ex. 3a) $\mathcal{L} = (P_n)_{n<\omega}$, $T$ - an $L$-theory of independent predicates. Show that $T$ admits elimination of quantifiers.

Proof Take any $\varphi(\bar{x}) \in \mathcal{F}_L$. Let $M \models T$.

We'll show that $M \models \varphi(\bar{x}) \longleftrightarrow \psi(\bar{x})$,

where $\psi(\bar{x})$ is q.f. formula.

First let's say that all predicate symbols that appear in $\varphi(\bar{x})$ are within $P_1, \dots, P_n$. We define:

- for $J \subseteq \{1, \dots, n\}$, $P_J(x) : \bigwedge_{i \in J} P_i(x) \wedge \bigwedge_{\substack{i \notin J \\ 1 \leq i \leq n}} \neg P_i(x)$
- for $\bar{J} = (J_1, \dots, J_k)$, $\bar{x} = (x_1, \dots, x_k)$

$$P_{\bar{J}}(\bar{x}) : \bigwedge_{1 \leq i \leq k} P_{J_i}(x_i)$$

- for a matrix $E \in 2^{k \times k}$ we

define $q_E(\bar{x})$ : "equalities and inequalities given by $E$"

for example if $E = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then

$$q_E(x_1, x_2) : x_1 = x_1 \wedge x_1 = x_2 \wedge x_2 = x_1 \wedge x_2 \neq x_2$$

Some of those matrices are "silly", but this doesn't concern us.

- finally, we define configuration for $\bar{J}, E, \bar{x}$ as $c_{\bar{J},E}(\bar{x}) : p_{\bar{J}}(\bar{x}) \wedge q_E(\bar{x})$.

<u>Fact</u> For any $\bar{a} \subseteq M$ there's $\bar{J}, E$ s.t. $M \models c_{\bar{J},E}(\bar{a})$

Let $M' = (M, P_1, \dots, P_n)$ be a structure similar to $M$, but interpreting only the predicate symbols $P_1, \dots, P_n$. Clearly, for any $\psi(\bar{x})$ built only from those symbols $M' \models \psi(\bar{x}) \iff M \models \psi(\bar{x})$.

<u>Lemma</u> For any $\bar{a}, \bar{b} \subseteq M$, $M \models c_{\bar{J},E}(\bar{a})$ and $M \models c_{\bar{J},E}(\bar{b})$ for some $\bar{J}, E$, there is $f \in \mathrm{Aut}(M')$ s.t. $f(a_i) = b_i$, $1 \leq i \leq k$.

Proof It's simple, so I'll leave it and attach it after the end of this excersice.

Now we can prove the main thesis.

Recall To show that $M \models \varphi(\bar{x}) \to \psi(\bar{x})$ it suffices to show that for any $\bar{a} \subseteq M$
$M \models \varphi(\bar{a}) \implies M \models \psi(\bar{a})$.

Let $\mathcal{J} = \{(\bar{J}, E) : M \models C_{\bar{J},E}(\bar{x}) \to \varphi(\bar{x})\}$.
We'll show that $M \models \varphi(\bar{x}) \leftrightarrow \underbrace{\bigvee_{(\bar{J},E)\in\mathcal{J}} C_{\bar{J},E}(\bar{x})}_{q.f.}$

1) $M \models \varphi(\bar{x}) \leftarrow \bigvee_{(\bar{J},E)\in\mathcal{J}} C_{\bar{J},E}(\bar{x})$

Take any $\bar{a} \subseteq M$ s.t. $M \models \bigvee_{(\bar{J},E)\in\mathcal{J}} C_{\bar{J},E}(\bar{a})$.

Then there are $(\bar{J}, E) \in \mathcal{J}$ s.t. $M \models C_{\bar{J},E}(\bar{a})$.
But $M \models C_{\bar{J},E}(\bar{a}) \to \varphi(\bar{a})$, so
$M \models \varphi(\bar{a})$.

2) $M \models \varphi(\bar{x}) \longrightarrow \bigvee_{(\bar{J},E)\in\mathcal{Y}} C_{\bar{J},E}(\bar{x})$

Take any $\bar{a} \subseteq M$ s.t. $M \models \varphi(\bar{a})$, thus $M' \models \varphi(\bar{a})$. Take $\bar{J}, E$ s.t. (by the Fact) $M \models C_{\bar{J},E}(\bar{a})$. We want to show that $(\bar{J},E) \in \mathcal{Y}$, i.e. $M \models C_{\bar{J},E}(\bar{x}) \to \varphi(\bar{x})$.

Take any $\bar{b} \subseteq M$ s.t. $M \models C_{\bar{J},E}(\bar{b})$. By our lemma there is $f \in Aut(M')$ s.t. $f(a_i) = b_i$, $1 \leq i \leq k$. By the definition of isomorphism

$$M \models \varphi(\bar{a}) \iff M' \models \varphi(\bar{a})$$
$$\iff M' \models \varphi(f(\bar{a}))$$
$$\iff M' \models \varphi(\bar{b})$$
$$\iff M \models \varphi(\bar{b}).$$

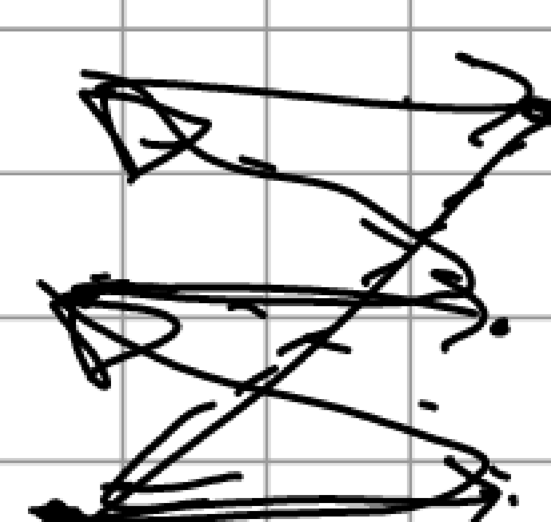
Thus $M \models C_{\bar{J},E}(\bar{x}) \longrightarrow \varphi(\bar{x})$, so $(\bar{J},E) \in \mathcal{Y}$. Thus $M \models \bigvee_{(\bar{J},E)\in\mathcal{Y}} C_{\bar{J},E}(\bar{a})$, so $M \models \varphi(\bar{x}) \longrightarrow \bigvee_{\bar{J},E} C_{\bar{J},E}(\bar{x})$.

By 1) and 2) $M \models \varphi(\bar{x}) \longleftrightarrow \bigvee_{(\bar{J},E)\in y} C_{\bar{J},E}(\bar{x})$.
We've chosen $M$ arbitrarly, so it follows that $T \vdash \varphi(\bar{x}) \longleftrightarrow \bigvee_{(\bar{J},E)\in y} C_{\bar{J},E}(\bar{x})$.

$\blacksquare$

<u>Proof of lemma</u> If $M \models C_{\bar{J},E}(\bar{a})$ then we know exactly which predicates are fulfilled and which elements of $\bar{a}$ are equal. The same follows for $\bar{b}$, so we can simply take $f : |M| \to |M|$,

$$f(x) = \begin{cases} b_i & \text{if } x = a_i \text{ for some } 1 \leq i \leq k \\ a_i & \text{if } x = b_i \quad \text{— " —} \\ x & \text{if } x \notin \bar{a},\ x \notin \bar{b} \end{cases}$$



this is wrong,
I'm not sure how to formalize this, but I think it's clear how to construct the iso.