Skolemization of:

$\begin{cases} \cdot \text{ language } L \\ \cdot \text{ theory } T \text{ (in } L) \\ \cdot \text{ structure } M \end{cases}$

① <u>Single step</u>: $L \rightsquigarrow L'$, $T \rightsquigarrow T'$, $M \rightsquigarrow M'$:

[here: $M \models T$, T: a theory in L].

For each formula $\varphi(\bar{x}, y) \in \mathcal{F}_L$ let $t_\varphi(\bar{x})$: a new function symbol.

$L' = L \cup \{t_\varphi : \varphi \in L\}$

$T' = Cn(T \cup \{\exists y\, \varphi(\bar{x}, y) \rightarrow \varphi(\bar{x}, t_\varphi(\bar{x})) : \varphi \in \mathcal{F}_L\}$

$M' = M$ expanded to an L'-structure:

t_φ^M such that $M' \models \exists y\, \varphi(\bar{x}, y) \rightarrow \varphi(t_\varphi(\bar{x}))$.

So: $M' \models T'$, hence T': consistent.

② Iteration:

$L^{(0)} = L$, $L^{(n+1)} = (L^{(n)})'$

$T^{(0)} = T$, $T^{(n+1)} = (T^{(n)})'$, $n = 0, 1, 2, \ldots$

$M^{(0)} = M$, $M^{(n+1)} = (M^{(n)})'$

③ Result: skolemization of:

L: $L^s = \bigcup_n L^{(n)}$, T: $T^s = \bigcup_n T^{(n)}$, M: $M^s =$

$= M$ expanded to an L^s-structure ... $M^{(n)}$, $n < \omega$.

Remark $M^s \models T^s$, T^s has Skolem functions.

Proof Let $\varphi(\bar{x}, y) \in \mathcal{F}_{L^s}$. Then $\varphi(\bar{x}, y) \in \mathcal{F}_{L^{(n)}}$ for some n,

hence $t_\varphi(\bar{x}) \in L^{(n+1)} \subseteq L^s$

and $T^{(n+1)} \vdash \exists y\, \varphi(\bar{x}, y) \to \varphi(\bar{x}, t_\varphi(\bar{x}))$

$\rotatebox{90}{$\subseteq$}$

$T^s \vdash \exists y\, \varphi(\bar{x}, y) \to \varphi(\bar{x}, t_\varphi(\bar{x}))$.

Fact. T^s is a conservative extension of T, that is:

If $\varphi \in \mathcal{F}_L$, then $T \vdash \varphi \iff T^s \vdash \varphi$.

Proof Wlog $\varphi = \bar{\varphi}$: a sentence. $\Rightarrow$ clear.

$\Leftarrow$: Suppose $T \nvdash \varphi$. So there is $M \models T \cup \{\neg\varphi\}$.

$\Downarrow$

$T^s \nvdash \varphi \quad \Leftarrow \quad M^s \models T^s \cup \{\neg\varphi\}$

Ramsey thm Assume $f : \underbrace{[\mathbb{N}]^k}_{\{X \in \mathcal{P}(\mathbb{N}) : |X| = k\}} \to \{0, 1\}$.
$k \geq 1$

Then $\exists \underset{\text{infinite}}{X \subseteq \mathbb{N}}\ \exists t \in \{0, 1\}\ f \restriction_{[X]^k} \equiv t$

(X is called "homogeneous for f").

Proof. Induction on k. $k = 1$: OK.
and $\mathbb{N}$ ~~may~~ replaced by any infinite set.

Induction step $k \mapsto k+1$:

Assume $f : [\mathbb{N}]^{k+1} \to \{0,1\}$.

We ~~define~~ define $a_n \in \mathbb{N}$ and $X_n \subseteq \mathbb{N}$ infinite (for $n \in \mathbb{N}$) s.t.

(1) $a_i \in X_{i-1} \setminus X_i$ $\quad$ [and $X_{-1} = \mathbb{N}$]

(2) $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$

(3) $\forall i \; \exists t_i \in \{0,1\} \; \forall Y \in [X_i]^k \; f(\{a_i\} \cup Y) = t_i$

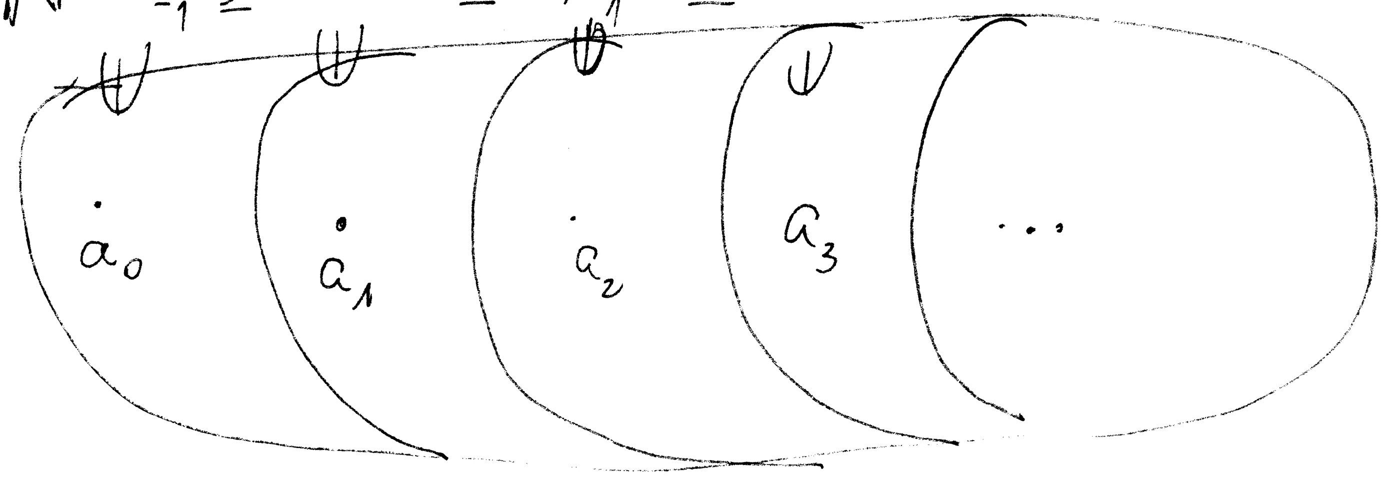
Constructuon:

- Let $a_i \in X_{i-1}$ <u>any</u>. Define $f' : [X_{i-1} \setminus \{a_i\}]^k \to \{0,1\}$ by $f'(Y) = f(Y \cup \{a_i\})$

by inductive assumption there is $X_i \subseteq X_{i-1} \setminus \{a_i\}$ infinite, homogeneous for f' ... $f' \upharpoonright_{[X_i]^k} \equiv t_i$

<u>~~Picture~~</u> Drawing:

$$\mathbb{N} = X_{-1} \supseteq X_0 \supseteq X_1 \supseteq X_2 \dots$$



Let $t \in \{0,1\}$ s.t. $\{i : t = t_i\}$ is infinite.

Then $X := \{a_i : t = t_i\}$ good for f

Why? Let $Y \in [X]^{k+1}$ then $\underline{\text{homogeneous}}$ $f(Y) = t$, because:

$$Y = \{a_{i_1}, a_{i_2}, \dots, a_{i_{k+1}}\} = \{a_{i_1}\} \cup Y', \quad Y' \in [X_{i_1}]^k$$

$$\text{so: } f(\{a_{i_1}\} \cup Y') = \underset{\substack{\uparrow \\ \text{step } i_1}}{f'}(Y') = t_{i_1} = t.$$

Corollary (Ramsey thm)

$$f : [\mathbb{N}]^k \longrightarrow \{0,1,\dots,l\} \Rightarrow \underset{\text{infinite}}{\exists X \subseteq \mathbb{N}} \; \exists t \in \{0,1,\dots,l\}$$

$$f\restriction_{[X]^k} \equiv t.$$

<u>many variants</u>, like:

Corollary (finite version of Ramsey theorem)

~~$\forall k, l \in \mathbb{N}^+ \; \exists n \in \mathbb{N}^+ \; \forall$~~

$$\forall k,l,m \in \mathbb{N}^+ \; \exists n \in \mathbb{N}^+ \; \underset{|X| \geq n}{\forall X} \; \forall f : [X]^k \longrightarrow \{0,\dots,l\}$$

$$\exists Y \in [X]^m \; f\restriction_{[Y]^k} \text{ constant}$$

<u>Order indiscernible sets</u>

<u>Def</u> Assume $(I, \leq)$: a linear ordering (e.g. $I = \mathbb{N}$)

and $A = \{a_i : i \in I\} \subseteq M$ (an I-indexed set)

We say that:

A is <u>order indiscernible</u> if $\forall n$ $\forall \varphi(x_1,\dots,x_n) \in F_L$

$\forall i_1 < \dots < i_n,\ j_1 < \dots < j_n \in I$

~~$M \models \varphi(a_{i_1},\dots,a_{i_n}) \Leftrightarrow M$~~

$$M \models \varphi(a_{i_1},\dots,a_{i_n}) \Longleftrightarrow \varphi(a_{j_1},\dots,a_{j_n}). \qquad (*)$$

<u>Thm</u> If T is a complete theory with infinite models and $(I, \leq)$ is a linear order, then

$\exists M \models T\ \exists A = \{a_i : i \in I\} \subseteq M$ A infinite, order indiscernible.

<u>Proof</u> Let $L' = L \cup \{c_i : i \in I\}$ new constant symbols.

$$T' = T \cup \bigcup_{\varphi} S_\varphi, \qquad \varphi = \varphi(x_1,\dots,x_n) \in F_L$$

where $S_\varphi = \{\varphi(c_{i_1},\dots,c_{i_n}) \Longleftrightarrow \varphi(c_{j_1},\dots,c_{j_n}) : i_1 < \dots < i_n \in I,\ j_1 < \dots < j_n \in I\}$

<u>It is enough to show that T' is consistent.</u>

Let $\varphi_1,\dots,\varphi_k$: formulas, $S'_{\varphi_t} \subseteq_{\text{finite}} S_{\varphi_t}$ for $t = 1,\dots,k$

We will show that $T \cup \bigcup_t S'_{\varphi_t}$ is consistent.

Let $N \models T$ infinite

Lemma. If $A = \{a_n, n \in \mathbb{N}\} \subseteq N$ and $\varphi(x_1, \ldots, x_k) \in \mathcal{F}_L$, then $\exists \cancel{X} \subseteq \mathbb{N}$ infinite $A' = \{a_n : n \in X\}$ is order φ-indiscernible [i.e. (*) holds for φ]

Proof For $Y = \{n_1 < \ldots < n_k\} \in [\mathbb{N}]^k$

let $$f(Y) = \begin{cases} 0, & \text{when } N \models \neg\varphi(a_{n_1}, \ldots, a_{n_k}) \\ 1, & \text{when } N \models \varphi(a_{n_1}, \ldots, a_{n_k}) \end{cases}$$

$$f : [\mathbb{N}]^k \longrightarrow \{0, 1\}.$$

Let $X \subseteq \mathbb{N}$: homogeneous for f infinite $\quad$ X : good. $\square$ Lemma.

Applying Lemma a few times we find

$A = \{a_n : n \in \mathbb{N}\} \subseteq N$ s.t. $\forall 1 \leq t \leq k$ (*) holds for φ_t

Assume S'_{φ_t} $(1 \leq t \leq k)$ use only constants $c_{i_1}, \ldots, c_{i_N}$, $i_1 < \ldots < i_N$

interpretation $\downarrow \quad \downarrow$

$a_1, \ldots, a_N \in N$

So : $N \models T \cup \bigcup_t S'_{\varphi_t}$.

Corollary (Ehrenfeucht – Mostowski)

Assume T is countable, consistent, with infinite models. Then $\exists M \models T$ ctble $\mathrm{Aut}(\mathbb{Q}, \leq) \hookrightarrow \mathrm{Aut}(M)$.

<u>Proof</u> ~~Proof~~

$T \rightsquigarrow T^s$. Let $M^s \models T^s$ ctble, containing

$A = \{a_q : q \in \mathbb{Q}\}$ order indiscernible

Let $N^s = \mathcal{H}(A) \prec M^s$

(†) For $f \in Aut(\mathbb{Q}, \leq)$ let $\hat{f} : N^s \longrightarrow N^s$ given by:

(a) $\hat{f}(a_q) = a_{f(q)}$

(b) If $t(\bar{x})$: a term of L^s, then

$\hat{f}(t^{N^s}(a_{q_1}, \dots, a_{q_n})) = t^{N^s}(\hat{f}(a_{q_1}), \dots, \hat{f}(a_{q_n}))$.

$\hat{f} \in Aut(N^s) \implies \hat{f} \in Aut(N)$, where $N = N^s \restriction_L$.

$f \longmapsto \hat{f}$

$Aut(\mathbb{Q}, \leq) \hookrightarrow Aut(N)$. N ~~particularly~~ ctble, $N \models T$

N^s is called an Ehrenfeucht-Mostowski model.

<u>Comments</u>

- the definition of $\hat{f}$ is correct:
  - each $a \in N^s$ is of the form $a = t^{N^s}(\bar{a}_{\bar{q}})$, $\bar{a}_{\bar{q}} \subseteq A$
  - if $a = t^{N^s}(\bar{a}_{\bar{q}}) = t'^{N^s}(\bar{a}_{\bar{q}'})$, then

$N^s \models t(\bar{a}_{\bar{q}}) = t'(\bar{a}_{\bar{q}'}) \implies N^s \models t(\bar{a}_{f(\bar{q})}) = t'(\bar{a}_{f(\bar{q}')})$

[A: order indiscernible, f preserves $\leq$]

- ~~$\hat{f} \in \mathrm{Aut}(N^s)$~~ $\cdot$ $f \in \mathrm{Aut}(N^s)$

$$N^s \models \varphi(a_{q_1}, \dots, a_{q_n}) \Longleftrightarrow \varphi(\hat{f}(a_{q_1}), \dots, \hat{f}(a_{q_n}))$$ [by order indiscernibility].

Ultraproducts

(a) Products. M_i, $i \in I$: L-structures.

$\rightsquigarrow M = \prod_I M_i$ product of structures M_i, $i \in I$.

$$|M| = \prod_{i \in I} |M_i| = \{ f : I \longrightarrow \bigcup_{i \in I} |M_i| : \forall i\ f(i) \in |M_i| \}.$$

L-structure on M:

- P: relational symbol of L.

$$P^M(f_1, \dots, f_n) \Leftrightarrow \forall i\ \ P^{M_i}(f_1(i), \dots, f_n(i))$$

- F: function symbol of L

$$F^M(f_1, \dots, f_n) = \langle F^{M_i}(f_1(i), \dots, f_n(i)) : i \in I \rangle \in |M|.$$

- constant symbol c of L

$$c^M = \langle c^{M_i} : i \in I \rangle.$$

Examples products of groups, linear spaces, fields, rings...

Troubles 1. Product $K_1 \times K_2$ of fields is not a field.

2. $\mathrm{Th}(\prod_I M_i)$ unrelated to $\mathrm{Th}(M_i)$, $i \in I$.

Solution ultraproducts (families of "large" sets)

Def.: $\mathcal{U}$ is an ultrafilter on $I \Leftrightarrow$
$\mathcal{U}$ is an ultrafilter in algebra $\mathcal{P}(I)$

- An ultrafilter $\mathcal{U}$ on I is principal if $\exists a \in I$
$$\mathcal{U} = \mathcal{U}_a = \{X \subseteq I : a \in X\}.$$
Otherwise: $\mathcal{U}$: non-principal.

Properties of ultrafilter $\mathcal{U}$:

(1) $\forall X \subseteq I \ (X \in \mathcal{U} \ \underline{\vee} \ X^c \in \mathcal{U})$

(2) $\forall X \in \mathcal{U} \ \forall Y \underset{X \subseteq}{\subseteq} I \ \ Y \in \mathcal{U}$

(3) Every proper filter on I extends to an ultrafilter on I.