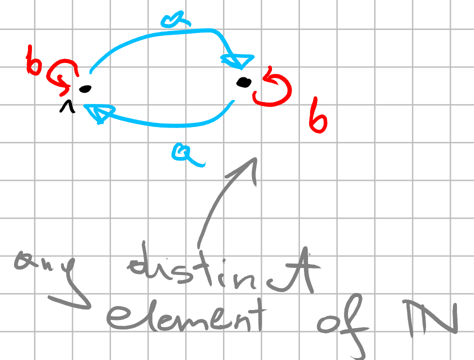
Ex. 9 $\quad G = \{ f \in Sym(\mathbb{N}) : \{ i : f(i) \neq i \} \text{ is finite} \}$

To show that $G^\omega/\mathcal{U}$ has a copy of $F(2)$ we need to show (by item 8) that for any $n \in \mathbb{N}$ there are $a_n, b_n \in G$ s.t. for any $w(x,y)$ irreducible $|w| \leq n$ we have $G \models w(a_n, b_n) \neq id$.

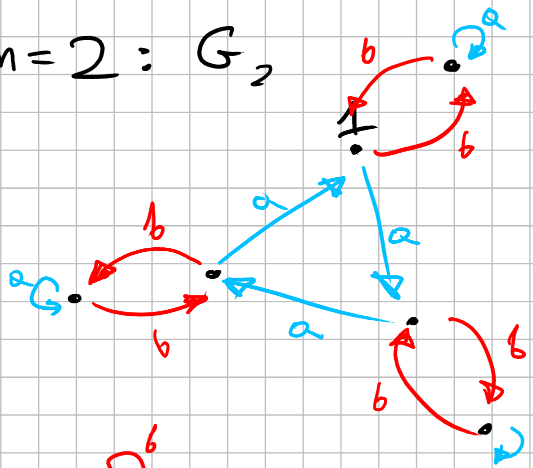
To show this we'll find such a_n, b_n that for any irreducible word $w(x,y)$ of length $\leq n$

$$G \models \underbrace{w(a_n, b_n)(1)}_{\text{this is a permutation}} \neq 1 \implies w(a_n, b_n) \neq id$$

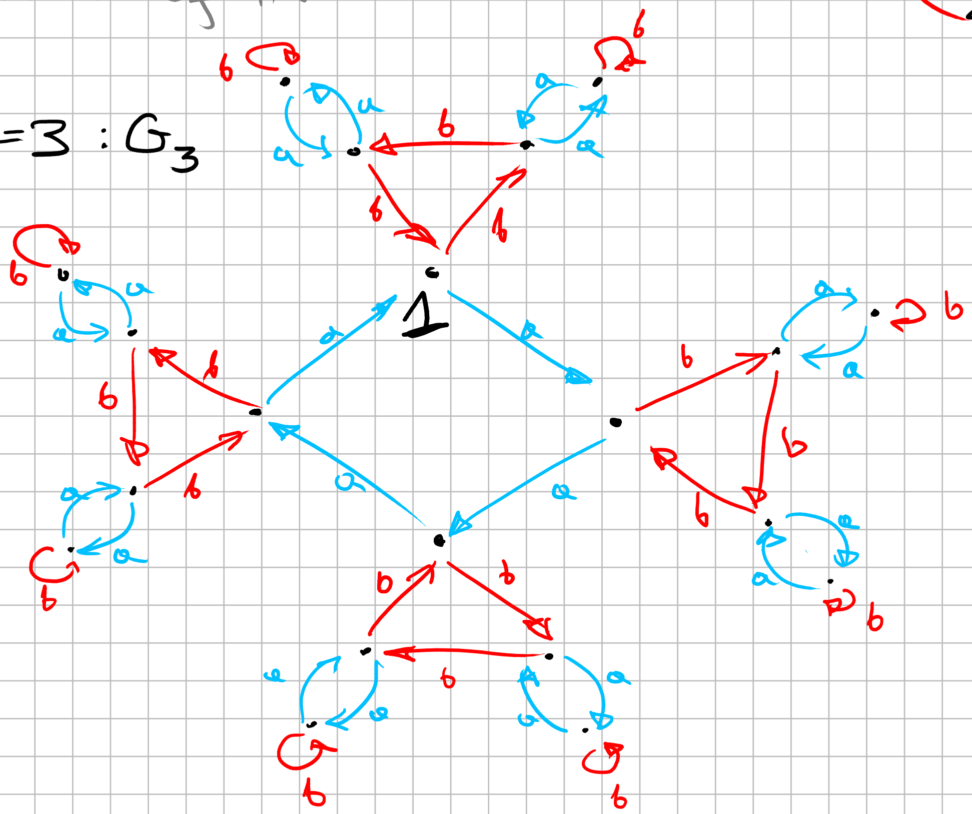
$n = 1 : G_1$



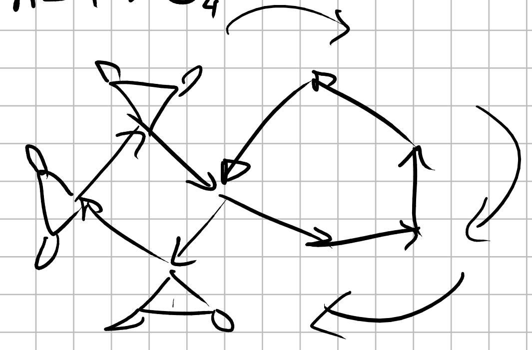
$n = 2 : G_2$



$n = 3 : G_3$

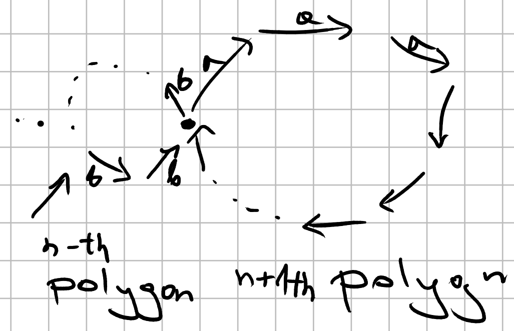


$n=4$: G_4



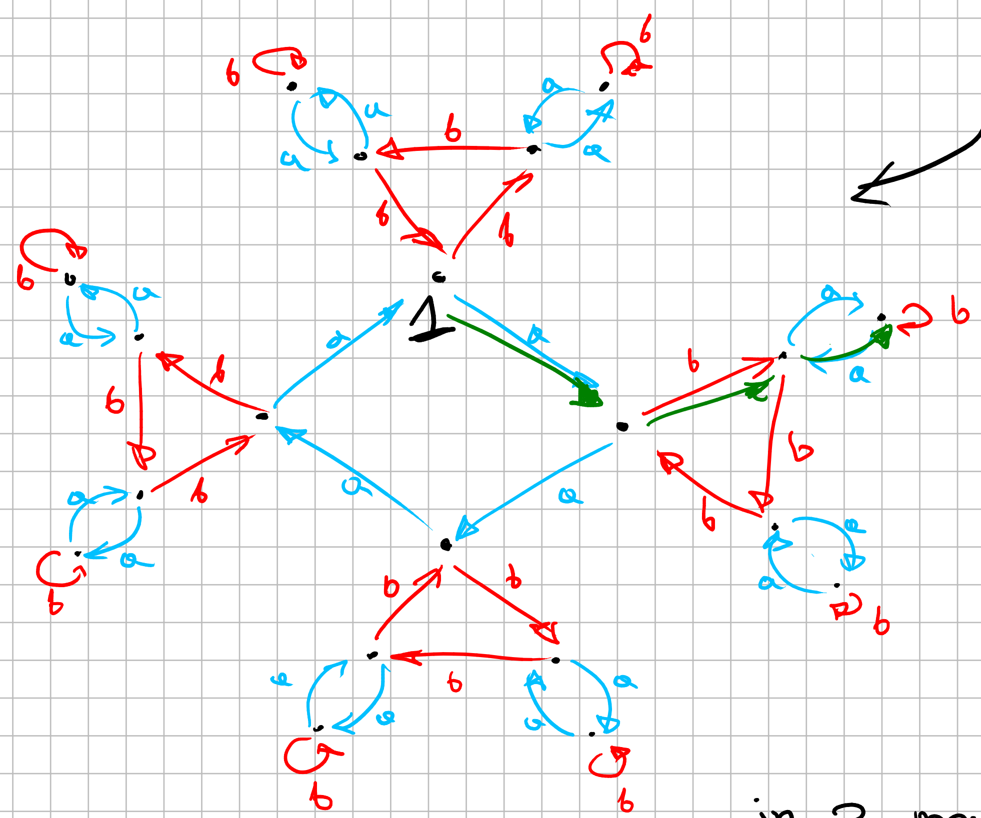
In general:

G_n



Let's focus on $n=3$. For example $w(x,y) = xyx^{-1}$

aba^{-1}



Only way to get 1 back in 3 moves is $w(x,y)=y^3$.

<u>Lemma</u> In G_n you need at least n moves to get 1 back and it happens only for $w(x,y)=y^n$. Otherwise you need $n+1$ moves.

Proof: We will prove it by induction.

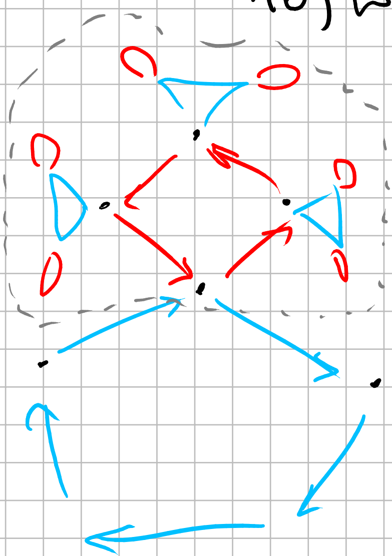
For $n=1$ we see it's true.

Suppose it works for G_{n-1}. Now look at G_n. How can we get 1 back to 1 with ≥ 1 moves?

1) $w(x,y) = y \cdot w'(x,y)$

1a) $w(x,y) = y^n$, then it is as in lemma statement

1b) $w'(x,y) \neq y^{n-1}$, then we can think that we are in G_{n-1} and look at word $v(x,y) = x \cdot \overline{w'(x,y)}$,



G_3 without one "leaf"

by ind. hyp. we need $(n-1)+1 = n$ moves to get to 1.

change x's to y's and vice versa

2) $w(x,y) = x \cdot w'(x,y)$, then

2a) $w(x,y) = x^{n+1}$, then OK

2b) $w(x,y) = \underbrace{x^k}_{\geq 1} \cdot \underbrace{y \cdot w'(x,y)}_{\substack{\geq n \\ \text{by ind. hyp.}}} \cdot \underbrace{x^{-k} x^{n+1-k}}_{\geq 1}$ OK