Logic R, list 4 (declarations by Sunday, Dec 12, 9:00, homework due by Monday, Dec 13, 14:00).

As usual, every item in a problem from the list is treated as potentially one of 3 items of the homework. Among the 3 items in the homework no two items may be from the same problem from the list. The discussion classess on Dec 13 and Dec 20 will be online. On Sunday morning I will assign ptoblems to the students, for presentation. Probably this list will take 2 meetings (or more). So if you are unable to deliver full homework for this list on Dec 13, you will be able add some homework on Dec 20, from the problems from the list that were not done yet in class. Also, you will be able to add declarations for these problems.

1. Let $\mathbb{R}^*$ be a non-standard model of analysis, i.e. an $\aleph_1$-saturated model of the theory

$$A = Th(\mathbb{R}, +, \cdot, f, P, x)_{f,P,x \text{ all functions, relations and reals}},$$

an elementary extension of $\mathbb{R}$.
(a) The number $a \in \mathbb{R}^*$ is called *bounded* if there is $r \in \mathbb{R}$ such that $\mathbb{R}^* \models -r < a < r$. The number $a \in \mathbb{R}^*$ is called infinitesimal if $a \neq 0$ and $\forall n \in \mathbb{N}$, $\mathbb{R}^* \models -1/n < a < 1/n$. Prove that if $a \in \mathbb{R}^*$ is bounded, then there is a unique number $st(a) \in \mathbb{R}$ such that $st(a) - a$ is equal to 0 or is infinitesimal. $st(a)$ is called the standard part of a.
(b) Assume that $f : \mathbb{R} \to \mathbb{R}$ is differenriable in $a \in \mathbb{R}$. Let $a' \neq 0$ be infinitesimal. Prove that $f'(a) = st(\frac{f(a+a')-f(a)}{a'})$.
(c) One of the relation symbols of L is $N(x)$ defining in $\mathbb{R}$ natural numbers. Prove that $(N^{\mathbb{R}^*}, +, \cdot)$ is an $\aleph_1$-saturated extension of $(\mathbb{N}, +, \cdot)$. Consequently, in $\mathbb{R}^*$ there are non-standard natural numbers .
(d) Assume that $f : I \to \mathbb{R}$ is continuous. $I = [0, 1]$. Let $n^* \in \mathbb{R}^*$ be a non-standard natural number. Prove that

$$\int_I f = st(\sum_{i=0}^{n^*-1} \frac{1}{n^*} f(\frac{i}{n})).$$

(formally: in $\mathbb{R}$: let $g(n) = \sum_{i=0}^{n-1} \frac{1}{n} f(\frac{i}{n})$, g is a function symbol of language L. Then in $\mathbb{R}^*$: $g^{R^*}(n^*)$ is fomally interpreted as $\sum_{i=0}^{n^*-1} \frac{1}{n^*} f(\frac{i}{n^*})$).
(e) Assume that $f : \mathbb{R} \to \mathbb{R}$. Prove that $\lim_{x\to 0^+} f(x) = a \iff \forall$ infinitesimal $b > 0,\ st(f(b)) = a$.
Think of similar examples (manifolds, Brownian motion, singularities etc.)

2. (a) Prove that if $M \equiv N$, N is countable, M is $\aleph_0$-saturated, then $\exists M' \prec M$, $M' \cong N$.

   (b) Prove that if $M \equiv N$ are countable and saturated, then $M \cong N$.

3. Let T be the theory of independent predicates $P_n(x), n < \omega$.
(a) Prove that T is q.e.,
(b) Describe types in $S_1(M), M \models T$.
(c) Prove that $\forall \kappa \geq 2^{\aleph_0}$, T is κ-stable.
(d) Describe a saturated model of T of arbitrary power $\geq 2^{\aleph_0}$.

4. (a) Prove that $DLO_0 = Th(Q, \leq)$ and BA_0 are not $\aleph_0$-stable (hint: in DLO_0: Dedekind cuts).
(b) Prove that these theories are not stable.
In this problem treat the case of each of the theories as a separate item. Hence this problem has 4 items.

5. Assume that M is an infinite countable saturated model. Prove that $Aut(M)$ is
(a) infinite,
(b) of power $2^{\aleph_0}$,
(c) homeomorphic with the Baire space ω^ω.
By the way, w this manner we gte any examples of topologicla groups, that is groups that are topological spaces T_1 (hence $T_{3\frac{1}{2}}$) and where group operations (group multiplication and taking inverse) are continuous.

6. Prove Scott's Theorem: (a) item (1), (b): item (2).

7. Find Scott sentences for the following structures: (a)$(\mathbb{Z}, \leq)$ (b)$(\mathbb{Q}, \leq)$, (c)$(\mathbb{N} \cup \mathbb{N}^*, \leq)$, (d) $(\mathbb{Z} \cup \mathbb{Z}, \leq)$ (here: $\mathbb{N}^* = \mathbb{N}$ with reverse order, $\mathbb{Z} \cup \mathbb{Z}$: two copies of $\mathbb{Z}$, one after another.

8. * Prove that for T countable complete $\kappa < 2^{\aleph_0}$:
If $\{p_\alpha, \alpha < \kappa\} \subseteq S(\emptyset)$ is a family of non-isolated types, then $\exists M \models T$ omitting all types $p_\alpha, \alpha < \kappa$.