ex. 2 Suppose T is complete and satisfies elimination of quantifiers, $M, N \models T$ and $M \subseteq N$, then $M \preccurlyeq N$.

Proof

First, let us recall the definition of a substructure.

$M \subseteq N \overset{def}{\iff}$ 1) $|M| \subseteq |N|$

2) For n-ary $f \in \mathcal{L}$: $f^M = f^N \restriction_{M^n}$

3) For n-ary $R \in \mathcal{L}$: $R^M \subseteq R^N \cap M^n$

4) For constant c: $c^M \cancel{\varphi}= c^N$.

Lemma. For every q.f. sentence $\varphi \in \mathcal{F}_{\mathcal{L}(M)}$ we have $M \models \varphi \iff N \models \varphi$ (where we think of M, N as $\mathcal{L}(M)$-structures).

Proof of lemma Induction on complexity of φ.

First we need to say that for every constant term $t \in \mathcal{T}_{\mathcal{L}(M)}$ we have $t^M = t^N$.

This is true by the definition of substructure.

Induction on complexity of φ:

- $\varphi : R_i(t_1, \dots, t_n)$:

$$M \models \varphi \iff R_i^M(t_1^M, \dots, t_n^M) \overset{\text{def. of substr.}}{\iff} R_i^N(t_1^N, \dots, t_n^N) \iff N \models \varphi$$

- $\varphi : t_1 = t_2$: $M \models \varphi \iff t_1^M = t_2^M \iff \underset{t_1^M}{\overset{\shortparallel}{t_1^N}} = \underset{t_2^M}{\overset{\shortparallel}{t_2^N}} \iff N \models \varphi$

- $\varphi : \varphi_1 \wedge \varphi_2, \neg \psi$: trivial.

lemma ■

Now, take any $\varphi(\bar{x}) \in \mathcal{F}_L$. Let $\psi(\bar{x}) \in \mathcal{F}_L$ be a formula without quantifiers s.t.

$$T \vdash \varphi(\bar{x}) \longleftrightarrow \psi(\bar{x}).$$

Now by our lemma we know that for every $\bar{a} \subseteq M$:

$$M \models \varphi(\bar{a}) \iff M \models \psi(\bar{a}) \overset{\text{Lemma}}{\iff} N \models \psi(\bar{a}) \iff N \models \varphi(\bar{a})$$

■

Ex. 3 B – BA.

1) $S(B)$ is compact Haussdorf
2) B ctble $\Longrightarrow$ $S(B)$ is homeomorphic to a closed subset of the Cantor set.

First we will prove two usefull lemmas:

Lemma 1 Every subset $A \subseteq B$ with FIP (Finite Intersection Property, i.e. $\forall \bar{a} \subseteq A$ finite $\wedge \bar{a} \neq 0$) can be extended to a proper filter.

Proof Take any subset $A \subseteq B$ with FIP.
Wlog suppose that $\forall \bar{a} \subseteq A$, finite $(\wedge \bar{a} \in A)$.
Now let $F = A \cup \{ b \in B : \exists a \in A \, (a \leq b) \}$.
$F \supseteq A$ and it's a proper filter:

- It is certeinly closed under $\leq$
- It is closed under finitie intersections and $0 \notin F$:

Take any finite $\bar{a} \subseteq F$. Wlog $a_1, \dots, a_k \in A$, $a_{k+1}, \dots, a_n \notin A$ for some k ($n = |\bar{a}|$).

By the construction of F there are $b_{k+1}, \dots, b_n \in A$ s.t. $b_i \leq a_i$, $i = k+1, \dots, n$.

Now because A has FIP, then

$$0 \neq a_1 \wedge \dots \wedge a_k \wedge b_{k+1} \wedge \dots \wedge b_n \leqslant$$

$$\leqslant a_1 \wedge \dots \wedge a_k \wedge a_{k+1} \wedge \dots \wedge a_n.$$

this is easy to show, I'll omit it.

$\blacksquare$

<u>Lemma 2</u> $F \in S(B)$, then $\forall a \in B \, (a \in F \,\dot{\vee}\, a' \in F)$

<u>Proof</u> (A.a.) Suppose $F \in S(B)$ and that there is $a \in B$ s.t. $a, a' \notin F$.

If $\forall b \in F$ $a \wedge b \neq 0$, then $F \cup \{a\}$ has FIP, thus it could be extended to a filter bigger then F, which contradicts maximality of F. Similarly, it can't be that $\forall b \in F$ $a' \wedge b \neq 0$.

So there is $b, c \in F$ s.t. $a \wedge b = a' \wedge c = 0$

But then $0 = (a \wedge b) \vee (a' \wedge c) =$

$$= a' \vee b' \vee a \vee c' = b' \vee c' = (b \wedge c)'$$

$$\Downarrow$$

$$b \wedge c = 0 \qquad \lightning$$

Now we can tackle the problems.

<u>Proof</u> 1) First, we'll show that $S(B)$ is Hausdorf. Take any $F_1, F_2 \in S(B)$, $F_1 \neq F_2$. Take $a \in F_1$ s.t. $a \notin F_2$ and $b \in F_2$ s.t. $b \notin F_2$ (there are such a, b because $F_1 \not\subseteq F_2$, $F_2 \not\subseteq F_1$).

Now by lemma 2 $b' \in F_1$, $a' \in F_2$, so $F_1 \in [a \wedge b']$, $F_2 \in [b \wedge a']$ and $[a \wedge b'] \cap [a' \wedge b] = \emptyset$. $\square$

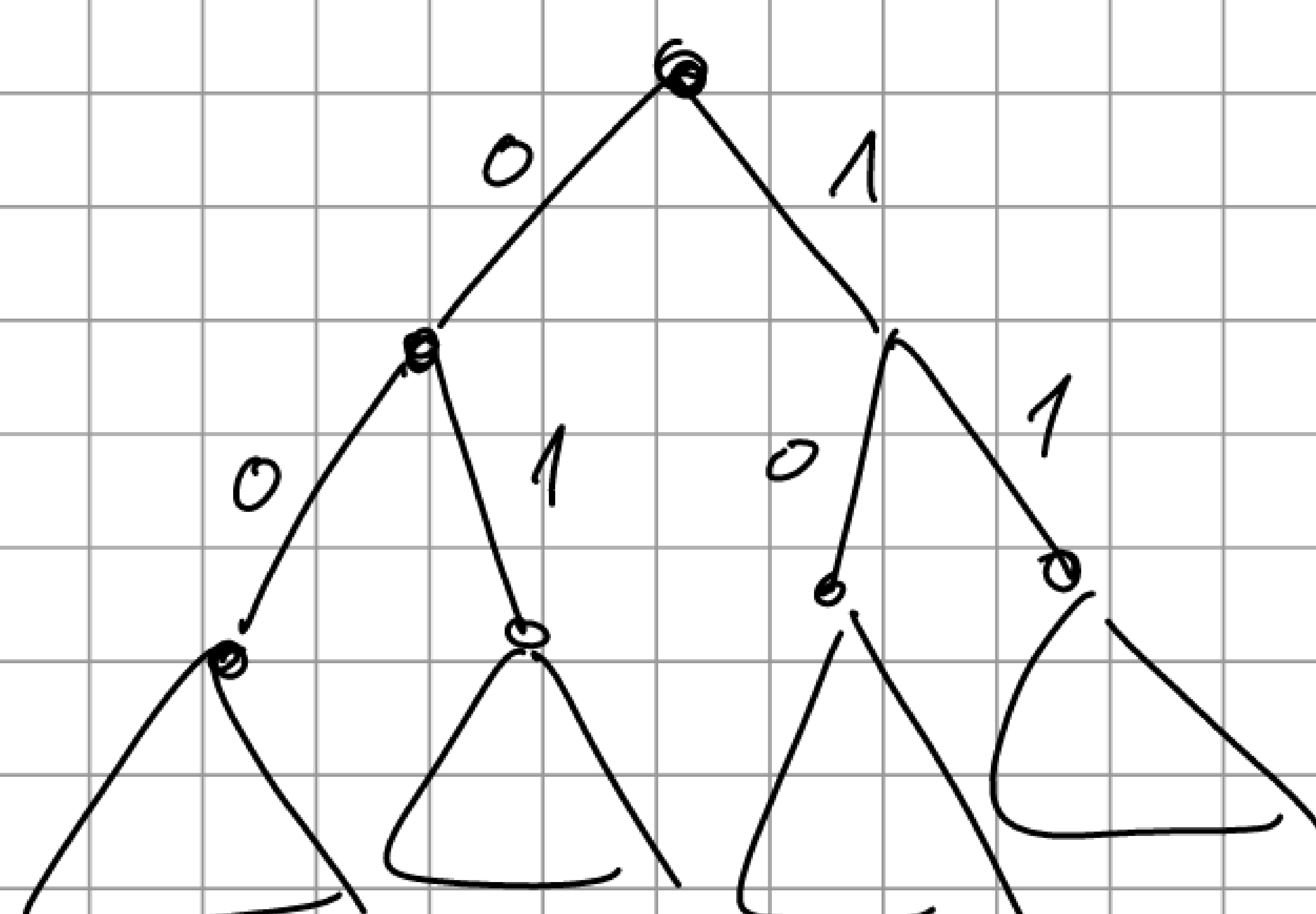
Now let's show that $S(B)$ is compact. It's sufficient to show that every cover consisting exclusively of base open sets has a finite subset that's also a cover of $S(B)$. Let's take any $A \subseteq B$ s.t. $\mathcal{U} = \{[a] : a \in A\}$ is a cover. (A.a.) suppose that there's no such finite subset of $\mathcal{U}$ that's also a cover. That is, for every finite $\bar{a} \subseteq A$ there is $F \in S(B)$ s.t. $F \notin \bigcup_{i=1}^{n} [a_i]$.

What it means is that $a_i \notin F$ for $i = 1, \dots, n$. From lemma 2 we have $a_i' \in F$ for $i = 1, \dots, n$, so $F \ni \bigwedge_{i=1}^{n} a_i' = \left(\bigvee_{i=1}^{n} a_i\right)' \neq 0$ (because F is a proper filter)

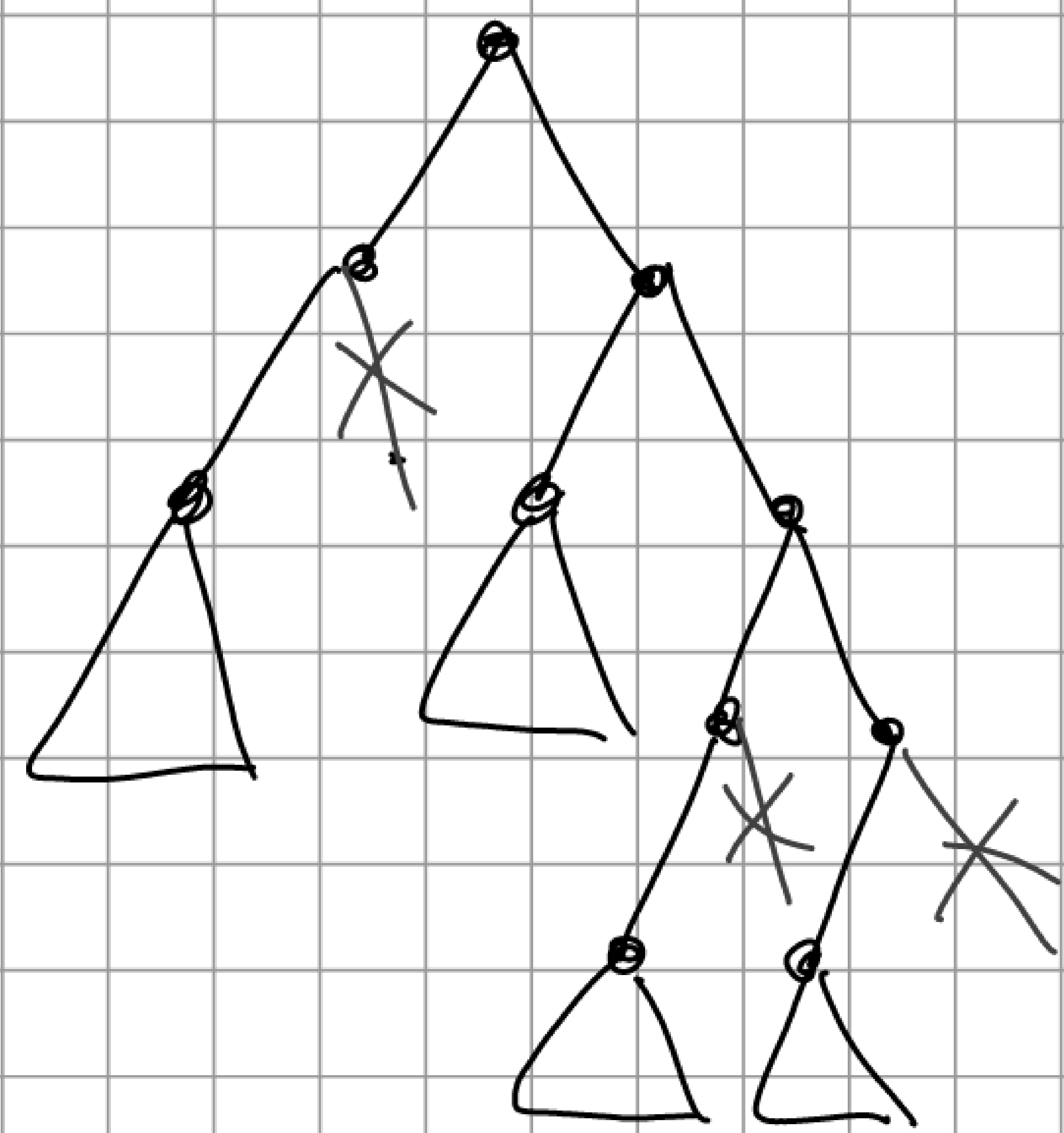
Let $G = \{a' : a \in A\}$. G has the FIP because of that. So by lemma 1 we can extend G to a proper filter which can be extended to an ultrafilter (by ex. 5). Let's call this ultrafilter $\mathcal{H}$. For every $a \in A$ $a' \in \mathcal{H}$, so $a \notin \mathcal{H}$, so $\mathcal{H} \notin \bigcup_{a \in A} [a] = S(B)$. ↯

Proof 2)

$B = \{b_1, b_2, \dots\}$. We want to construct a homeomorphism $f: B \to \overline{D}$, where $D \subseteq \mathfrak{C} = \{0,1\}^{\mathbb{N}}$. We can think of Cantor set as a binary tree:



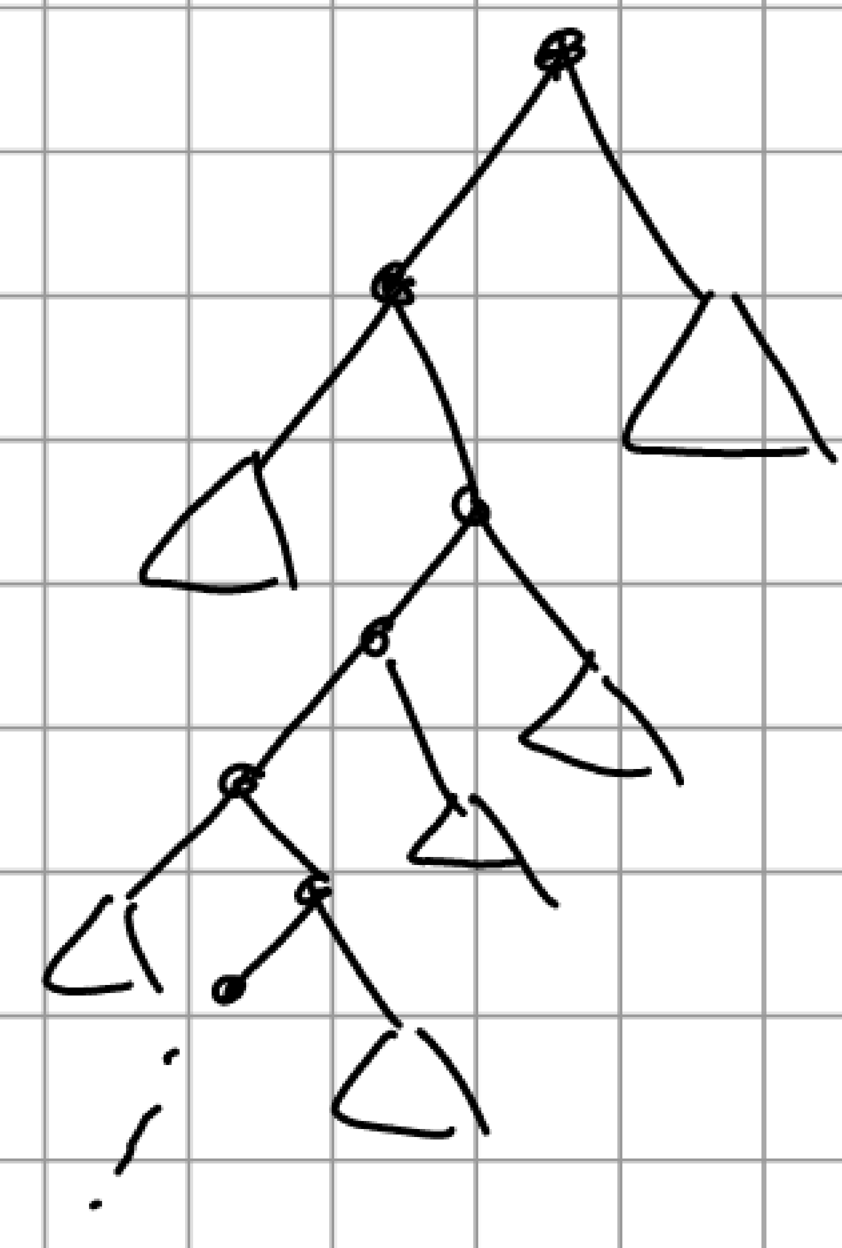
A point in $\mathfrak{C}$ is an infinite branch and a closed set is a tree with some subtrees "cut":



Let's define f. For $F \in S(B)$ define an auxilary function $\sigma_F : B \to \{0\ 1\}$ with a formula $\sigma_F(b) = \begin{cases} 1 & b \in F \\ 0 & b' \in F \end{cases}$
By lemma 2 this is a correct definition. Now:

$$f(F) = \langle \sigma_F(b_1), \sigma_F(b_2), \dots \rangle$$

First let's see that $Rng(f)$ is a closed subset of $\mathbb{C}$.
To see that let's take any $x \in \overline{Rng(f)}$. x is some branch

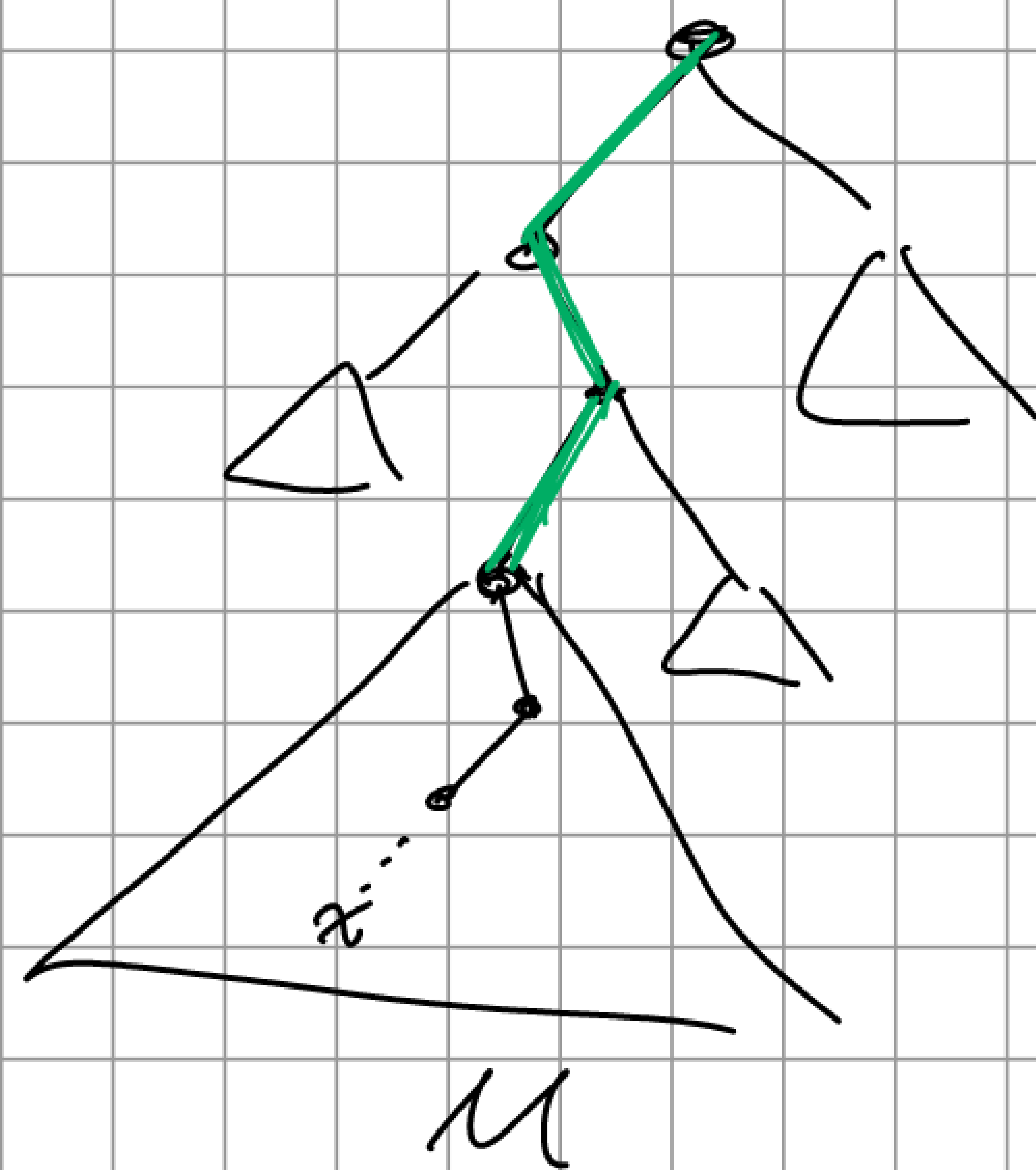


$X = (x_1, x_2, \dots)$

We want to show that $x \in \mathrm{Rng}(f)$,
i.e. there is some $F \in S(B)$ s.t.

$\sigma_F(b_i) = x_i$ for $i = 1, 2, \dots$ (A.a.)

suppose there's no such F. What it means is that for some prefix $x_1, \dots, x_n$ we have $\bigwedge\limits_{i:\, x_i = 1}^{n} b_i = 0$ (if there's no such prefix, then by lemma 1 there is F s.t. $f(F) = x$). But "prefix" is a subtree, which is a neighbourhood of x:

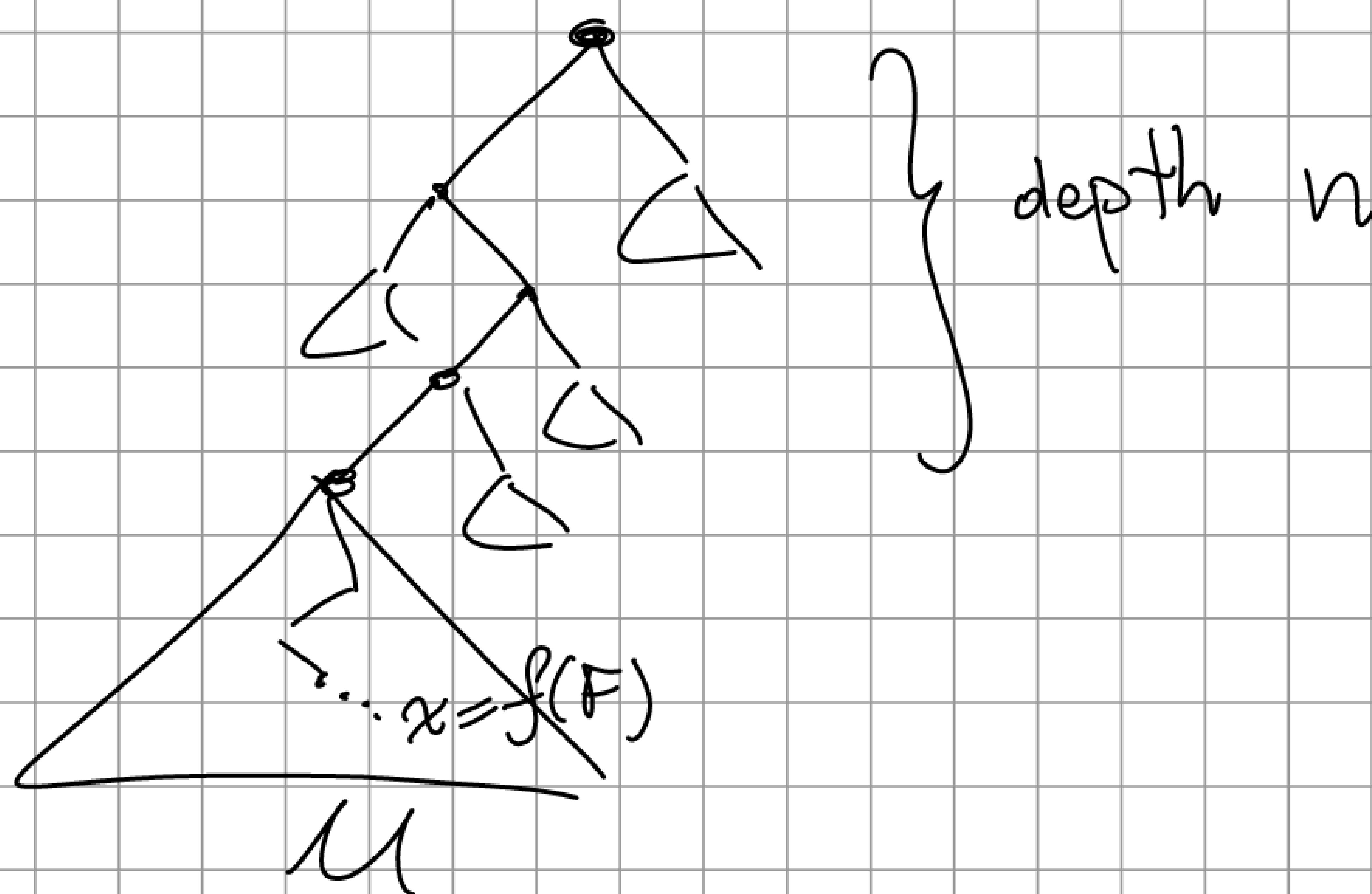


Moreover this neighbourhood has an empty intersection with $Rng(f)$! ↯

Sidenote: I just realised that $S(B)$ is compact, so it's sufficient to show that f is a continous 1-1 function, then we know that $f[S(B)]=D$ is compact and f is a homeomorphism. I'll leave the proof because it's nice in my opinion.

We'll show that f is continous (1-1 is obvious).

Take $F \in S(B)$ and any base open neighbourhood $\mathcal{U}$ of $f(F) = x$



Now let $b = \bigwedge_{i : b_i \in F}^{n} b_i$, $\mathcal{V} = [b]$.

Then $$f[\mathcal{V}] \subseteq \mathcal{U}.$$

This is trivial, because for every $F' \in \mathcal{V}$ we have $\sigma_{F'}(b_i) = x_i$ where, $i = 1, \dots, n$, which means that $f(F')$ is in the subtree $\mathcal{U}$. ∎

Ex. 5 Every proper filter F in BA B extends to an ultrafilter.

Proof Let F be a proper filter in B.
Let $\mathcal{F} = \{F' : F \subseteq F', F' \text{ is a prop. filter}\}$.
Take any chain $C \subseteq \mathcal{F}$. Then $\bigcup C$ is a proper filter. Take any $a, b \in C$. Then there is some filter $F \in C$ s.t. $a, b \in F$, so $a \wedge b \in F \Rightarrow a \wedge b \in C$. Of course $0 \notin C$ because 0 is not in any $F \in C$.
It is closed under supersets because every filter in C is.
By Zorn lemma we conclude that there is a maximal $\mathcal{U} \in \mathcal{F}$. It is an ultrafilter. Suppose not: then there's a prop. fil. $\mathcal{U}' \supsetneq \mathcal{U}$, but also $F \subseteq \mathcal{U}'$, so $\mathcal{U}' \in \mathcal{F}$ which contradicts maximality of $\mathcal{U}$. ∎