorner) \rightarrow \neg Con(T)$

and $PA \vdash Con(T) \rightarrow \neg Prov_T(\ulcorner \underline{F} \urcorner)$

$PA \vdash Con(T) \rightarrow F$.

$\Leftarrow$:

$PA \vdash \neg Con(T) \rightarrow Prov_T(\ulcorner \underline{F} \urcorner)$

$PA \vdash \neg Con(T) \rightarrow \neg F$

$PA \vdash F \rightarrow Con(T)$.

<u>Corollary</u> (Gödel's 2nd incompleteness thm)

(1) If T is consistent, then $T \cup \{\neg Con(T)\}$ is consistent.

(2). If $\mathbb{N} \models T$, then $T \cup \{Con(T)\}$ is consistent.
(i.e. $T \subseteq TA$)

Proof (1)

$T \vdash F \leftrightarrow Con(T)$ and $T \nvdash F$, so $T \nvdash Con(T)$
$\Downarrow$
$T \cup \{\neg Con(T)\}$
consistent

(2) Since T is consistent, $\mathbb{N} \models Con(T)$

so if $\mathbb{N} \models T$ then $\mathbb{N} \models \underbrace{T \cup \{Con(T)\}}_{\text{consistent}}$,

consistent.

Corollary Assume $\mathbb{N} \models T$.

Let $A(x)$ = "x is a proof of ~~$0 \neq 1$~~ $0=1$ in T".

so $\underline{PA \vdash \neg A(\underline{n})}$ for every $n \in \mathbb{N}$

but $PA, \underline{T \nvdash \forall x \neg A(x)}$; because

$T \cup \{\neg Con(T)\}$ consistent,

If $M \models T \cup \{\neg Con(T)\}$ then $M \models \exists x \neg A(x)$

non-standard "proof"
of $0 = 1$ in T.

Corollary

Similarly if ZFC is consistent,

then $ZFC \cup \{\neg Con(ZFC)\}$ is consistent.

On (*):

Σ_1-formulas in L_{PA}:

of the form $\exists \bar{x}\, \psi$, where in ψ only bounded quantifiers:

$\exists x \leq y$, $\forall x \leq y$.

<u>Lemma</u> Assume $D(x_1, \ldots, x_n) \in F_{L_{PA}}$ is Σ_1-formula.

Then $PA \vdash D(x_1, \ldots, x_n) \longrightarrow Prov_{PA}(\underline{\ulcorner D(\underline{x_1}, \ldots, \underline{x_n})\urcorner})$.

<u>Explanation:</u>

For $x_1, \ldots, x_n \in \mathbb{N}$

$$(x_1, \ldots, x_n) \xrightarrow[\text{recursive}]{f_D} \ulcorner D(\underline{x_1}, \ldots, \underline{x_n})\urcorner \in \mathbb{N}$$

$\Downarrow$

represented by $\cancel{\Phi}\ \varphi_{f_D}(\bar{x}, y)$

$Prov_{PA}(\ulcorner D(\underline{x_1}, \ldots, \underline{x_n})\urcorner = \exists y\, (\varphi_{f_D}(\bar{x}, y) \wedge Prov_{PA}(y))$

<u>Corollary</u>. Let $n \in \mathbb{N}$

$PA \vdash D(\underline{n}) \longrightarrow Prov_{PA}(\underline{\ulcorner D(\underline{n})\urcorner})$.

Apply this to $D(x) = Prov_T(x)$: a Σ_1-formula.