Logic R, List 2 (homework due on Nov. 8, declarationes: by 13:00 on Nov 8).

1. We define the notion of semantical consequence: for a set of formulas T and a formula φ of a given language L:

$$T \models \varphi \iff (\forall M \models T)\ M \models \varphi.$$

   Prove that $T \vdash \varphi \Leftrightarrow T \models \varphi$. (This result justifies that the definition of $\vdash$ is correct.)

2. Prove that $M \cong N \Rightarrow M \equiv N$.

3. Prove that if $M \equiv N$ and L, M are finite, then $M \cong N$. Moreover, there is a sentence σ such that $M \models \sigma$ and $\forall N \models \sigma,\ N \cong M$.

4. Assume that L is a finite relational language. Prove that $M \equiv N \Leftrightarrow \forall n\ D$ has a winning strategy in the game $\Gamma_n(M, N)$. This problem splits into two parts:
   (a) $\Rightarrow$
   (b) $\Leftarrow$.

5. Prove that if T is complete and has a recursively enumerable set of axioms, then T is decidable.

6. Assume the sentence φ is built exclusively from unary relational symbols and logical symbols. Prove that $\exists n < \omega$ (n depends only in the length of φ) such that if φ has a model, then, it has a model of size $< n$.

7. Deduce from this that there is an algorithm deciding, if φ is a tautology for φ as in Problem 5.

8. Let

$$FLO = \{\varphi : \forall M \models LO,\ M \text{ finite} \Rightarrow M \models \varphi\},$$

$$FLO_\infty = \{\varphi : \exists n \forall M \models LO,\ M \text{ finite and } \|M\| > n \Rightarrow M \models \varphi\}.$$

   (a) Prove that the theories FLO, FLO_∞ are decidable (hint: use Ehrenfeucht games).
   (b) Notice that the theory FLO is not complete , the theory FLO_∞ is complete and $FLO_\infty = Th(?, \leq)$.

9. (a) Prove that there is no sentence φ such that $\forall M$ ($M \models \varphi \iff M$ is finite).
   (b) Prove that there is no theory $T \supseteq LO$ with infinite models such that all its infinite models are well ordered.