Logic R, list 3. (homework due Nov 29, declarations due by 13:00 on Nov 29)

1. Let $BA_0$ be the theory of atomless Boolean algebras. Prove that $BA_0$ is $\aleph_0$-categorical and admits elimination of quantifiers.

2. Prove that if $T$ is complete and admits elimination of quantifiers, then for $M, N \models T$, $M \subseteq N$ implies $M \prec N$.

3. Check that the Stone space $S(B)$ of Boolean algebra $B$ is compact Hausdorff. Prove that if $B$ is countable, then $S(B)$ is homeomorphic to a closed subset of the Cantor set.

4. Assume that $M$ is a model (structure) for language $L$ and $N$ is a model of the theory $Th(M, n)_{m \in M}$ (in language $L(M)$, with new constant symbols naming all elements of $|M|$ added). Let $M' = \{m^N : m \in M\}$. Prove that
(a) $M'$ is a model for $L$ (with the structure induced from $N$).
(b) $M' \cong M$ and $M' \prec N$.

5. Prove tha every proper filter in a Boolean algebra $B$ extends to an ultrafilter.

6. (a) Prove that every two Boolean algebras of the same size are isomorphic.
(b) Prove that a Boolean algebra $B$ is finite $\iff$ $S(B)$ is finite.
(c) Prove that the number of elements of a finite Boolean algebra is a power of 2.

7. Prove that in some elementary extension $\mathbb{Z}'$ of the group $(\mathbb{Z}, +)$ there is an element $a \neq 0$ that is divisible (i.e. for every $n > 0$, $\mathbb{Z}' \models \exists x \underbrace{x + \cdots + x}_{n} = a$).

8. (a) Describe all countable models of the theory of $(\mathbb{Z}, S)$, where $S$ is the successor function (up to isomorphism).
(b) A cycle in a $k$-element set $X = \{x_1, \ldots, x_k\}$ is a structure $(X, S)$, where $S$ is a permutation of $X$ such that $S(x_i) = x_{i+1}$ for $i < k$, and $S(x_k) = x_1$. Let $T$ be the theory of finite cycles in the language $L = \{S\}$, that is, the set of formulas of $L$ true in all finite cycles $(X, S)$. Prove that $T$ and $Th(\mathbb{Z}, S)$ have the same infinite countable models.

9. Assume that $G$ is a group of permutations of an non-empty finite set $X$ (i.e. $G < S(X)$). Prove that for certain language $L$, there is an $L$-structure on the set $X$ such that $G = Aut(X)$.