Ex. 2b) M, N ctble saturated L-structures, $M \equiv N$.

Then $M \cong N$.

<u>Proof</u> First let's state this simple fact:

<u>Fact 1</u> If M is saturated, then $\forall A \subseteq M$, $|A| < \|M\|$ we have $S_1(A) = \{tp^M(a/A) : a \in M\}$.

$\square$

Enumerate $|M| = \{a_1, a_2, \ldots\}$, $|N| = \{b_1, b_2, \ldots\}$

We will construct sequences of models, languages and theories recursively while maintaining those properties:

- $M_0 = M$, $N_0 = N$, $L_0 = L$, $T_0 = Th(M) = Th(N)$
- $L_{n+1} = L_n \cup \{c_{n+1}\}$ a new constant symbol
- M_n, N_n are L_n-structures, $M_n = (M, c_k)_{1 \le k \le n}$, $c_k^{M_n} = c_k^{M_{n-1}}$, $k = 1, \ldots, n-1$ similarly with N_n
- $M_n \equiv N_n$, i.e. $T_n = Th(M_n) = Th(N_n)$

- $c_{2n+1}^{M_{2n+1}} = a_n$, $c_{2n+2}^{N_{2n+2}} = b_n$

Fact 2 M_n, N_n are saturated.

Proof $S_1^{T_n}(\bar{a}) = S_1^{T_0}(\bar{a} \cup \{c_1^{M_n}, \dots, c_n^{M_n}\})$

$$\underset{\mathcal{F}_{L_n}}{\varphi(\bar{a}, x)} \longleftrightarrow \underset{\mathcal{F}_L}{\psi(\bar{a}, \bar{c}^{M_n}, x)}$$

all types here are realisable in M_n

$\square$

Now suppose $L_{2n}, M_{2n}, N_{2n}, T_{2n}$ satisfy our conditions.

Let $c_{2n+1}^{M_{2n+1}} = a_n$. By facts 1 & 2 we know that $S_1^{T_{2n}}(\emptyset) = \{tp^{M_{2n}}(a) : a \in |M|\}$ $= \{tp^{N_{2n}}(b) : b \in N\}$. So we can find $b \in N$ s.t. $tp^{M_{2n}}(a_n) = tp^{N_{2n}}(b)$.

Let $c_{2n+1}^{N_{2n+1}} = b$.

Now take any sentence $\varphi \in \mathcal{F}_{L_{2n+1}}$ ($\varphi = \psi(c_{2n+1})$ for some ψ in $\mathcal{F}_{L_{2n}}(x)$)

$M_{2n+1} \models \varphi \iff M_{2n+1} \models \psi(c_{2n+1})$

$\iff N_{2n+1} \models \psi(c_{2n+1}) \iff N_{2n+1} \models \varphi$.

Thus $M_{2n+1} \equiv N_{2n+1}$.

Similarly $c_{2n+2}^{N_{2n+2}} = b_n$ and $c_{2n+2}^{M_{2n+2}} = a$ for some a s.t. $tp^{N_{2n+1}}(b_n) = tp^{M_{2n+1}}(a)$.

Now Let $M_\infty = (M, c_n)_{n \in \mathbb{N}^+}$, $c_n^{M_\infty} = c_n^{M_n}$, similarly N_∞.

<u>Fact</u> $M_\infty \equiv N_\infty$ (easy)

Now
$$f : M \xrightarrow{\equiv} N .$$
$$c_n^{M_\infty} \longmapsto c_n^{N_\infty}$$

Why is this definition correct?

Suppose for some $n < m$ $c_n^{M_\infty} = c_m^{M_\infty}$.

Then $\varphi : x = c_n \in tp^{M_m}(c_m^{M_m})$.

This means that $\varphi \in tp^{N_m}(c_m^{N_m})$

$\Rightarrow c_m^{N_\infty} = c_n^{N_\infty}$

It follows that f is a bijection.

Fact Elementary bijection is isomorphism.
(easy)

So $f: M \xrightarrow{\cong} N$. $\blacksquare$

ex. 1

a) Take $a \in \mathbb{R}^*$. Suppose there is no $r \in \mathbb{R}$ s.t. $a - r$ is 0 or infinitesimal. This means that there is $n \in \mathbb{N}$ s.t.

$$a - r > \frac{1}{n} \iff a > \frac{1}{n} + r$$

for all $r \in \mathbb{R}$, which implies that a is not bounded. Thus if a is bounded, then there's $r \in \mathbb{R}$ s.t. $a - r$ is 0 or infinitesimal.

Take $s \in \mathbb{R}$ s.t. — " —

Then $\forall n \in \mathbb{N} \left(-\frac{1}{n} < a - r < \frac{1}{n} \wedge -\frac{1}{n} < a - s < \frac{1}{n} \right)$

$$\Updownarrow$$

$$\forall n \in \mathbb{N} \left(-\frac{1}{n} < a - r < \frac{1}{n} \wedge -\frac{1}{n} < s - a < \frac{1}{n} \right)$$

$$\Downarrow$$

$$\forall n \in \mathbb{N} \left(-\frac{2}{n} < s - r < \frac{2}{n} \right) \Rightarrow s = r.$$

This implies that r is unique.

$\blacksquare$