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$\mathcal{U}$: ultrafilter on I

$\{M_i\}_{i\in I}$: a family of L-structures

<u>Def</u> <u>Ultraproduct</u> $M = \prod_I M_i / \mathcal{U}$

(a) $\sim$: an equivelance relation on $\prod_{i\in I} |M_i|$

$$f \sim g \iff \{i \in I : f(i) = g(i)\} \in \mathcal{U}$$

(b) Let $|M| = \prod_I |M_i| / \sim$. We define an L-structure on M:

1) P - predicate symbol, then

$$P^M([f_1]_\sim, \dots, [f_n]_\sim)$$
$$\Updownarrow$$
$$\{i \in I : M_i \models P^{M_i}(f_1(i), \dots, f_n(i)\}$$

2) F: func. symbol of L:

$$F^M([f_1]_\sim, \dots, [f_n]_\sim) = [g]_\sim$$

where $g(i) = F^{M_i}(f_1(i), \dots, f_n(i))$

3) C: constant symbol of L:

$$c^M = [\langle c^{M_i} \rangle_{i\in I}]_\sim$$

This definition is correct.

Let's see for 1):

Suppose $f_1 \sim f_1', \dots, f_n \sim f_n' \Rightarrow$

$A := \{ i \in I : f_1(i) = f_1'(i) \wedge \dots \wedge f_n(i) = f_n'(i) \} \in \mathcal{U}$

Let $B = \{ i \in I : P^{M_i}(f_1(i), \dots, f_n(i)) \}$

$C = \{ i \in I : P^{M_i}(f_1'(i), \dots, f_n'(i)) \}$

$B \in \mathcal{U} \Leftrightarrow B \cap A \in \mathcal{U}$

$C \in \mathcal{U} \Leftrightarrow C \cap A \in \mathcal{U}$, but $B \cap A = C \cap A$!

Let's see for 2):

$$A \subseteq \{ i \in I : F^{M_i}(f_1(i), \dots, f_n(i)) = F^{M_i}(f_1'(i), \dots, f_n'(i)) \}$$

$$\underset{\mathcal{U}}{\cap} \Longrightarrow \underset{\mathcal{U}}{\cap} \Rightarrow \langle F^{M_i}(f_1(i), \dots, f_n(i)) \rangle_{i \in I} \overset{\sim}{} \langle F^{M_i}(f_1'(i), \dots, f_n'(i)) \rangle_{i \in I}$$

Remark

If $\underset{\substack{\shortparallel \\ \mathcal{U}_a,\ a \in I}}{\mathcal{U}}$ is principal then $M \cong M_a$

Proof
$$\begin{aligned} M &\xrightarrow{\cong} M_a \\ [f]_\sim &\longmapsto f(a) \end{aligned}$$ $\blacksquare$

Theorem (Łoś)

For $\varphi(x_1, \dots, x_n) \in \mathcal{F}_L$ and $[f_1]_\sim, \dots, [f_n]_\sim \in |M|$,
where $M = \prod_{i \in I} M_i / \mathcal{U}$:

$$M \models \varphi([f_1]_\sim, \dots, [f_n]_\sim) \iff \underbrace{\{ i \in I : M_i \models \varphi(f_1(i), \dots, f_n(i)) \}}_{\substack{\text{\rotatebox{90}{$\in$}} \\ \mathcal{U}}}$$

An auxiliary fact

If $t(x_1, \dots, x_n) \in \mathcal{T}_L$ then

$$t^M([f_1]_\sim, \dots, [f_n]_\sim) = [\langle t^{M_i}(f_1(i), \dots, f_n(i)) \rangle_{i \in I}]_\sim$$

Proof: exc.

Proof (Łoś)

Induction on $|\varphi|$

(a) φ atomic, $\overline{x} = (x_1, \dots, x_n)$

(i) $\varphi : t(\overline{x}) = t'(\overline{x})$

$$M \models \varphi([\overline{f}]_\sim) \iff t^M([\overline{f}]_\sim) = t'^M([\overline{f}]_\sim)$$

$$\iff \langle t^{M_i}(\overline{f_i(i)}) : i \in I \rangle \sim \langle t'^{M_i}(\overline{f_i(i)}) : i \in I \rangle$$

aux. fact

$\Leftrightarrow \{i \in I : M_i \models t(\bar{f}(i)) = t'(\bar{f}(i))\} \in \mathcal{U}$

(ii) φ: $P(\underbrace{t_1(\bar{x}), \dots, t_n(\bar{x})}_{\bar{t}(\bar{x})})$

$M \models \varphi([\bar{f}]_\sim) \Leftrightarrow M \models P^M\left([\bar{t}^M([\bar{f}]_\sim)]_\sim\right)$

$\Leftrightarrow \{i \in I : P^{M_i}(\bar{t}^{M_i}(\bar{f}(i)))\} \in \mathcal{U}$

$\Leftrightarrow \{i \in I : M_i \models \underbrace{P(\bar{t}(\bar{f}(i)))}_{\varphi(\bar{f}(i))}\}$

(b) $\neg$, $\wedge$

(i) φ: $\neg\psi$. $M \models \varphi([\bar{f}]_\sim) \Leftrightarrow M \not\models \psi(\bar{f})$

$\overset{\text{ind.}}{\Leftrightarrow} \{i \in I : M_i \models \psi(\bar{f}(i))\} \notin \mathcal{U}$

$\overset{\mathcal{U}:\text{uf}}{\Leftrightarrow} \{i \in I : M_i \not\models \psi(\bar{f}(i))\} \in \mathcal{U}$

$\Leftrightarrow \{i \in I : M_i \models \varphi(\bar{f}(i))\} \in \mathcal{U}$

(ii) φ: $\varphi_1 \wedge \varphi_2$ easy

(iii) φ: $\exists y\, \psi(\bar{x}, y)$

"$\Rightarrow$" $M \models \varphi([\bar{f}]_\sim) \Rightarrow \exists g \in |M|\ M \models \psi([\bar{f}]_\sim, [g]_\sim)$

$\overset{\text{ind.}}{\Leftrightarrow} \{i \in I : M_i \models \psi(\bar{f}(i), g(i))\} \in \mathcal{U}$

$\cap$ |

$$A := \{ i \in I : M_i \models \underbrace{\exists y\, \psi(f(i), y)}_{\varphi(\bar{f}(i))} \} \in \mathcal{U}$$

„$\Leftarrow$" Assume $A \in \mathcal{U}$

We define a witness $[g]_\sim \in M$ for y

in $\varphi([f]_\sim) = \exists y\, \psi([f]_\sim, y)$

- when $i \in A$ $g(i) \in |M_i|$ s.t.

$$M_i \models \psi(f(i), g(i))$$

- when $i \notin A$ then $g(i) \in |M_i|$ arbitrary

So:

$$\underset{\mathcal{U}}{\underset{\cap}{A}} \subseteq \underset{\mathcal{U}}{\underset{\cap}{\{ i}} : M_i \models \psi(f(i), \bar{g}(i)) \}$$

$$\mathcal{U} \Rightarrow \mathcal{U}$$

by the ind. hypothesis for ψ.

$$M \models \psi([f]_\sim, [g]_\sim) \Rightarrow M \models \varphi([f]_\sim).$$

$\blacksquare$

A special case

$M_i = N$ for all $i \in I$

$$M = \prod_I M_i / \mathcal{U} = \underbrace{N^I / \mathcal{U}}_{\text{Ultrapower}}$$

Let $f : N \to N^I / \mathcal{U}$

$$f(a) = [f_a]_\sim \,, \; f_a : I \to |N|$$

$$f_a \equiv a$$

(a diagonal embedding)

Remark

(a) f is 1-1

(b) $f[N] < N^I/\mathcal{U}$

(c) $f : N \xrightarrow{\equiv} N^I/\mathcal{U}$

Proof (c) Let $\varphi(\bar{x}) \in \mathcal{F}_L$, $\bar{a} \subseteq N$.

$$N \models \varphi(\bar{a}) \iff \{ i \in I : N \models \varphi(\underbrace{\bar{f_{\bar{a}}}(i)}_{\overset{\shortparallel}{\bar{a}}}) \} \in \mathcal{U}$$

$$\overset{\text{Łoś thm}}{\iff} N^I/\mathcal{U} \models \varphi(f(\bar{a}))$$

Compactness thm (new proof)

Assume T is a set of sentencess s.t. every finite $T_0 \subseteq T$ has a model. Then T has a model.

Proof Let $I = [T]^{<\aleph_0} = \{T_0 \subseteq T : T_0 \text{ is finite}\}$

For $i \in I$ let $M_i \models i$

Let $\varphi \in T$. $\underset{\cup\!|}{\{i \in I : M_i \models \varphi\}}$

$$X_\varphi := \{i \in I : \varphi \in i\}$$

$\{X_\varphi : \varphi \in T\} \subseteq \mathcal{P}(I)$

has f. i. p. $\longrightarrow \{\varphi_1, \dots, \varphi_n\} \in X_{\varphi_1} \cap \dots \cap X_{\varphi_n} \neq \emptyset$

(finite intersection property)

so $\exists\, \mathcal{U}$: an uf on I st. $\forall \varphi \in T \; X_\varphi \in \mathcal{U}$

Let $M = \prod_I M_i / \mathcal{U}$

- $M \models T$

Let $\varphi \in T$. $M \models \varphi \iff \underset{\cup\!|}{\{i \in I : M_i \models \varphi\}} \in \mathcal{U}$

$X_\varphi \in \mathcal{U}$ ✓

Example Let $\mathcal{U}$: a non-principal ultrafilter on $\mathbb{N} = \omega$ (a Frechet ultrafilter)

If $\{M_i : i < \omega\}$ s.t. $\|M_i\| = k < \omega$ then $\|\prod_\omega M_i/\mathcal{U}\| = k$

Proof Look at φ_k: "there're exactly k-many elements"

Special case Let $\mathcal{R} = (R, +, \cdot, r, f, P)_{r, f, P: \text{all}}$

Let $\mathcal{U}$: a Frechet ultrafilter on ω.

$$\begin{array}{c} R^* = N^\omega/\mathcal{U} \\ \curlyvee \\ R \end{array}$$

Properties of R^*:

(a) $\|R^*\| = 2^{\aleph_0}$

(b) Every ctble consistent type $p(x)$ over R^* is realised in R^*

Set theory as "meta theory" for mathematics.

- What exists?
  - for sure: $\emptyset$ (in style of Descartes)

Set theory:

(1) describes procedures of creating sets

(2) describes properties of sets.

For simplicity: "every individual is a set"

- Language of set theory: $L = \{\in\}$

relational binary symbol

The first try: "META-META SET THEORY"

Informal: If $\varphi(x) \in$ "F_L" then there is $y = \{x : \varphi(x)\}$

Formally: $\exists y \forall x \, (x \in y \leftrightarrow \varphi(x))$

Russel antynomy:

Let $\varphi(x)$: $x \notin x$

$\exists y \forall x (x \in y \leftrightarrow x \notin x)$ is contradictory, it has no model.

If $(M, E) \models \exists y \forall x (x \in y \leftrightarrow x \notin x)$, $E = \in^M$

$a \in M$ $\quad$ $a \in M$ $\quad$ $a \in a \leftrightarrow a \notin a$

Therefore: need care when constructing sets.

- General rule: allow as many sets as possible (by Gödel, freedom principal)

Axioms of ZFC (Zermelo-Frenkel set theory with choice axiom)

A1 (Existence) "$\exists \phi$"

$$\exists x \forall y \; y \notin x$$

A2 (Extensionality) If x, y have the same elements, then $x = y$

A3 (Pair axiom)

$$(\forall x, y) \exists z \forall t (t \in z \leftrightarrow t = x \vee t = y)$$

A4 (Comprehension)

If $\varphi(x, \bar{y})$ is a formula, X : a set then

$\forall \bar{y} \, \exists Y = \{x \in X : \varphi(x, \bar{y})\}$

A5 (Union) $\forall X \; \bigcup X$ exists

A6 (Power set) $\forall X \; \mathcal{P}(X)$ exists

A7 (Replacement)