ex. 17 $\quad rk : V \to Ord$

$$rk(x) = \min\{\alpha \in Ord : \forall y \in x \ rk(y) < \alpha\}$$

show that $rk(x) \leq \alpha \iff x \in V_{\alpha+1}$

<u>Proof</u> We'll prove it with induction on α. For $\alpha = 0$ OK. Now

- $\alpha+1$: Take x s.t. $rk(x) \leq \alpha+1$. Then

$$
\begin{aligned}
& rk(x) \leq \alpha+1 \\
\iff & \forall y \in x \quad rk(y) < \alpha+1 \\
\iff & \forall y \in x \quad rk(y) \leq \alpha \\
\overset{IND}{\iff} & \forall y \in x \quad y \in V_{\alpha+1} \\
\iff & x \subseteq V_{\alpha+1} \iff x \in V_{\alpha+2}
\end{aligned}
$$

- $\lambda \in$ Limit. Take x s.t. $rk(x) \leq \lambda$. Then

$$
\begin{aligned}
& rk(x) \leq \lambda \\
\iff & \forall y \in x \quad rk(y) < \lambda \\
\iff & \forall y \in x \quad rk(y) \leq \alpha_y \text{ for some } \alpha_y < \lambda \\
\iff & \forall y \in x \quad y \in V_{\alpha_y+1} \\
\iff & \forall y \in x \quad y \in V_\lambda \quad \text{because } V_{\alpha_y+1} \subseteq V_\lambda \\
\iff & x \subseteq V_\lambda \iff x \in V_{\lambda+1}
\end{aligned}
$$

$\Leftarrow$ also holds

$\blacksquare$