Logic R, list 6 (declarations: Jan 30, 9:00, homework: Jan 31)

Problems and items thereof marked with minus are excluded from homework. "recursive function" by default means "partial recursive function". You can use the fact that a function is recursive iff it is TM-computable.

1. - Let $\Sigma = \{0, 1\}$. Find 1-tape Turing machines recognizing language $L \subseteq \Sigma^*$: (a) $L = \{01, 0101, 010101, \ldots\}$, (b) $L = \{0^n 1^n : n \in N\}$ (here $\sigma^n = \sigma\sigma \ldots \sigma$ (n times))

2. (a) Prove that if $A, B \subseteq \mathbb{N}$ are recursive, then $N \setminus A$, $A \cup B$, $A \cap B$ are recursive, too (that is, recursive sets are an algebra of subsets of $\mathbb{N}$).
(b) Prove that if $f(x, y)$ is recursive, then $g(x) = \sum_{y \leqslant x} f(x, y)$ is recursive.
(c) Prove that if a (total) function $f : \mathbb{N} \to \mathbb{N}$ is increasing and $Rng(f)$ is a recursive set, then f is recursive.
(d) Prove that an infinite set $X \subseteq \mathbb{N}$ is recursive $\iff$ $X = Rng(f)$ for some total recursive increasing function f.

3. (a) Prove that the range and domain of a recursive function $f : \mathbb{N} \to \mathbb{N}$ are recursively enumerable.
(b) Prove that if $A \subseteq \mathbb{N}$ is recursively enumerable, then there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that A is its domain.
(c) Assume that $f, g : \mathbb{N} \to \mathbb{N}$, $A = f[\mathbb{N}]$, $B = g[\mathbb{N}]$ oraz $A \cap B \neq \emptyset$. Using f, g, basic recursive functions and operartions of composition, recursion and minimum define a function $h : \mathbb{N} \to \mathbb{N}$ with $Rng(h) = A \cap B$.

4. (a) Prove that the set $A \subseteq \mathbb{N}$ is recursively enumerable $\iff$ A is the projection of a recursive set $B \subseteq \mathbb{N} \times \mathbb{N}$.
(b) Prove that the projection of a recursively enumerable set $A \subseteq \mathbb{N} \times \mathbb{N}$ to the first coordinate is recursively enumerable.
(c) Assume that $A \subseteq \mathbb{N}$ is recursively enumerable and infinite. Prove that there is a recursively enumerable set $B \subseteq A$ such that the set $A \setminus B$ is not recursively enumerable.

5. The set $A \subseteq N$ is called spectrum when there is a sentence σ in a language L such that $A = \{\|M\| : M \models \sigma \text{ i } M \text{ skończony}\}$. Prove that if A is spectrum, then A is recursive.(Note an open problem of Fagin, 1970: Assume A is a spectrum. Is the complement of A a spectrum?).

6. Prove that $A \subseteq \mathbb{N}$ is recursively enumerable $\iff$ there is a Turing machine M such that for the input $x \in \mathbb{N}$: M stops $\iff$ $x \in A$.

7. Prove in PA that
(a) $x + y = y + x$; (b)– $x \cdot y = y \cdot x$; (c)– $(x + y) + z = x + (y + z)$ etc.

8. Write in the language $L = \{+, \cdot, <\} \cup \{\underline{n} : n \in N\}$ the following formulas about natural numbers:

(a)- x is a prime number;
(b)- x is a power of 2;
(c)- x is a product of two different primes;
(d) x is a power of 10;
(e) x is the number of non-isomorphic graphs with y vertices;
(f) x is the number of distinct bases in a y-dimensional vector space over Z_2.

9. * Primitive recursive functions $Prec$ arise from the basic ones by application of composition and simple recursion. Prove that there is a total recursive function that is not primitive recursive. (hint: find a total recursive function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ universal for $Prec$, that is such that $\forall n \; f(n, \cdot) \in Prec$ and $\forall g : \mathbb{N} \to \mathbb{N}, \; g \in Prec \Rightarrow \exists n \; f(n, \cdot) = g(\cdot)$)