Ex. 1 $T \models \varphi \overset{def.}{\iff} \forall M \models T \ (M \models \varphi)$

Prove that $T \vdash \varphi \iff T \models \varphi$

Proof "$\Rightarrow$" We'll prove it by induction on the length of φ's proof in T.

Inductive basis: proof is of length 1, which means:

$1°$ φ is a KRL axiom, $\vdash \varphi$. We know that $\vdash \varphi \Rightarrow \models \varphi$, so $\forall M \models T$ we have $M \models \varphi$

$2°$ $\varphi \in T$, so $M \models \varphi$ follows from definition of „$\models$".

Inductive step: suppose $\alpha_1, \dots, \alpha_n = \varphi$ is φ's proof in T and that our thesis is true for $\alpha_1, \dots, \alpha_{n-1}$. Then we have three cases:

$1°$ φ is a KRL axiom or $\varphi \in T$. This we already covered.

2° φ inferes from MP for $i,j<n$ s.t.
$\alpha_j : \alpha_i \to \varphi$. Also α_j is equivalent to $\varphi \vee \neg\alpha_i$. By inductive assumption $M \models \alpha_i$ and $M \models \alpha_j$. Moreover $M \models \varphi$ or $M \models \neg\alpha_i$. We know $M \not\models \neg\alpha_i$, so it is true that $M \models \varphi$.

3° φ inferes from $\forall$-rule, i.e. $\varphi : \forall v \alpha_i$ for some $i<n$. By inductive assumption $M \models \alpha_i$. But $M \models \overline{\alpha_i}$ (elementall closure), but this is also elementol closure of $\forall v \alpha_i$, so $M \models \forall v \alpha_i$ (by our assumption that all formulas are equiv. to e.c.).

"$\Leftarrow$" (A.a.) suppose $T \models \varphi$, but $T \nvdash \varphi$.
This means that $T \cup \{\neg\varphi\} =: T'$ is consistent, which means there is some model M s.t. $M \models T' \Rightarrow M \models T$

$$\begin{array}{ccc} M \models T' & \Rightarrow & M \models T \\ \Downarrow & & \Downarrow \\ M \not\models \varphi & & M \models \varphi \end{array} \quad \lightning$$

Ex. 2 „$\cong \Rightarrow \equiv$"

Proof Suppose $F: M \xrightarrow{\cong} N$ is an iso. for models M and N. We'll show that for all $t \in T_L$ we have $F(t^M) = t^N$. We'll do it by induction on the complexity of t:

- $t = c$ for some constant symbol $c \in L$.

  Then, by the definition of isomorphism:

  $F(t^M) = F(c^M) = c^N$.

- $t = f_i(t_1, \dots, t_k)$. Then $F(t^M) = F(f_i^M(t_1^M, \dots, t_k^M))$

  $\overset{\text{def. of} \cong}{=} f_i^N(F(t_1^M), \dots, F(t_k^M)) \overset{\text{ind.}}{=} f_i^N(t_1^N, \dots, t_k^N) =$

  $= t^N$

Now we'll show that $\forall \varphi \in F_L \; (M \models \varphi \Leftrightarrow N \models \varphi)$.

Once again, we'll use induction on the complexity of formula φ:

- $\varphi: t_1 = t_2$. Then $M \models \varphi \Leftrightarrow t_1^M = t_2^M$

  $\Leftrightarrow F(t_1^M) = F(t_2^M) \Leftrightarrow t_1^N = t_2^N$

- $\varphi : P_i(t_1, \dots, t_k)$. Then $\mathcal{M} \models \varphi \iff \langle t_1^{\mathcal{M}}, \dots, t_k^{\mathcal{M}} \rangle \in P_i^{\mathcal{M}}$

$\overset{\cong}{\underset{\text{by def.}}{\iff}} \langle F(t_1^{\mathcal{M}}), \dots, F(t_k^{\mathcal{M}}) \rangle \in P_i^{\mathcal{N}} \iff \langle t_1^{\mathcal{N}}, \dots, t_k^{\mathcal{N}} \rangle \in P_i^{\mathcal{N}}$

$\iff \mathcal{N} \models \varphi$.

- $\varphi : \varphi_1 \wedge \varphi_2, \neg \psi, \exists v \psi$ follows easily from the definition of $\models$.

$\blacksquare$

Ex. 46. We'll prove the thesis by transposison: Suppose $M \not\equiv N$, i.e. there is φ s.t. $M \models \varphi$ but $N \not\models \varphi$. Wlog. φ is a formula in prenex normal form: $Q_1 v_1 \dots Q_n v_n \psi$, where ψ is a formula with no quantifiers and $Q_i \in \{\forall, \exists\}$.

Now we'll play $\Gamma_n(M, N)$ as spoiler. At step i:

- if $Q_i = \exists$, then take $a_i \in M$ s.t.
$$M \models Q_{i+1} v_{i+1} \dots Q_n v_n \psi(v_1/a_1, \dots, v_i/a_i)$$
- if $Q_i = \forall$, then take $b_i \in N$ s.t.
$$N \not\models Q_{i+1} v_{i+1} \dots Q_n v_n \psi(v_1/a_1, \dots, v_i/a_i)$$

We can always pick like this. At step 1 we know that $M \models \varphi$, $N \not\models \varphi$. If $Q_1 = \exists$ then we can of course find $a_1 \in M$ s.t. $Q_2 v_2 \dots Q_n v_n \psi(v_1/a_1) \models M$. Observe, that $N \not\models \varphi$, so no matter what prover chooses, $N \not\models Q_2 v_2 \dots Q_n v_n \psi(v_1/b_1)$. When $Q_1 = \forall$

then we need to take b_1 that is a wittness of falseness of φ in N. Once again, no matter what prover picks, $Q_2 v_2 \dots Q_n v_n \varphi(^{v_1}/_{a_1})$ is true in M. This way we ensure that on every step our formula is true in M, but not in N. That means that $f: \{a_1, \dots, a_n\} \longrightarrow \{b_1, \dots, b_n\}$ cannot be an isomorphism because of φ being true in structure induced by $\{a_1, \dots, a_n\}$ and false in structure induced by $\{b_1, \dots, b_n\}$.

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